Thu, Oct 17, 2019 **MATH 255A Midterm 2, Version 2** Prof. V. Panferov

Name: (print) ______________________________

CSUN ID No. : Solutions.

This test includes 7 questions (40 points in total), on 7 pages. Page 8 is a formula sheet. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. No electronic devices except an approved model of graphing calculator. All cellphones must be off and put away completely for the duration of the exam. Show all your work.

1. (4 points) Estimate the derivative $f'(a)$ numerically, using a table of values:

$$f(x) = 5^x, \quad a = 0.$$

Compare the obtained estimate with the exact value obtained using Derivative Rules.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{5^h - 1}{h}$$

L_1 (=h)	$(5^{L_1} - 1)/L_1$
0.1	1.7462
0.01	1.6225
0.001	1.6107
0.0001	1.6096
−0.0001	1.6093
−0.001	1.6081
−0.01	1.5966
−0.1	1.4866

Estimate: (half-sum of 1.6096 and 1.6093)

limit $\approx$ 1.60945

Exact:

$$f'(x) = (\ln 5) 5^x$$

$$f'(0) = \ln 5 \approx 1.60944$$

The estimate was accurate to 4 decimal places.

2. (6 points) Find the limits *exactly*. Show work. If the limits do not exist provide reasons for your answer.

(a) $\lim_{x\to 0} \frac{|5x|}{x}$

$$f(x) = \frac{|5x|}{x} = \begin{cases} \frac{5x}{x}, & x>0 \\ \frac{-5x}{x}, & x<0 \end{cases} = \begin{cases} 5, & x>0 \\ -5, & x<0 \end{cases}$$

$$\lim_{x\to 0+} f(x) = 5, \quad \lim_{x\to 0-} f(x) = -5$$

One sided limits are different, the limit does not exist.

(b) $\lim_{x\to+\infty} \sqrt{x^2+3x} - x$

$$\sqrt{x^2+3x} - x = \frac{(\sqrt{x^2+3x} - x)(\sqrt{x^2+3x} + x)}{\sqrt{x^2+3x} + x}$$

$$= \frac{x^2+3x-x^2}{\sqrt{x^2+3x} + x}$$

$$= \frac{3x}{\sqrt{x^2+3x} + x} = \frac{3}{\sqrt{1+\frac{3}{x}} + 1}$$

$\frac{3}{x} \to 0$

$$\longrightarrow \frac{3}{2} \text{ as } x \to +\infty$$

(c) $\lim_{x\to-\infty} \frac{100e^x}{e^x+e^{-1}}$.

$e^x \to 0$ as $x \to -\infty$

$$\frac{100e^x}{e^x+e^{-1}} \longrightarrow \frac{100 \cdot 0}{0+e^{-1}} = 0$$

(d) (bonus: 2 points) In the last example, how large and negative does x need to be so that $f(x) = \frac{100e^x}{e^x+e^{-5}}$ is within 0.01 from its limit as $x \to -\infty$?

e^{-1}

$$\frac{100e^x}{e^x+e^{-1}} < 0.01$$

$$\frac{e^x}{e^x+e^{-1}} < 10^{-4}$$

$$e^x < 10^{-4}(e^x + e^{-1})$$

$$(1-10^{-4})e^x < 10^{-4}e^{-1}$$

$$e^x < \frac{10^{-4}e^{-1}}{1-10^{-4}}$$

$$x < \ln \frac{10^{-4}e^{-1}}{1-10^{-4}}$$

$$x \lesssim -10.21$$

Continued...

3. (6 points) Given $f(x) = x^2 + x$.

(a) Find the derivative $f'(x)$ by definition.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - x^2 - x}{h}$$

$$\lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{x} + h - \cancel{x^2} - \cancel{x}}{h}$$

$$\lim_{h \to 0} 2x + h + 1 = 2x+1$$

(b) Find the average rate of change of f over the interval $[0, 2]$.

$$ARC = \frac{f(2) - f(0)}{2 - 0} = \frac{4 + 2 - 0}{2} = \frac{6}{2} = 3$$

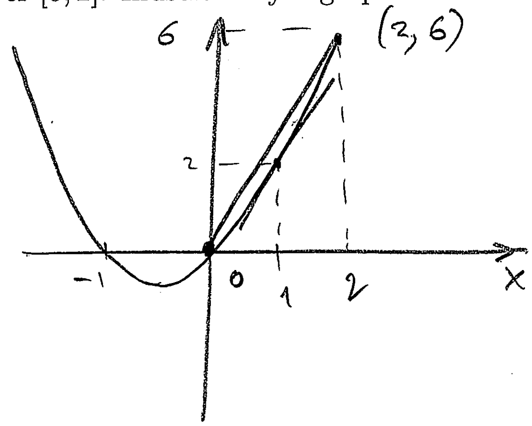
(c) Find the value c in the interval $(0, 2)$ at which the instantaneous rate of change of f is equal to the average rate of change over $[0, 2]$. Illustrate by a graph.

$$f'(x) = 3$$

$$2x+1 = 3$$

$$2x = 2$$

$$x = 1$$



secant through $(0,0)$ and $(2,6)$ is parallel to the tangent line through $(1, 2)$

Continued...

4. (6 points) An enviromental study suggests that the level of NO_2 pollution in the air is modelled by the function

$$P(t) = 78.2 + 3.2t - 0.04t^2 \quad [\text{ parts per billion (ppb) }]$$

t days after the start of observation.

(a) Find the derivative $\frac{dP}{dt}$.

$$\frac{dP}{dt} = 3.2 - 0.08t \qquad \left(\frac{ppb}{day}\right)$$

(b) Find the value $\left.\frac{dP}{dt}\right|_{t=30}$, specify units, and write a sentence to describe the meaning of the obtained value.

$$\left.\frac{dP}{dt}\right|_{t=30} = 3.2 - 0.08 \cdot 30 = 3.2 - 2.4 = 0.8 \quad \left[\frac{ppb}{day}\right]$$

The concentration of the pollutant is increasing at a rate of 0.8 ppb per day after 30 days since the beginning of observation.

(c) When will the level of pollution start to decline?

$$\frac{dP}{dt} = 0$$

$$3.2 - 0.08t = 0$$

$$t = \frac{3.2}{0.08} = 40 \ [\text{days}]$$

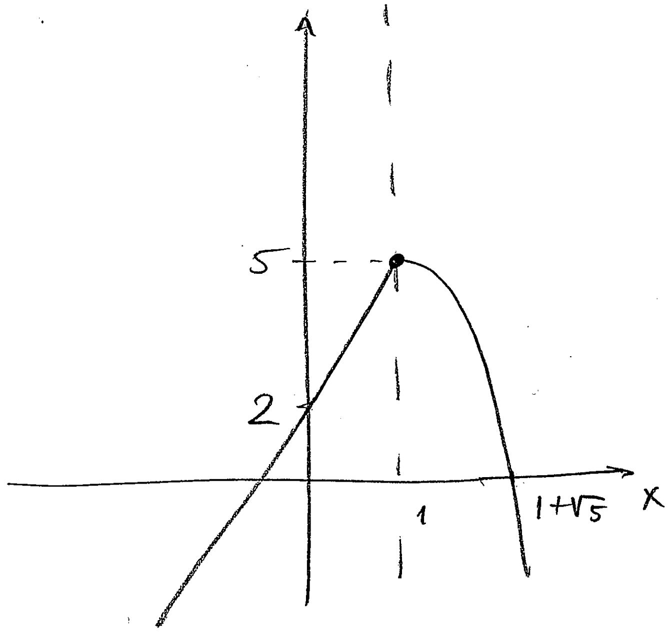
$P(t)$ - parabola with negative coef. at t^2

$\Rightarrow P(t)$ incr. on $[0, 40]$, decr on $(40, \infty)$.

5. (6 points) (a) Determine the values that need to be assigned to k and to $f(1)$ so that the function $f(x)$ would be continuous at $x = 1$:

$$f(x) = \begin{cases} 2 + kx, & x < 1 \\ 5 - (x-1)^2, & x \geq 1. \end{cases}$$

Sketch a graph of f for this value k.



$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 2 + kx$

$= 2 + k$

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} 5 - (x-1)^2$

$= 5.$

$2 + k = 5$

$k = 5 - 2$

$k = 3$

$f(1) = 5$

(b) Is the function f obtained in part (a) differentiable at $x = 1$? Write a sentence to justify your answer.

No. Differentiable means there's a well-defined tangent line; here the slope of the graph on the left is 3, while on the right it is 0.

6. (6 points) Find the derivative of the function $f(x)$ and use it to determine on which intervals is the function increasing or decreasing:

$$f(x) = 70 + 30x - e^x.$$

$$f'(x) = 0 + 30 - e^x$$
$$= 30 - e^x.$$

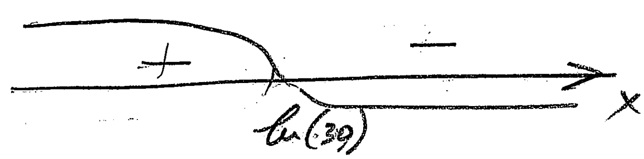
f increasing: $f'(x) > 0$

f decreasing: $f'(x) < 0$.

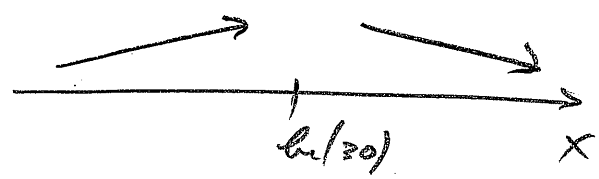
$f'(x) = 0$: $\quad 30 - e^x = 0$

$$e^x = 30$$
$$x = \ln(30) \simeq 3.40$$

f'



f:



incr: $(-\infty, \ln(30))$

decr: $(\ln(30), \infty)$.

7. (6 points) Use the Intermediate Value Theorem to prove that the following equation has at least one solution:

$$x^3 - 2x^2 - 8x = 3.$$

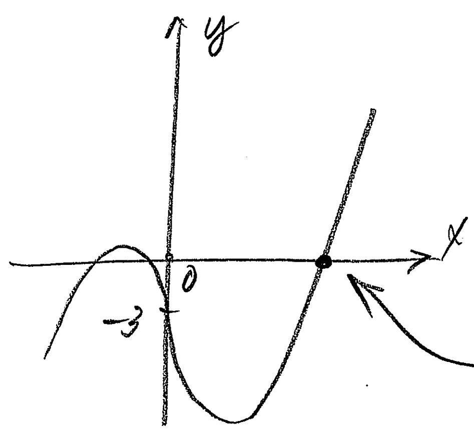
$f(x) = x^3 - 2x - 8x - 3$

$f(0) = -3$

$f(4.255) = 3.7962$ (used calculator, trace)

Since f is continuous on $[0, 4.255)$ by IVT there must be a value c on $[0, 4.255]$ such that $f(c) = 0$.

(b) Find the largest positive solution of the equation in part (a) accurate to three decimal places. Show your work!



Graph;

[Calc] → zero

returns

$\begin{cases} x = 4.1190267 \\ y = 0 \end{cases}$

Table of formulas

Average Rate of Change over $[a, b]$: $\text{ARC} = \dfrac{f(b) - f(a)}{b - a}$.

Instantaneous Rate of Change at a: $\text{IRC} = \lim_{b \to a} \frac{f(b)-f(a)}{b-a}$.

Derivative at x: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Derivative Notations:

$$f'(x) = \frac{dy}{dx}; \quad f'(a) = f'(x)\Big|_{x=a} = \frac{dy}{dx}\Big|_{x=a}$$

Derivative Rules:

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$f_1(x) \pm f_2(x)$	$f_1'(x) \pm f_2'(x)$
$cf_1(x)$	$cf_1'(x)$
e^{mx}	me^{mx}
b^x	$(\ln b)b^x$

Intermediate Value Theorem: If $f(x)$ is continuous on $[a, b]$, and L is a value strictly between $f(a)$ and $f(b)$ then for some c in (a, b) we must have $f(c) = L$.

Mean Value Theorem: If $f(x)$ is differentiable on (a, b), continuous on $[a, b]$ then for some c in (a, b) we must have

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The end.