Midterm 2: Study Guide

Textbook coverage:

2.1: Rates of Change and Tangent Lines: Average and Instantaneous Rates of Change, Units, Equations of Tangent Lines. Examples 1, 4-6.

2.2: Limits. Techniques: numerical esimation, graphing calculator zoom-in; One-sided limits. Matching Limits Theorem. Examples 1, 4, 5, 7.

2.3: Limit laws and Continuity. Continuity of Elementary Functions (Plug-in Principle – Theorem 2.2); Algebra techniques; Intermediate Value Theorem. Examples 1, 2, 4, 6-9, 10 (use trace function rather than bisection).

2.4: Asymptotes and Infinity; Vertical and Horizontal Asymptotes. Examples 1, 2, 4, 6.

2.6: Derivative at a Point; Examples 1, 3, 5-8.

2.7: Derivative as Function; Mean-Value Theorem. Intervals of Increase and Decrease. Examples 3, 4, 5, 6, Problems 29-32.

3.1: Derivatives of Powers (including radicals), sums/differences, constant multiples (including polynomials) and Exponentials; Examples 1, 2, 5, 7, online and paper homework.

Review Questions

1. (see also **2.2**: 13-15) Estimate the limit numerically, using a table of values:

$$\lim_{t\to 0} \frac{e^{-3x}-1}{x}.$$

2. (see also **2.1**: 7-12, 23-36) (a) Estimate the derivative $f'(a)$ numerically, using a table of values:

$$f(x) = \sin(x), \quad a = \frac{\pi}{4}$$

(b) Find the equation for the tangent line to the graph at $(a, f(a))$. Graph both the function and the tangent line on the same plot.

3. (see also **2.7**: 1-7) Use the definition of derivative to find $f'(x)$:

(a) $f(x) = 3x + 7$ (b) $f(x) = \dfrac{1}{x^2}$ (c) $f(x) = \sqrt{3+x}$

For each of the cases above find $f'(1)$.

4. (see also **2.5**: 25-38) (a) Find the limit L of the function $f(x)$ as $x \to \infty$:

$$f(x) = \frac{6x}{x-3}.$$

Determine how large x needs to be to ensure that $|f(x) - L| \le 0.01$.

(b) Show that $\lim_{x\to\infty} f(x) = \infty$. Determine how large x needs to be to ensure that $f(x) \ge 10,000,000$:

$$f(x) = \frac{x^2}{1+x}.$$

5. (see also **2.2**: 17, 21, 22, 23, 24) Determine the limits in the examples below. If the limits do not exist, explain why. Show your work!

(a) $\lim_{x\to -3+} \frac{|x+3|}{x+3}$ (c) $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$ (e) $\lim_{x\to\infty} \frac{3e^x - e^{-x}}{e^x + e^{-x}}$.

(b) $\lim_{x\to -3} \frac{|x+3|}{x+3}$ (d) $\lim_{x\to 3} \frac{2x^2-5x-3}{x^2-8x+15}$ (f) $\lim_{x\to -\infty} \frac{3e^x - e^{-x}}{e^x + e^{-x}}$.

6. (see also **2.3**: 1, 2, 15; **2.4**: 13, 14)

Let $f(x) = \begin{cases} 2-2x, & \text{if } x < 1 \\ \frac{1-x}{x-4}, & \text{if } x > 1 \text{ and } x \neq 4. \end{cases}$

Find the following limits:

(a) $\lim_{x\to 1-} f(x)$ (c) $\lim_{x\to 4-} f(x)$ (e) $\lim_{x\to -\infty} f(x)$.

(b) $\lim_{x\to 1+} f(x)$ (d) $\lim_{x\to 4+} f(x)$ (f) $\lim_{x\to\infty} f(x)$.

Without calculating the derivative $f'(x)$ answer the following: On which intervals within the domain is the function $f(x)$ increasing or decreasing?

7. (see also **2.3**: 41, 42) Determine the values that need to be assigned to k and to $f(4)$, in order for the function f to be continuous everywhere:

$$f(x) = \begin{cases} kx+1, & x < 4 \\ 10, & x > 4. \end{cases}$$

Sketch a graph of the function for this value k.

8. (see also **2.6**: 27, 29, 30) Determine the value that needs to be assigned to k, if any,

in order for the function f to be (i) continuous (ii) differentiable for all x:

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x-1), & x > 1. \end{cases}$$

9. (see also **2.3**: 23-28)

   (a) Use the intermediate value theorem to prove that the following equation has at least one solution:
   $$1 + \sin(x) + x^3 = 0.$$

   (b) Use a root finding procedure in a graphing calculator to find the solution from part (a), accurate to four decimal places.

10. Given
   $$f(x) = e^{5x}, \quad a = 0.1.$$

   (a) Estimate the value $f'(a)$ numerically, using a table of values. Give answer accurate to two decimal places.

   (b) Compare your answer in part (a) to the exact value $(e^{5x})'|_{x=0.1}$.

   (c) Find the equation of the tangent line to the graph $y = f(x)$ about $x = a$; sketch the tangent line and the graph.

11. (see also **3.1**: 5-14) Find the derivatives of the functions. (Use Derivative Rules from Section 3.1. If necessary, simplify function before differentiation.)

   (a) $y = 1 + 5x + 3x^7$

   (b) $P = 10.2\,Q^{1/3}$

   (c) $N = 8.3(1.33)^t$

   (d) $y = 5\sqrt[3]{x} + \dfrac{7}{x\sqrt{x}}$

   (e) $y = \dfrac{3x - 5x^2}{x}$

   (f) $y = \dfrac{3 + e^{2x}}{e^x}$.

12. The function $P(t) = 8.33(1.33)^t$ models the population of the United States (expressed in millions of people) t decades after 1815.

   (a) Find the average rates of change of $P(t)$ over the intervals $[0, 2]$ and $[2, 4]$.

   (b) Find the instantaneous rate of change of $P(t)$ at $t = 2$.

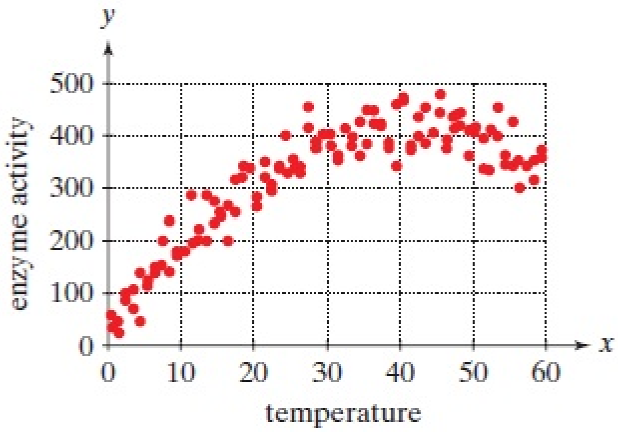
   (c) Determine the units of the quantities computed in parts (a) and (b) and discuss their meaning.

13. (see also **2.7**: 17-22) Determine at what point in the interval $[0.5, 1.5]$ the instantaneous rate of change of the function $f(x) = x^2 - x$ equals the average rate of change over that interval. Provide a sketch that illustrates this relationship.

14. The data in the figure below represents a set of measurements relating enzyme activity to temperature in degrees Celsius; the quadratic equation

$$A(x) = 11.8 + 19.1x - 0.2x^2$$

provides a good fit to this data.



(a) Using the definition of the derivative find $A'(50)$.

(b) Find the same value using the rules of derivatives (sums/differences/powers, etc.)

(c) Discuss the meaning of the value $A'(50)$ in the context of this problem.

15. An enviromental study performed in a suburban community determines that the average level of carbon monoxide (CO) in the air t years after beginning of 2014 is modeled by the formula

$$L(t) = 0.05t^2 - 0.1t + 3.4 \quad \text{[parts per million]}.$$

(a) Find the instantaneous rate of change of L with respect to time after two years. [OK to use derivative rules from Section 3.1]

(b) State the units for the value obtained in part (a). Is the level of CO increasing or decreasing after two years?

(c) By how much will the level of CO change during the third year?

16. (see also **3.1**: 15-19) Find the derivatives of the functions and <u>use them</u> to determine the intervals where the functions are increasing or decreasing:

(a) $f(x) = 100x - e^x$

(b) $g(x) = 6 + 3x - x^3$.

Answers:

1. -3.

2. $f'(\frac{\pi}{4}) \approx 0.7071$; $y = 0.7071x - 0.15175$.

3. (a) 3; (b) $-2/x^3$; $1/(2\sqrt{3+x})$. Plug in $x = 1$: (a) 3; (b) -2; (c) $1/4$.

4. (a) $L = 6$, $n \geq 1803$; (b) $n \geq 1 + 10^7$.

5. (a) 1; (b) limit does not exist (left and right limits are different); (c) $1/6$; (d) -3.5; (e) 3; (f) -1.

6. (a) 0; (b) 0; (c) $+\infty$ (d) $-\infty$ (e) $-\infty$ (f) -1.

7. $k = 2.25$, $f(4) = 10$.

8. (a) any k works; (b) $k = 2$.

9. (a) For instance: $f(0) > 0$, $f(-\frac{\pi}{2}) = -\left(\frac{\pi}{2}\right)^3 < 0$; by the intermediate value theorem there must be a value c in $(-\frac{\pi}{2}, 0)$ such that $f(c) = 0$; (b) $x \approx -0.705694$.

10. (a)

h	0.01	-0.01	0.001	-0.001	0.0001	-0.0001
$\frac{e^{5(0.1+h)}-e^{0.5}}{h}$	8.45	8.04	8.26	8.25	8.24	8.24

$\Rightarrow f'(0.1) \approx 8.24$.

(b) $f'(0.1) = 5e^{0.5} \approx 8.2436$; the estimate in part (a) is indeed correct to two decimal places; (c) $y = e^{0.5}(5x + 0.5)$.

11. (a) $5 + 21x^6$; (b) $3.4\,Q^{-2/3}$; (c) $8.3\ln(1.33)(1.33)^t$; (d) $\frac{5}{3}x^{-\frac{2}{3}} - \frac{21}{2}x^{-\frac{1}{2}}$; (e) -5; (f) $-3e^{-x} + e^x$.

12. (a) 3.20, 5.66; (b) 4.20; (c) millions of people per year; these are average rates of growth from 1815 to 1835, from 1835 to 1855, and the estimate of the instantaneous rate of growth at the end of 1835.

13. $x = 1$.

14. (a) $A'(50) = \lim_{h\to 0} \frac{A(50+h)-A(50)}{h} = -0.9$; (b) $A'(50) = 19.1 - 0.4 \cdot 50 = -0.9$.
(c) The enzyme activity decreases at a rate ≈ 0.9 units per °C as the temperature passes through 50°C.

15. 0.1 ppm per year; (b) the level of CO is increasing; (c) 0.15 ppm.

16. (a) increasing on $(-\infty, 4.6052)$ decreasing on $(4.6052, \infty)$; (b) decreasing on $(-\infty, -1)$ and on $(1, \infty)$; increasing on $(-1, 1)$.