MATH 265A 10/10/2019

Derivatives of Powers, Sums / Differences, Constant multiples, and Exponentials.

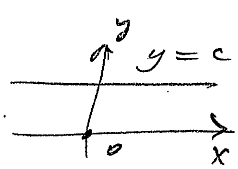
Objective: To establish a set of rules so that given a formula,

(for instance $f(x) = \sqrt{\dfrac{3x - \sin x^3}{e^{5x} + 2\ln x}} \cdots$)

we can produce a reasonable expression for the derivative $f'(x)$.

Examples Simplest cases

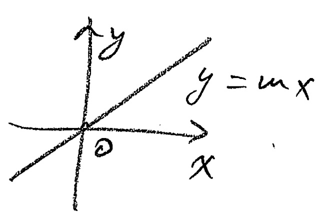
1) $f(x) = c$ (a const)



$\Rightarrow f'(x) = 0 \quad \left(= \lim\limits_{h\to 0} \dfrac{c-c}{h} \quad \dfrac{\text{"zero"}}{\text{nonzero}} \right)$

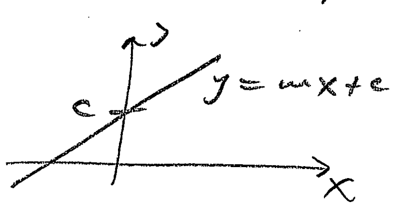
2) $f(x) = mx$ (linear)

$f'(x) = m \quad \left(= \lim\limits_{h\to 0} \dfrac{m(x+h) - mx}{h} \right.$



$= \lim\limits_{h\to 0} \dfrac{mh}{h} = m$

3) $f(x) = mx + c$



$f'(x) = m$ (check!)

4) $f(x) = x^2 \Rightarrow f'(x) = 2x$

(computed previously)

5) $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

(computed a similar case!)

General case of a power:

$$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$$

works for $n = 0, 1, 2, 3$ (checked above)

Applies generally for any exponent n integer, fractional, positive, or negative (even for something like $n = -\sqrt{2}$!!!)

<u>Derivation</u> when n - integer, > 0.

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + h^n.$$

— Binomial formula

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + nx^{n-1}h + \dots + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + \underbrace{\frac{n(n-1)}{2}x^{n-2}h}_{\to 0} + \dots + \underbrace{h^{n-1}}_{\to 0}$$

$$= nx^{n-1}$$

Example: $n = \frac{1}{2}$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2} \frac{1}{x^{\frac{1}{2}}} =$$

$$= \frac{1}{2} x^{-\frac{1}{2}}$$

So the rule ("pull the exponent out, reduce the exponent by 1")

works: $\left(x^{\frac{1}{2}}\right)' = \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-\frac{1}{2}}$.

$n = -1$

$$f(x) = x^{-1} = \frac{1}{x}$$

$f'(x) = -\frac{1}{x^2}$ (checked in one of the examples before).

Sums, differences, and constant multiples.

<u>Example</u>: $f(x) = x^3 - 3x$

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)] - [x^3 - 3x]}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} - 3 \lim_{h\to 0} \frac{(x+h) - x}{h}$$

$$= \dots = 3x^2 - 3$$

Sums / Differences can be split, constant multiples can be factored out:

$$(f_1 \pm f_2)' = f_1' + f_2'$$

$$(cf_1)' = cf_1'. \qquad (c\text{ – constant.})$$

<u>Example</u>: $f(x) = 9x^3 - \underbrace{6x\sqrt{x}} + 5\underbrace{\sqrt[3]{x}}$

Identify exponents: $x^1 \cdot x^{\frac{1}{2}}$ $\quad x^{\frac{1}{3}}$

$$f(x) = 9x^3 - 6x^{\frac{3}{2}} + 5x^{\frac{1}{3}}$$

Apply rules:

$$f'(x) = 9(x^3)' - 6(x^{\frac{3}{2}})' + 5(x^{\frac{1}{3}})'$$

$$= 9 \cdot 3 \cdot x^2 - 6 \cdot \tfrac{3}{2} \cdot x^{\frac{3}{2}-1} + 5 \cdot \tfrac{1}{3} x^{\frac{1}{3}-1}$$

$$= 27x^2 - 9x^{\frac{1}{2}} + \frac{5}{3}x^{-\frac{2}{3}}$$

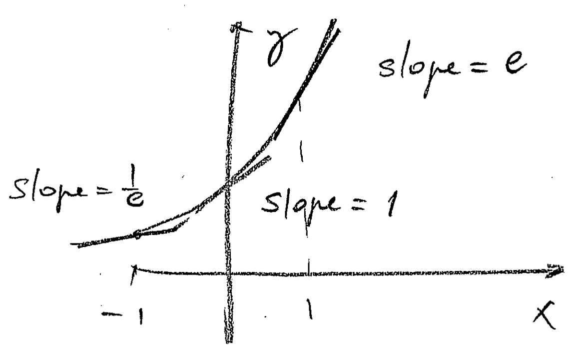
$$\left(= 27x^2 - 9\sqrt{x} + \frac{5}{3\sqrt[3]{x^2}} . \right)$$

Guideline for simplification: be able to use the final formula to compute values with a calculator.

$x^{\frac{1}{2}}$ is OK; $\sqrt{x}$ is a little simpler since it only requires 2 buttons [$\sqrt{\ }$] and [x].

Derivatives of Exponentials.

$$f(x) = e^x \Rightarrow f'(x) = e^x$$



For the exp. function
$y = e^x$
the "slope" of the graph is the same as the y-value (e^x).

Generally

$$f(x) = e^{mx} \Rightarrow f'(x) = m e^{mx}$$

Not the same as the power rule!!

The exponent "mx" is not pulled out, only "m" is!

The exponent "mx" is not reduced by 1!

Derivation $\quad f(x) = e^{mx}$

$$f'(x) = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{e^{m(x+h)} - e^{mx}}{h}$$

$$= \lim_{h \to 0} \frac{e^{mx} e^{mh} - e^{mx}}{h}$$

$$= \lim_{h \to 0} e^{mx} \frac{e^{mh} - 1}{h}$$

$$= e^{mx} \lim_{h \to 0} \frac{e^{mh} - 1}{h}$$

Use $mh = t \to 0 \Rightarrow h = t/m$

$$\lim_{h \to 0} \frac{e^{mh} - 1}{h} = \lim_{t \to 0} \frac{e^{t} - 1}{t/m}$$

$$= m \lim_{t \to 0} \frac{e^{t} - 1}{t}.$$

Estimate the limit $\lim_{t \to 0} \frac{e^t - 1}{t}$

numerically:

t	$(e^t-1)/t$
0.01	1.005
0.001	1.0005
0.0001	1.00005
−0.0001	0.99995
−0.001	0.9995
−0.01	0.995

$\Rightarrow \quad \frac{e^t - 1}{t} \longrightarrow 1$ as $t \to 0$.

Thus $f'(x) = e^{mx} \cdot m \lim_{h \to 0} \frac{e^t - 1}{t}$

$= m \cdot e^{mx}$.

Example: #13 $f(x) = Cx^2 + 5x + e^{-2x}$

$f'(x) = 2Cx + 5 - 2e^{-2x}$.

(C is a constant.)

#24: $f(x) = (e^x - 1)(e^x + 1)$

"product rule" not necessary!

$f(x) = (e^x)^2 - 1^2 = e^{2x} - 1$

$f'(x) = 2e^{2x}$

Exponentials with arbitrary base

$$f(x) = b^x$$

$$f(x) = \left(e^{\ln b}\right)^x = e^{(\ln b)x}$$

$$m = \ln b \Rightarrow f'(x) = (\ln b)\, e^{(\ln b)x}$$

$$= (\ln b)\, b^x.$$

Example: $(2^x)' = (\ln 2)\, 2^x$

$$P(t) = 8.3 \cdot (1.33)^t$$

$$\frac{dP}{dt} = P'(t) = 8.3 \cdot (\ln 1.33) \cdot (1.33)^t$$

$$P'(4) = 8.3 \cdot \ln(1.33) \cdot 1.33^4$$

$$\cong 7.4063$$

Example

(Ex. 3 in 3.1)

$$L(t) = -2.318 + 0.2356t - 0.002674\,t^2$$

L – ventricular length [cm] of the heart in a fetus

t – age in [weeks].

$t = 18$ – means "at the end of week 18"

(a) find $L'(t)$ for $18 \le t \le 38$

(b) Discuss and interpret $L'(t)$

(c) During which week is the ventricular length growing most rapidly?

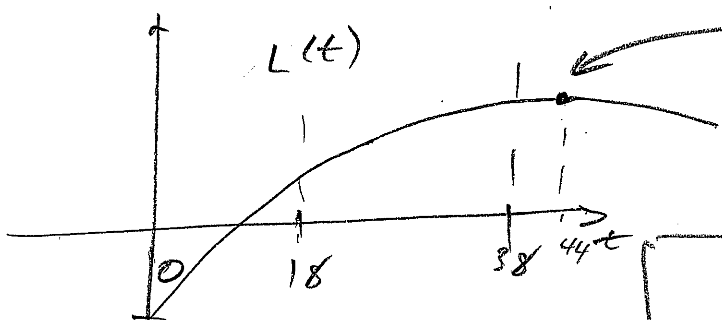
(a) $L'(t) = 0.2356 - 0.005348\,t$

(b) The units are $\frac{cm}{week}$

$L'(t)$ - is the rate of growth [cm per week].

(c) L is growing most rapidly when $L'(t)$ is the greatest, which on the interval $18 \le t \le 38$ is at $t = 18$;

$L'(18) = 0.1393$ cm/week.



L(t) reaches a maximum

Note:

$L'(t) = 0$ when $t = \frac{0.2356}{0.005348} \approx 44.05$ [weeks]

$L(t) = 0$ when $t \cong 11.28$ (weeks)

[Rmk: solved with graphing calc.]