Midterm 2: Study Guide

Textbook coverage:

1.7: Sequences and Difference Equations: Computation, Equilibria, Cobwebbing. Examples 3, 4, 6, 8, 9.

2.1: Rates of Change and Tangent Lines: Average and Instantaneous Rates of Change, Units, Equations of Tangent Lines. Examples 1, 4-6.

2.2: Limits. Techniques: numerical esimation, graphing calculator zoom-in; One-sided limits. Matching Limits Theorem. Examples 1, 4, 5, 7.

2.3: Limit laws and Continuity. Continuity of Elementary Functions (Plug-in Principle – Theorem 2.2); Algebra techniques; Intermediate Value Theorem. Examples 1, 2, 4, 6-9, 10 (use trace function rather than bisection).

2.4: Asymptotes and Infinity; Vertical and Horizontal Asymptotes. Examples 1, 2, 4, 6.

2.5: Sequential Limits. Choosing sufficiently large n; Monotone Convergence Theorem. Examples 1-4, 6, 8 (with corrections; see problem 29 done in class).

2.6: Derivative at a Point; Examples 1, 3, 5-8.

2.7: Derivative as Function; Mean-Value Theorem. Intervals of Increase and Decrease. Examples 3, 4, 5, 6, Problems 29-32.

3.1: Derivatives of Polynomials, Fractional Powers (including radicals) and Exponentials; Examples 1, 2, 5, 7, online homework examples.

Review Questions

1. Find and graph the first five terms for the sequences. Find the limits of the sequences if they exist:

   (a) $\frac{1}{n}\cos\left(\frac{\pi n}{2}\right)$; (b) $1 - \frac{(-1)^n}{n+1}$.

2. (a) Find the limit L of the sequence a_n, as $n \to \infty$:

$$a_n = \frac{2n}{3+n}.$$

   Determine how large n needs to be to ensure that $|a_n - L| \leq 0.01$.

(b) Show that $\lim_{n\to\infty} a_n = \infty$. Determine how large n needs to be to ensure that $a_n \geq 10,000,000$:

$$a_n = \frac{n^2}{1+n}.$$

3. Consider the sequence defined by the difference equation:

$$a_{n+1} = f(a_n), \quad a_1 = 1.1, \quad \text{where } f(x) = 3x^2/(2+x^2).$$

(a) Find the first five terms of the sequence a_n.

(b) Find all equilibria and sketch a cobwebbing diagram for the given value a_1.

(c) Is the function $f(x)$ continuous and increasing? Justify your answer.

(d) Is the sequence a_n increasing or decreasing? Justify your answer.

(e) Use the monotone convergence theorem to determine the limit of the sequence: find a closed interval $[A, B]$ that is transformed by f into itself, contains the initial data a_1 and the relevant equilibrium value.

4. Determine the limits in the examples below. If the limits do not exist, explain why. Show your work!

(a) $\lim_{x\to-3+} \frac{|x+3|}{x+3}$ (c) $\lim_{x\to2} \frac{\sqrt{x+2}-2}{x-2}$ (e) $\lim_{x\to\infty} \frac{3x-x^2}{2-3x+x^2}$.

(b) $\lim_{x\to-3} \frac{|x+3|}{x+3}$ (d) $\lim_{x\to3} \frac{2x^2-5x-3}{x^2-8x+15}$ (f) $\lim_{x\to-\infty} \frac{3e^x-e^{-x}}{e^{2x}+e^{-x}}$.

5. (a) Find the value that needs to be assigned to c, if any, to guarantee that f will be continuous for all $x > 0$:

$$f(x) = \begin{cases} \frac{50}{1+x}, & x < c \\ 2, & x \geq c. \end{cases}$$

(b) Determine the value that needs to be assigned to k, if any, in order for the function f to be (i) continuous (ii) differentiable for all x:

$$f(x) = \begin{cases} x^2-1, & x \leq 1 \\ k(x-1), & x \geq 1. \end{cases}$$

6. (a) Use the intermediate value theorem to prove that the following equation has at least one solution:

$$1 + sin(x) + x^3 = 0.$$

(b) Use a root finding procedure in a graphing calculator to find the solution from part (a), accurate to four decimal places.

7. (a) Find the derivative $f'(a)$ by definition ("the long way") or using derivative rules ("the short way"):

   (i) $f(x) = x^3 + 2$, $a = 2$; (ii) $f(x) = \sqrt{x}$, $a = 9$.

   (b) For the examples in part (a) find the tangent lines to the graphs at $x = a$.

8. Given
$$f(x) = e^{5x}, \quad a = 0.1.$$

   (a) Estimate the value $f'(a)$ numerically, *using the definition of the derivative.* [Give answer accurate to two decimal places.]

   (b) Compare your answer in part (a) to the exact value $(e^{5x})'|_{x=0.1}$.

   (c) Find the equation of the tangent line to the graph $y = f(x)$ about $x = a$; sketch the tangent line and the graph.

9. Find the derivatives of the functions (use Derivatives Rules from Section 3.1):

   (a) $y = 1 + 5x + 3x^7$

   (b) $P = 10.2\,Q^{1/3}$

   (c) $N = 8.3(1.33)^t$

   (d) $y = 5\sqrt[3]{x} + \dfrac{7}{x\sqrt{x}}$

   (e) $y = \dfrac{3x - 5x^2}{x}$

   (f) $y = \dfrac{3 + e^{2x}}{e^x}$.

10. The function $P(t) = 8.33(1.33)^t$ models the population of the United States (expressed in millions of people) t decades after 1815.

   (a) Find the average rates of change of $P(t)$ over the intervals $[0, 2]$ and $[2, 4]$.

   (b) Find the instantaneous rate of change of $P(t)$ at $t = 2$.

   (c) Determine the units of the quantities computed in parts (a) and (b) and discuss their meaning.

11. The number of children newly infected with a particular pathogen has been modeled by the function
$$N(t) = -0.2t^3 + 3.04t^2 + 44.05t + 200.29,$$
where $N(t)$ is measured in thousands of individuals per year, and t is the number of years since the beginning of 2000.

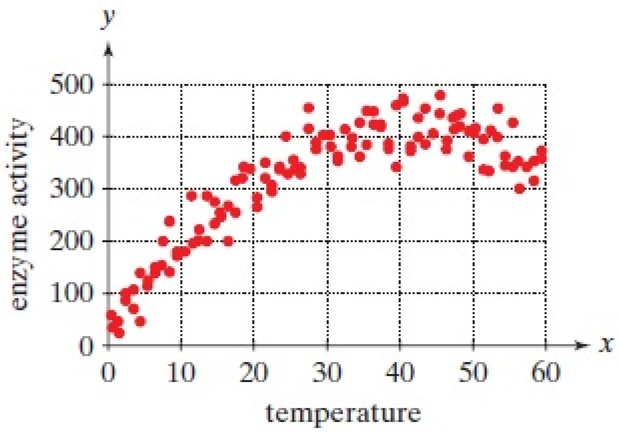
(a) At what rate is the function $N(t)$ changing with respect to time at the beginning of the year 2010?

(b) When will the number of infected children start to decline?

12. The data in the figure below represents a set of measurements relating enzyme activity to temperature in degrees Celsius; the quadratic equation

$$A(x) = 11.8 + 19.1x - 0.2x^2$$

provides a good fit to this data.



(a) Using the definition of the derivative find $A'(50)$.

(b) Find the same value using the rules of derivatives (sums/differences/powers, etc.)

(c) Discuss the meaning of the value $A'(50)$ in the context of this problem.

13. Find the derivatives of the functions and determine the intervals where the the functions are increasing and decreasing:

(a) $f(x) = x^3 - x^2 + 1$ (b) $g(x) = x - x^3$.

Answers:

1. (a) The limit is 0, the sequence is squeezed between the graphs of $y = 1/x$ and $y = -1/x$ $(x > 0)$. (b) The limit is 1, the sequence is squeezed between the graphs of $y = 1 + 1/(x + 1)$ and $y = 1 - 1/(x + 1)$.

2. (a) $L = 2$, $n \geq 597$; (b) $n \geq 1 + 10^7$.

3. (a) $a_n = 1.1, 1.1308, 1.1701, 1.2191, 1.2789$; (b) equilibria $a = 0, 1, 2$; hint: solve $a = 3a^2/(2+a^2)$; graph $y = x$ and $y = 3x^2/(2+x^2)$ using window $[0,3] \times [0,3]$; (c) continuous: rational function with denominator non-zero, increasing: graphing calculator, or solve the inequality $f(x_1) < f(x_2)$; (d) a_n is increasing; (e) $I = [1.1, 2]$ is transformed into itself (cobwebbing diagram is helpful!), the sequence a_n is contained in I and by the monotone convergence theorem a_n must converge to the equilibrium $a = 2$.

4. (a) 1; (b) limit does not exist (left and right limits are different); (c) 1/4; (d) -3.5; (e) -1; (f) -1.

5. (i) $c = 24$; (ii) (a) any k works; (b) $k = 2$.

6. (a) For instance: $f(0) > 0$, $f(-\frac{\pi}{2}) = -\left(\frac{\pi}{2}\right)^3 < 0$; by the intermediate value theorem there must be a value c in $(-\frac{\pi}{2}, 0)$ such that $f(c) = 0$; (b) $x \approx -0.705694$.

7. (a) 12; (b) 1/6.

8. (a)

h	0.01	-0.01	0.001	-0.001	0.0001	-0.0001
$\frac{e^{5(0.1+h)}-e^{0.5}}{h}$	8.45	8.04	8.26	8.25	8.24	8.24

$\Rightarrow f'(0.1) \approx 8.24$.

(b) $f'(0.1) = 5e^{0.5} \approx 8.2436$; the estimate in part (a) is indeed correct to two decimal places; (c) $y = e^{0.5}(5x + 0.5)$.

9. (a) $5 + 21x^6$; (b) $3.4\,Q^{-2/3}$; (c) $8.3\ln(1.33)(1.33)^t$; (d) $\frac{5}{3}x^{-\frac{2}{3}} - \frac{21}{2}x^{-\frac{1}{2}}$; (e) -5; (f) $-3e^{-x} + e^x$.

10. (a) 3.20, 5.66; (b) 4.20; (c) millions of people per year; these are average rates of growth from 1815 to 1835, from 1835 to 1855, and the estimate of the instantaneous rate of growth at the end of 1835.

11. (a) 44,850 of individuals per year; (b) 2014.

12. (a) $A'(50) = \lim_{h\to 0} \frac{A(50+h)-A(50)}{h} = -0.9$; (b) $A'(50) = 19.1 - 0.4 \cdot 50 = -0.9$.
(c) The enzyme activity decreases at a rate ≈ 0.9 units per °C as the temperature passes through 50°C.

13. (a) increasing on $(-\infty, 0)$ and $(\frac{2}{3}, \infty)$; decreasing on $(0, \frac{2}{3})$; (b) increasing on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$; decreasing on $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$.