MATH 255A HOMEWORK – SECTION 3.5

#6. Find the linear approximation; graph the function and the L.A. determine whether L.A. is over- or under-estimate.

$$y = xe^{-x}; \quad x_0 = \ln 2 \approx 0.69315$$

$$y_0 = \ln 2 \cdot e^{-\ln 2} = \frac{1}{2}\ln 2 \approx 0.34657$$

$$\frac{dy}{dx} = (xe^{-x})' = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

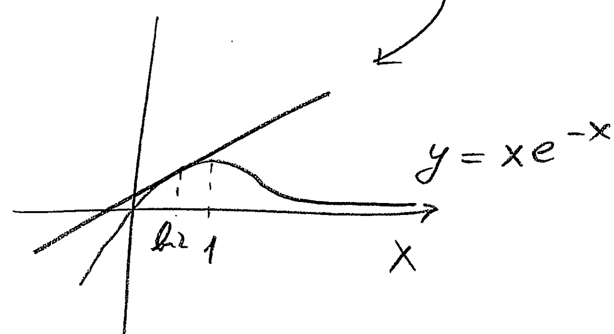
$$m = \left.\frac{dy}{dx}\right|_{x=\ln 2} = (1-\ln 2)e^{-\ln 2} = \frac{1}{2}(1-\ln 2) \approx 0.15343$$

Linear approximation:

$$y = y_0 + m(x-x_0) = \frac{1}{2}\ln 2 + \frac{1}{2}(1-\ln 2)(x-\ln 2)$$

$$= 0.34657 + 0.15343(x-0.69315)$$

Graph:



$y = L(x)$ is an over-estimate.

#10. Estimate using linear approx. near $x = x_0$:

$$\cos\left(\frac{\pi}{2} + 0.01\right); \quad x_0 = \frac{\pi}{2}; \quad y_0 = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f(x) = \cos x; \quad f'(x) = -\sin x; \quad m = f'(x_0) = -\sin\frac{\pi}{2} = -1.$$

$$\cos\left(\frac{\pi}{2} + 0.01\right) \approx y_0 + m(x - x_0) = 0 + (-1)\cdot(0.01) = -0.01$$

Calculator: ≈ -0.0099998.

#20. Find elasticity of y to x and use it to estimate the percent error for y:

$$y = \sqrt{2x^2+1}\,; \quad x_0 = -2; \quad \delta x = 8\%$$

$$f'(x) = \frac{2x}{\sqrt{2x^2+1}}\,; \quad f'(x_0) = \frac{-4}{3}$$

$$E = f'(x_0)\,\frac{x_0}{f(x_0)} = -\frac{4}{3}\cdot\frac{(-2)}{3} = \frac{8}{9}.$$

$$\delta y \approx E\,\delta x = \frac{8}{9}\cdot 8\% \approx 7.11\%$$

#24

$$y = \frac{1}{1+x}\,; \quad x_0 = 0\,; \quad \delta x = 12\%$$

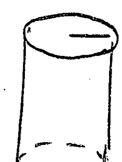
$$f'(x) = -\frac{1}{(1+x)^2}\,; \quad f'(x_0) = -1$$

$$E = f'(x_0)\,\frac{x_0}{f(x_0)} = (-1)\,\frac{0}{1} = 0$$

$$\delta y \approx E\cdot\delta x = 0.$$

Actually, this problem has a flaw, since $x_0 = 0 \Rightarrow \delta x = \frac{\Delta x}{x_0}$ is undefined. "12% error" is meaningless when the x-value measured is zero.

#28. r - accurate within 1%: $\delta r = \frac{\Delta r}{r_0} = 1\%$.



$h = 4.75''$

$$V = \pi\cdot h\cdot r^2 = 4.75\cdot\pi\, r^2$$

$$V'(r) = 9.5\cdot\pi\cdot r$$

$$E = \frac{9.5\cdot\pi\cdot r_0\cdot r_0}{4.75\cdot\pi\cdot r_0^2} = \frac{9.5}{4.75} = 2$$

(3)

$$\delta V \simeq E \delta r = 2 \cdot 1\% = 2\%$$

The obtained estimate of the volume is 2% accurate.

#4.

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

etc.

$$f^{(103)}(x) = \left(f^{(100)}(x)\right)''' = (\cos x)''' = \sin x.$$

#10.

$f(x) = \dfrac{4}{\sqrt{x}}$; $f'(x) = \left(4x^{-\frac{1}{2}}\right)' = 4\cdot\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$

$f''(x) = 4\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$

$f'''(x) = 4\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-\frac{7}{2}}$

$f^{(4)}(x) = 4\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-\frac{9}{2}} = \dfrac{3\cdot 5\cdot 7}{2\cdot 2}x^{-\frac{9}{2}}$

$= \dfrac{105}{4}\cdot x^{-\frac{9}{2}}.$

#12.

$y = (x^2+4)(1-3x^3)$

$= -3x^5 - 12x^3 + x^2 + 4$

$\dfrac{dy}{dx} = -15x^4 - 36x^2 + 2x$

$\dfrac{d^2y}{dx^2} = -60x^3 - 72x + 2$

Same answer obtained using product rule:

$\dfrac{dy}{dx} = 2x(1-3x^3) - 9x^2(x^2+4)$

$\dfrac{d^2y}{dx^2} = 2(1-3x^3) - 9x^2\cdot 2x - 18x(x^2+4) - 9x^2\cdot 2x$

$= -60x^3 - 72x + 2$

#24. Find the linear approximation around $x_0 = 1$

Use $f''(x)$ to determine whether the linear approximation overestimates or underestimates y.

$f(x) = xe^{-x}$

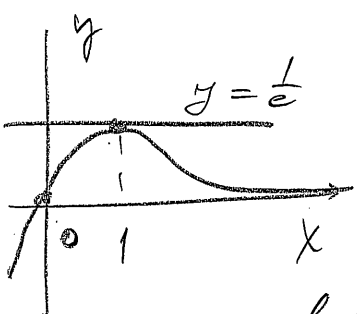
$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

$m = f'(x_0) = 0$

$y_0 = f(x_0) = 1 \cdot e^{-1} = \frac{1}{e}$

Linear approximation: $y = y_0 + m(x - x_0)$

$= \frac{1}{e} + 0(x-1) = \frac{1}{e}$.



$f''(x) = \left((1-x)e^{-x}\right)' = -e^{-x} - (1-x)e^{-x}$

$= (x-2)e^{-x}$

$f''(x_0) = -1 \cdot e^{-1} < 0$

$f(x)$ is concave down at $x = 1$

$\Rightarrow$ tangent line is above the graph

$\Rightarrow$ lin. approx is an overestimate.

#38. $y = \sec x$, $x_0 = 0$

Find the first (linear) and second (quadratic) approximations.

$$y_0 = \sec 0 = \frac{1}{\cos 0} = 1$$

$$f'(x) = \frac{dy}{dx} = (\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x$$

$$f''(x) = \frac{d^2y}{dx^2} = (\sec x \tan x)' = \sec x \cdot \tan^2 x + \sec x \cdot \sec^2 x$$

$$= \sec x \cdot \tan^2 x + \sec^3 x.$$

Linear approx:

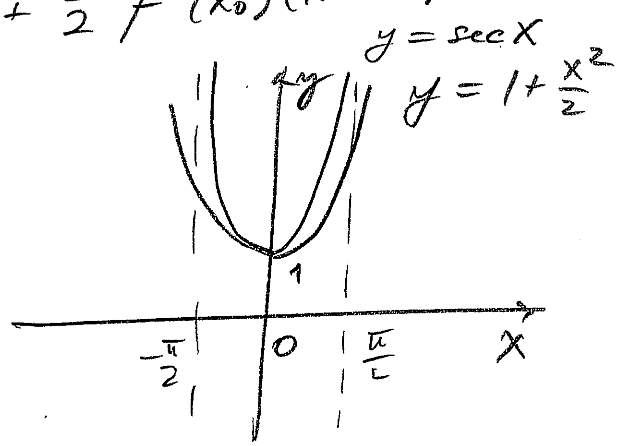
$$y = y_0 + m(x - x_0) = 1 + 0(x-0) = 1$$

since $m = f'(x_0) = 0$

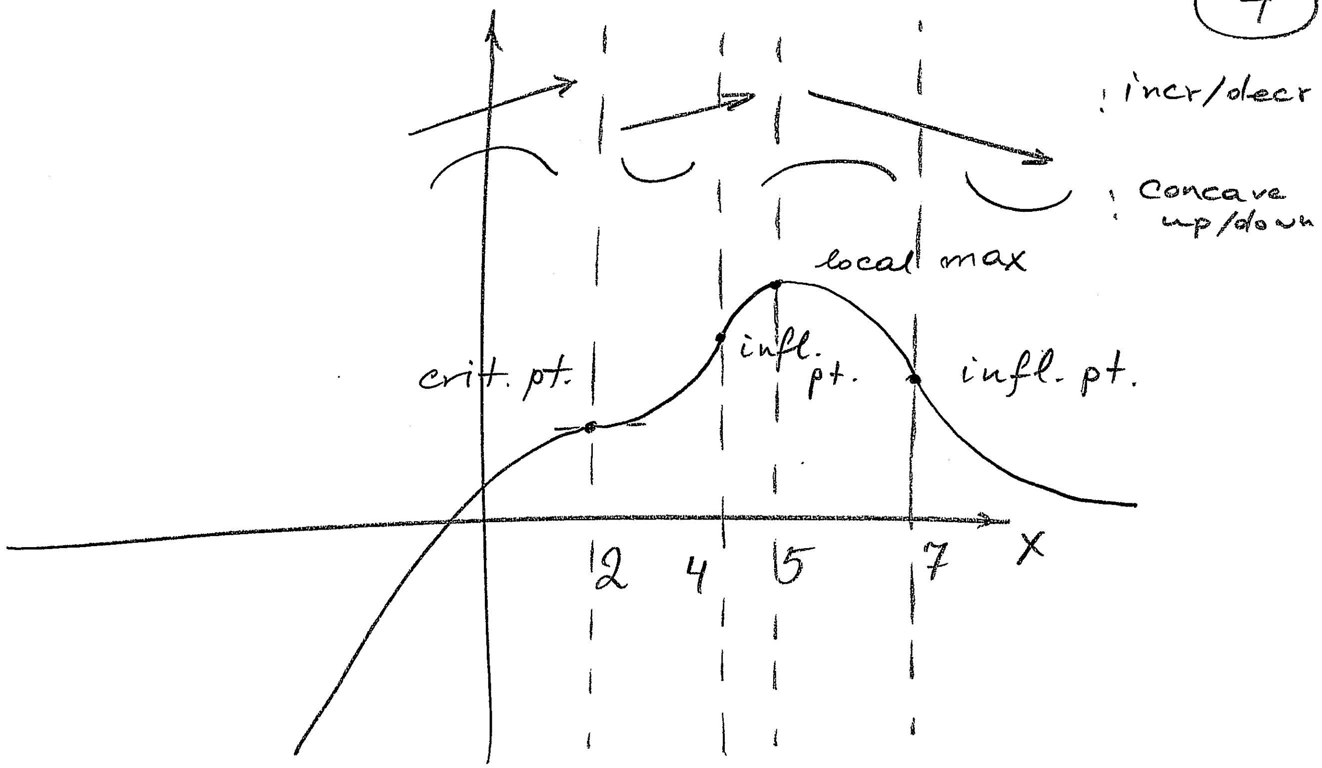
Quadratic approx: $f''(x_0) = 1$

$$y = L(x) + \frac{1}{2} f''(x_0)(x-x_0) = 1 + \frac{x^2}{2}$$

Graph:



#46.


incr/decr
Concave up/down
local max
crit. pt.
infl. pt.
infl. pt.
2
4
5
7
x