MATH 255A

HOMEWORK - SECTION 2.4

#14.

$$\lim_{x\to\infty} \frac{2x^2-5x+7}{x^2-9} = \lim_{x\to\infty} \frac{x^2\left(2-\frac{5}{x}+\frac{7}{x^2}\right)}{x^2\left(1-\frac{9}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{2-\frac{5}{x}+\frac{7}{x^2}}{1-\frac{9}{x^2}} = \frac{2-0+0}{1-0} = 2.$$

#16.

$$\lim_{Q\to\infty} \frac{aQ^2+Q}{1-Q^2} = \lim_{Q\to\infty} \frac{Q^2\left(a+\frac{1}{Q}\right)}{Q^2\left(\frac{1}{Q^2}-1\right)}$$

$$= \lim_{Q\to\infty} \frac{a+\frac{1}{Q}}{\frac{1}{Q^2}-1} = \frac{a+0}{0-1} = -a.$$

#36. How large does x need to be?

$$f(x) = e^x+5 \quad ; \quad f(x) > 1{,}000{,}000 = 10^6$$

$$e^x+5 > 10^6$$

$$e^x > 10^6-5$$

$$x > \ln(10^6-5) = 13.816$$

#40. Height of beer froth:

$$H(t) = 17\cdot(0.99588)^t \quad [\text{cm}]$$

(a) $0.99588 < 1 \Rightarrow$ decaying exponential fn

$$\Rightarrow \lim_{t\to\infty} H(t) = 0.$$

(b) $H(t) > 0$, decays; $H(t) < 0.1$

$$0.99588^t < \frac{0.1}{17} = \frac{1}{170}$$

$$t > \ln\frac{1}{170} / \ln 0.99588 \approx 1243.98$$

(c) $H(t) < 0.01$; $t > \ln\frac{1}{1700} / \ln 0.99588 \approx 1801.71$

#18. How large does n need to be so $a_n \geq 1{,}000{,}000$

$$a_n = \frac{n^2}{1+n} \geq 10^6 \Rightarrow n^2 \geq 10^6(1+n)$$

$$n^2 - 10^6 n - 10^6 \geq 0$$

$$n \geq \frac{10^6}{2} + \sqrt{\left(\frac{10^6}{2}\right)^2 + 10^6} \simeq 1{,}000{,}001$$

#20. $a_{n+1} = f(a_n)$; $f(x) = \frac{x}{3}$; $a_1 = 27$

n	1	2	3	4	5	6	---
a_n	27	9	3	1	1/3	1/9	---

$$a_n = \frac{27}{3^{n-1}} = 81\left(\frac{1}{3}\right)^n$$

$$\lim_{n\to\infty} a_n = 81 \underbrace{\lim_{n\to\infty}\left(\frac{1}{3}\right)^n}_{=0 \text{; decaying exp. fn.}} = 0$$

#24. $a_{n+1} = f(a_n)$; $f(x) = 4x^2$, $x_1 = 1$

n	1	2	3	4	5	6	---
a_n	1	4	4^3	4^7	4^{15}	4^{31}	---

$$a_n = 4^{2^{n+1}-1}; \quad \lim_{n\to\infty} 2^{n+1}-1 = \infty \text{ (growing exp. fn.)}$$

$$\Rightarrow \lim_{n\to\infty} 4^{2^{n+1}-1} = \infty$$

#26.

$$a_{n+1} = \frac{1}{5-a_n}, \quad a_1 = 1$$

Equilibria: $\frac{1}{5-x} = x$

$$1 = 5x - x^2$$

$$x^2 - 5x + 1 = 0$$

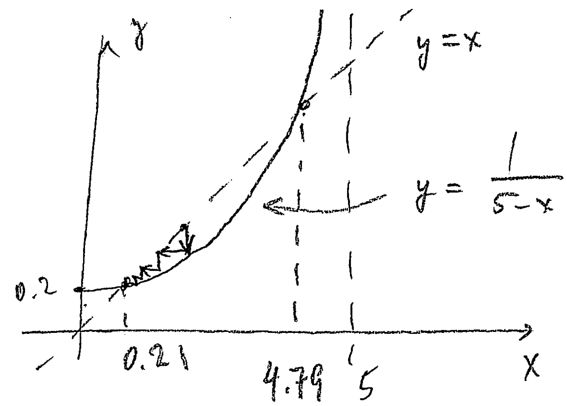
$$x = \frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 1} = \frac{5}{2} \pm \frac{\sqrt{21}}{2}$$

$$x = 0.20871 \text{ or } x = 4.79129$$

n	
1	1
2	0.25
3	0.210526
4	0.208791
5	0.208716
6	0.208712

Appears to converge to

$$\frac{5}{2} - \frac{\sqrt{21}}{2}.$$



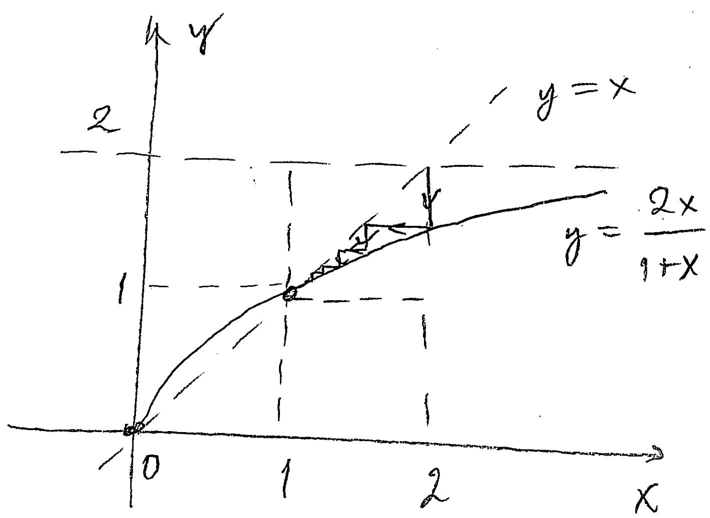
#30. $a_{n+1} = \frac{2a_n}{1+a_n}$, $a_1 = 2$.

Use the Monotone Convergence Theorem to determine the limit.

Equilibria: $f(x) = \frac{2x}{1+x}$; $x = \frac{2x}{1+x}$

$x = 0$ or $1 = \frac{2}{1+x}$

$1 + x = 2$; $x = 1$



* The interval $I = [1, 2]$ is transformed into itself
* $f(x)$ is continuous on $[1, 2]$ (denom. $\neq 0$)
* $f(x)$ is incr. on $[1, 2]$

MCT: The sequence must converge to an equilibrium in $[1, 2]$ $(a = 1)$

Thus, $\lim_{n \to \infty} a_n = 1$.