Thu, Mar 14, 2019 **MATH 255A Midterm 2, Version 1** Prof. V. Panferov

Name: (print) ______________________

CSUN ID No. : Solutions.

This test includes 8 questions (46 points in total), on 9 pages. Last page is a formula sheet. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: No electronic devices except an approved model of graphing calculator. All cellphones must be off and put away completely for the duration of the exam. Show all your work.

1. (4 points) (a) If $f(x) = x^2 - 3x$ find the derivative $f'(x)$ *by defintion.*

$$f'(x) = \lim_{h\to 0} \frac{((x+h)^2 - 3(x+h)) - (x^2-3x)}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{x^2}+2xh+h^2-\cancel{3x}-3h)-(\cancel{x^2}-\cancel{3x})}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2-3h}{h} = \lim_{h\to 0} 2x+h-3$$

$$= (2x+h-3)\Big|_{h=0} = 2x-3.$$

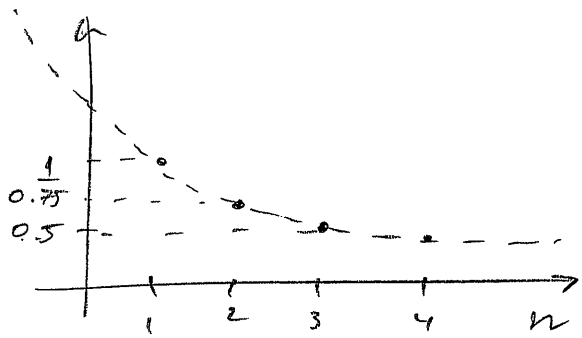
(b) Verify the answer in part (a) by using the Derivative Rules.

$$(x^2-3x)' = (x^2)' - 3(x)' =$$

$$= 2x^1 - 3x^0 = 2x-3$$

2. (6 points) Consider the sequence $a_n = \dfrac{3}{2+n}$.

(a) Illustate by a graph, showing the values a_1, a_2, a_3, a_4.



$a_1 = \frac{3}{3} = 1$

$a_2 = \frac{3}{4} = 0.75$

$a_3 = \frac{3}{5} = 0.6$

$a_4 = \frac{3}{6} = 0.5$

(b) Find $\lim\limits_{n\to\infty} a_n$. Show complete work using the limit rules and the fact that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

$$\lim_{n\to\infty} \frac{3}{2+n} = \lim_{n\to\infty} \frac{3/n}{2/n + 1}$$

$$= \frac{\lim 3/n}{\lim 2/n + 1} = \frac{3 \lim 1/n}{2 \lim 1/n + 1}$$

$$= \frac{3 \cdot 0}{2 \cdot 0 + 1} = 0$$

(c) Determine how large n needs to be to ensure that a_n is within 0.01 from the limit.

$a_n \leq 0.01$ (since $a_n > 0$ always)

$\frac{3}{2+n} \leq 0.01$

$\frac{300}{2+n} \leq 1$

$300 \leq 2+n$

$298 \leq n$

$n \geq 298$

3. (6 points) Given the function

$$f(x) = \begin{cases} 4 - x^2, & x \le 1 \\ k + m(x-1), & x > 1. \end{cases}$$

(a) Determine all values k and m such that $f(x)$ is *continuous* at $x = 1$. Show all work.

$$\lim_{x \to 1-} f(x) = 4 - x^2 \Big|_{x=1} = 3$$

$$\lim_{x \to 1+} f(x) = k + m(x-1) \Big|_{x=1} = k$$

$\Rightarrow \quad k = 3;$ no restriction on m.

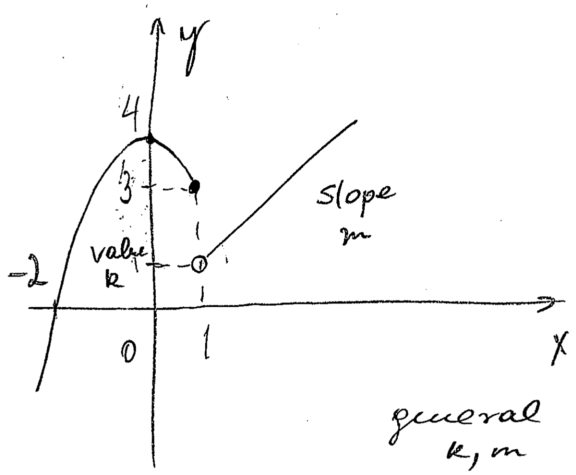
(b) Determine all values k and m such that $f(x)$ is *differentiable* at $x = 1$. Show all work.

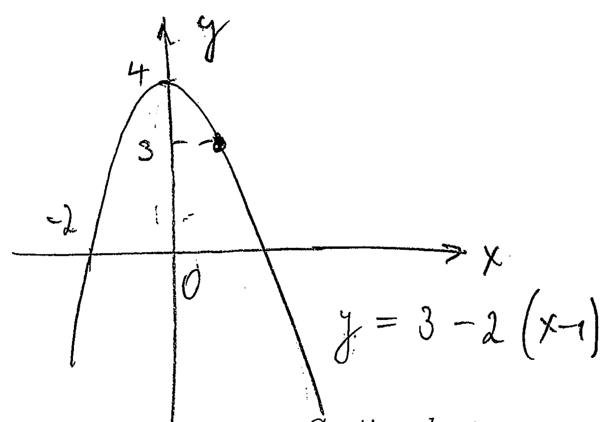
$$\lim_{x \to 1-} f'(x) = -2x \Big|_{x=1} = -2$$

$$\lim_{x \to 1+} f'(x) = m \Big|_{x=1} = m$$

$\Rightarrow \quad m = -2;$ $k = 3$ since f needs to be continuous in order to be differentiable.

(c) Sketch a graph of f for the values k, m in part (b).





4. (6 points) An enviromental study suggests that the level of NO_2 pollution in the air is modelled by the function

$$P(t) = 78.2 + 3.2t - 0.04t^2 \quad [\text{ parts per billion (ppb) }]$$

t days after the start of observation.

(a) Find the derivative $\dfrac{dP}{dt}$.

$$\frac{dP}{dt} = \left(78.2 + 3.2t - 0.04t^2\right)'$$
$$= 0 + 3.2 - 0.08t$$
$$= 3.2 - 0.08t$$

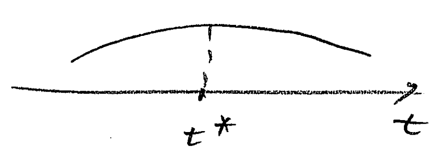
(b) Find the value $\left.\dfrac{dP}{dt}\right|_{t=45}$, specify units and interpret the meaning of the obtained value.

$$\left.\frac{dP}{dt}\right|_{t=45} = 3.2 - 0.08 \cdot 45 = 3.2 - 3.6 = -0.4 \ \frac{ppb}{day}$$

After 45 days the level of pollution is decreasing at a rate of a 0.4 $\frac{ppb}{day}$.

(c) When will the level of pollution start to decline? Use the derivative to justify your answer.

$P(t)$:



quadratic fn with leading order coefficient < 0.

$$\frac{dP}{dt} = 3.2 - 0.08t = 0$$

$$t = \frac{3.2}{0.08} = 40 \quad [\text{days}]$$

Continued...

$P(t)$ starts to decline after 40 days.

5. (6 points) (a) Estimate the limit numerically, filling in the values in the table:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}.$$

x	0.1	-0.1	0.01	-0.01	0.001	-0.001	0.0001	-0.0001
y	0.4996	0.4996	0.5	0.5	0.5	0.5	0.5	0.5

Based on the data in the table, the limit is estimated to be 0.5

(b) Find the limit using algebra:

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}.$$

$$\frac{\sqrt{x+1}-1}{x} = \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} = \frac{(x+1)-1}{x(\sqrt{x+1}+1)}$$

$$= \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}$$

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} =$$

$$= \left.\frac{1}{\sqrt{x+1}+1}\right|_{x=0} = \frac{1}{1+1} = \frac{1}{2}.$$

6. (6 points) (a) Use the Intermediate Value Theorem to prove that the equation

$$x^3 + \sin x - 1 = 0$$

has at least one real solution. Show your reasoning completely.

$f(x) = x^3 + \sin x - 1$

$f(0) = -1 < 0$

$f(1) = 1 + \sin(1) - 1 = \sin(1) > 0$

$f(x)$ is continuous for x in $[0, 1]$

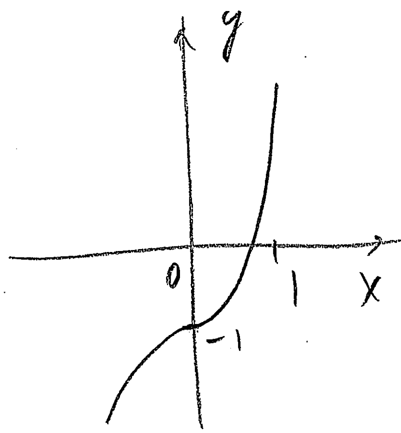
$f(0) < 0$, $f(1) > 0$

Therefore for a certain value c in $(0, 1)$

$f(c) = 0$.

(b) Find the solution using calculator, accurate to three decimal places. Show the steps.

Graph



Zoom in

Trace

x	y
0.70248	-0.00723
0.70592	$5.17 \cdot 10^{-4}$
0.70561	$-1.89 \cdot 10^{-4}$
0.70584	$3.4 \cdot 10^{-4}$

$\Rightarrow c \cong 0.7057$

Calc zero $\Rightarrow$ $c \cong 0.7056937$

Continued...

7. (6 points) Use the Derivative Rules to find the derivatives of the functions. (Simplify before differentiating if possible.)

(a) $y = x^2(4 - x)$

$$y = 4x^2 - x^3$$

$$\frac{dy}{dx} = 8x - 3x^2$$

(b) $y = \dfrac{3x^{1/3}}{\sqrt{x^2}}$

$$y = \frac{3x^{\frac{1}{3}}}{x^1} = 3x^{\frac{1}{3}-1} = 3x^{-\frac{2}{3}}$$

$$\frac{dy}{dx} = 3 \cdot \left(-\frac{2}{3}\right) \cdot x^{-\frac{2}{3}-1} = -2x^{-\frac{5}{3}}$$

(c) $L = 1.12\, W^{0.95}$

$$\frac{dL}{dW} = 1.12 \cdot 0.95 \cdot W^{0.95-1}$$

$$= 1.064 \cdot W^{-0.05} = \frac{1.064}{W^{0.05}}$$

8. (6 points) Given the difference equation:

$$a_{n+1} = 0.5a_n + 100, \quad a_1 = 0.$$

(a) Find a_2, a_3, a_4.

$a_1 = 0$

$a_2 = 0.5 \cdot 0 + 100 = 100$

$a_3 = 0.5 \cdot 100 + 100 = 150$

$a_4 = 0.5 \cdot 150 + 100 = 175$

(b) Find all equilibria and sketch a cobwebbing diagram for the sequence.

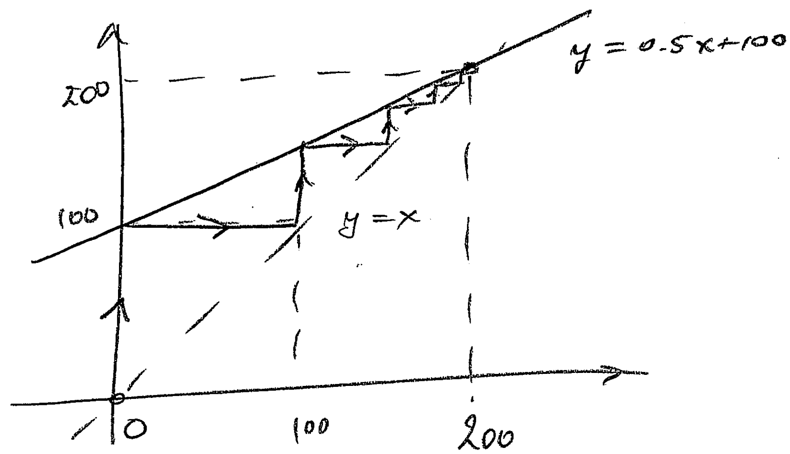
$a = 0.5a + 100$

$0.5a = 100$

$a = \frac{100}{0.5}$

$a = 200$

equilibrium.



Geometrically, $a_n \to 200$ as $n \to \infty$.

(c) Use the Monotone Convergence Theorem to determine $\lim_{n\to\infty} a_n$ (fill in the blanks):

The function $f(x) = 0.5x + 100$ on the interval $[A, B] =$ $[0, 200]$ is

✓ i. continuous

✓ ii. increasing

iii. transforms the interval $[A, B]$ into $[0, 200]$. (or "onto itself")

The only equilibrium in $[A, B]$ is 200.

Therefore, the limit of a_n must be 200.

Table of formulas

Difference Equations: $\begin{cases} a_{n+1} = f(a_n), \\ a_1 - \text{given.} \end{cases}$ Equilibrium: value a such that $a = f(a)$.

Cobwebbing diagram: On the same set of axes:

(i) Graph $y = x$.
(ii) Graph $y = f(x)$.
(iii) Label all points of equilibria.
(iv) Start at (a_1, a_1).
(v) Go vertically to the graph.
(vi) Go horizontally to the line $y = x$.
(vii) Repeat Steps (v) and (vi).

Limit Rules:

$$\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} f(x)$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \left(\lim_{x\to a} g(x) \neq 0\right)$$

Derivative at x: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Derivative Notations: $f'(x) = \frac{dy}{dx}$; $f'(a) = f'(x)\Big|_{x=a} = \frac{dy}{dx}\Big|_{x=a}$

Continuous at $x = a$: $\lim_{x\to a} f(x) = f(a)$. Differentiable at $x = a$: $f'(a)$ exists.

Intermediate Value Theorem: If $f(x)$ is continuous on $[A, B]$, and L is a y-value strictly between $f(A)$ and $f(B)$ then for some c in (A, B) we must have $f(c) = L$.

Derivative Rules:

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$f_1(x) \pm f_2(x)$	$f_1'(x) \pm f_2'(x)$
$cf_1(x)$	$cf_1'(x)$
e^{mx}	me^{mx}
b^x	$(\ln b)\, b^x$

The end.