Derivative as a Function (2.7)

Def. (Deriv. as a function)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

(provided the limit exists.)

Example: $f(x) = x^3 - 3x$

Find $f'(x)$; find $f'(-1)$.

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)] - [x^3 - 3x]}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} - 3\lim_{h\to 0} \frac{(x+h)-x}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} - 3\lim \frac{h}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2) - 3\cdot 1$$

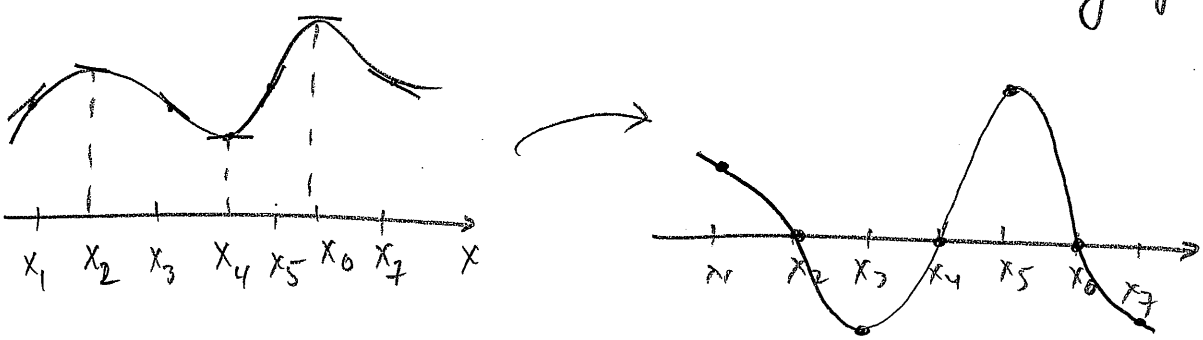
$$= 3x^2 - 3.$$

$$f'(-1) = 3x^2 - 3\Big|_{x=-1} = 3\cdot 1 - 3 = 0$$

No need to redo calculation with $x=-1$, because we already did it for general x!

Example:

$y = f(x)$ $\qquad\qquad\qquad$ $y = f'(x)$



Tangent horizontal $\Rightarrow$ $f'(x) = 0$

Tangent has pos. slope $\Rightarrow$ $f'(x) > 0$

Tangent has neg. slope $\Rightarrow$ $f'(x) < 0$.

(all for a particular x-value! such as $x_1, x_2, \ldots x_7$.)

Example $\qquad$ $f(x) = C$ (constant)

(Ex. 1, 2.7) $\qquad$ $f'(x) = 0$ $\left(= \lim\limits_{h\to 0} \dfrac{c-c}{h} \right)$

$f(x) = mx$ (linear)

$$f'(x) = m \quad \left(= \lim_{h\to 0} \frac{m(x+h) - mx}{h} \right)$$

$f(x) = x^2$ - (quadratic)

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} (2x+h) = 2x$$

For a sum of functions, $f_1 + f_2$
The derivative = the sum of the derivatives : $f_1' + f_2'$.

For the cube, $f(x) = x^3$ as calculation on Example 1 shows,

$$f'(x) = 3x^2.$$

So for $x^3 - 3x$ the deriv. is $3x^2 - 3$.

<u>Notations for the Derivative:</u>

$f'(x)$, $\frac{dy}{dx}$ (reminds of $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$), $\frac{d}{dx} f$

$f'(a)$, $\left.\frac{dy}{dx}\right|_{x=a}$; $\left.\frac{d}{dx} f\right|_{x=a}$

Ant biodiversity

<u>Example</u> (Ex. 4, 2.7)

$$S(x) = -10.3 + 24.9x - 7.7x^2$$

species of ants at elevation x (km)

(a) Find $\frac{dS}{dx}$.

(b) Discuss the meaning of $\frac{dS}{dx}$. Specify units.

$$\frac{dS}{dx} = S'(x) = \lim_{h\to 0} \frac{S(x+h) - S(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[-10.3 + 24.9(x+h) - 7.7(x+h)^2\right] - \left[-10.3 + 24.9x - 7.7x^2\right]}{h}$$

Split the sum; factor the constants out:

$$= -10.3 \underbrace{\lim_{h\to 0} \frac{1-1}{h}}_{\text{const}} + 24.9 \underbrace{\lim_{h\to 0} \frac{(x+h)-x}{h}}_{\text{linear}} - 7.7 \underbrace{\lim_{h\to 0} \frac{(x+h)^2 - x}{h}}_{\text{quadratic}}$$

$$= -10.3 \cdot 0 + 24.9 \cdot 1 - 7.7 \cdot 2x$$

$$= 24.9 - 15.4x$$

(b) Units: $\left[\frac{\text{species}}{\text{km of elevation}}\right] = \left[\frac{1}{\text{km}}\right]$

For $x < \frac{24.9}{15.4} \approx 1.6$

$S'(x) > 0$ (more species at higher elevation)

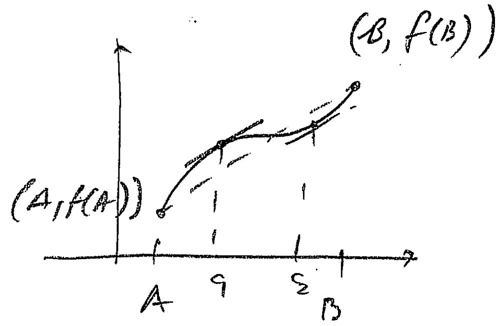
For $x > \frac{24.9}{15.4} \approx 1.6$

$S'(x) < 0$ (fewer species at higher elevation.)

Mean-Value Theorem

Thm 2.7 (Sect. 2.7)

Suppose f is continuous on $[A, B]$ and differentiable on (A, B). Then, for certain c on (A, B) we must have



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

[The average rate of change over an interval equals the instantaneous rate of change for at least one value c.]

Example: $f(x) = \frac{1}{x}$; $[A, B] = [1, 3]$

$$A.R.C = \frac{1/3 - 1/1}{3 - 1} = \frac{-2/3}{2} = -\frac{1}{3}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{x - (x+h)}{(x+h)x}$$

$$= \lim_{h \to 0} \frac{-h}{h(x+h)x} = \lim_{h \to 0} \frac{-1}{(x+h)x}$$

$$= -\frac{1}{x^2} \quad (x \neq 0).$$

Is there a c on (1,3) such that

$$f'(c) = -\frac{1}{3} \quad ?$$

Solve $\quad f'(x) = -\frac{1}{3}$

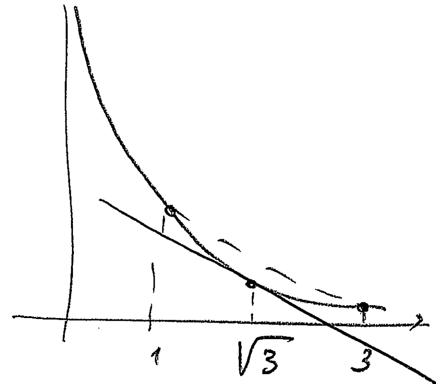
$$-\frac{1}{x^2} = -\frac{1}{3}$$

$$\frac{1}{x^2} = \frac{1}{3}$$

$$x^2 = 3$$

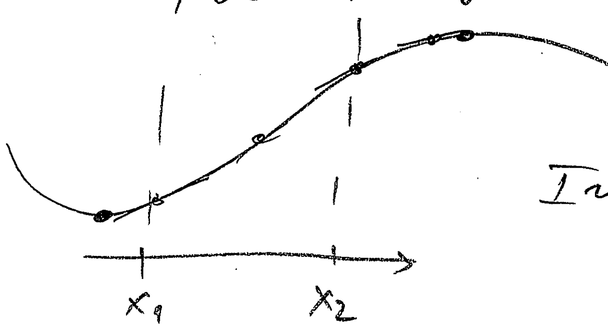
$$x = \pm\sqrt{3} \longrightarrow x = +\sqrt{3} \approx 1.73$$

since $(-\sqrt{3})$ is not in (1,3)



Derivatives and graphs. (increase/Decrease)

* If $f'(x) > 0$ on (A,B)
then the function $f(x)$ is increasing



Indeed if $x_2 > x_1$
then

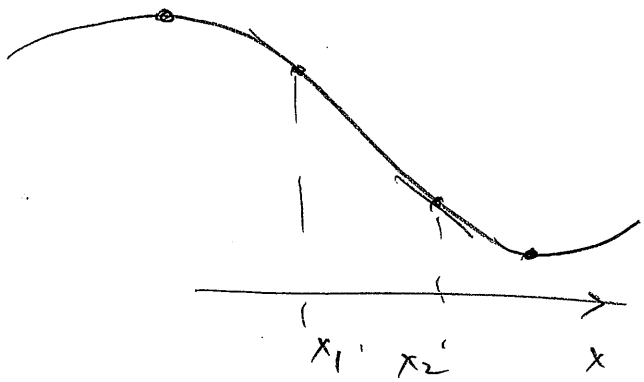
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

$$\Rightarrow f(x_2) - f(x_1) > 0$$

$$\Rightarrow f(x_2) > f(x_1).$$

* If $f'(x) < 0$ on (A, B)

then the function $f(x)$ is decreasing.



If $x_2 > x_1$

then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) < 0$$

$\Rightarrow f(x_2) - f(x_1) < 0$

$\Rightarrow f(x_2) < f(x_1)$

<u>Example</u>: $y = f(x) = x^3 - 3x$

Computed previously: $f'(x) = 3x^2 - 3$

Where is $f(x)$ increasing or decreasing?

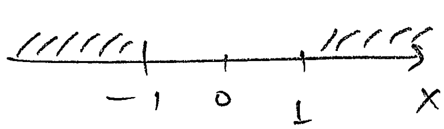
<u>incr</u>: $f'(x) > 0$

$3x^2 - 3 > 0$

$3x^2 > 3$

$x^2 > 1$

$|x| > 1$



$(-\infty, -1)$ or $(1, \infty)$

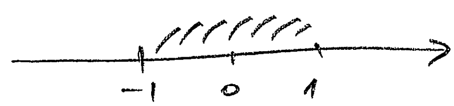
<u>decr</u>: $f'(x) < 0$

$3x^2 - 3 < 0$

$3x^2 < 3$

$x^2 < 1$

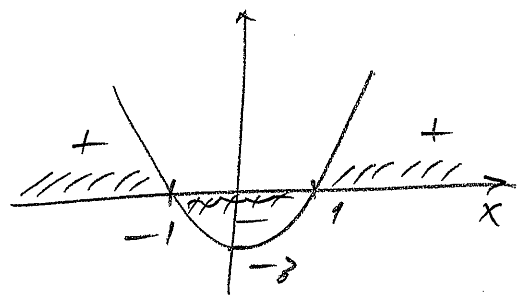
$|x| < 1$



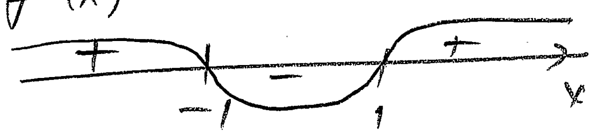
$(-1, 1)$

Remark: It is instructive to sketch a (rough) graph of the derivative $f'(x)$, or a sign diagram.

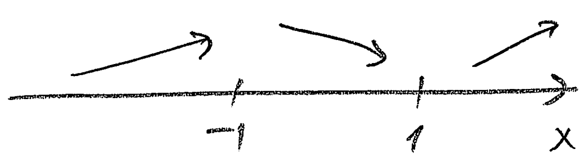
$$f'(x) = 3x^2 - 3 = 3(x^2-1)$$



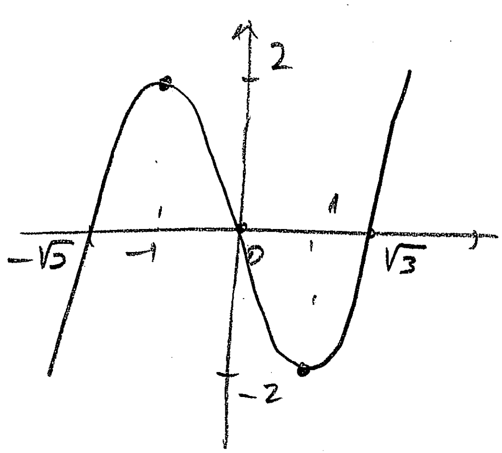
Sign of $f'(x)$:



$f(x)$:



More accurately, the graph of $f(x)$ is as follows:



$f(1) = -2$, $f(-1) = 2$

roots are:

$x(x^2-3) = 0$

so $x = 0$ or $x^2 = 3$

$x = \pm\sqrt{3}$