MATH 255A 10/3/2019 (1)

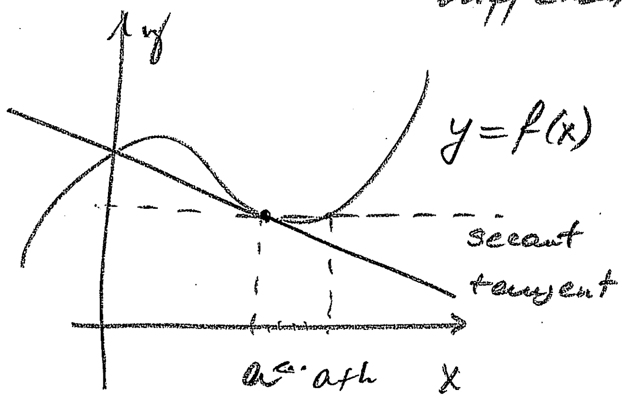
Derivative at a point (2.6)

Def: (Derivative at a point)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

(provided the limit exists.)

If $f'(a)$ is defined we say f is differentiable at 'a'



a - point on the real axis

$(a, f(a))$ - point on the graph

$$\left[\begin{array}{l} h\to 0 \Rightarrow \\ b=a+h \to a \end{array}\right]$$

$$\left(\begin{array}{l} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ = \lim_{b\to a} \frac{f(b)-f(a)}{b-a} \\ = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} \end{array}\right)$$

$f'(a)$ is:

* the inst. rate of change of y with respect to x
* the slope of the tangent line to the graph at $x=a$ (informally, the slope of the graph.)

This is the same quantity that we studied in Section 2.1.

Example: (Ex 2 (e) 2.6)

$$f(x) = \frac{1}{1+x}$$

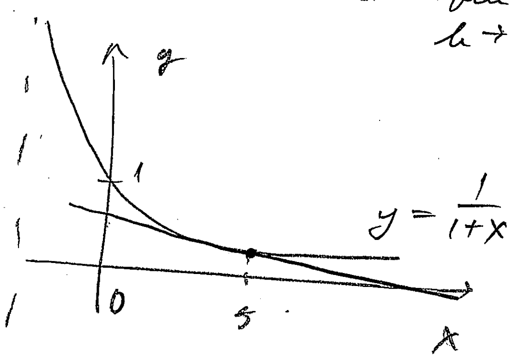
Find $f'(5)$ by definition

$$f'(5) = \lim_{h\to 0} \frac{\frac{1}{1+(5+h)} - \frac{1}{1+5}}{h} = \lim_{h\to 0} \frac{\frac{1}{6+h} - \frac{1}{6}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{6+h} - \frac{1}{6}\right) = \lim_{h\to 0} \frac{1}{h}\,\frac{6-(6+h)}{(6+h)\cdot 6}$$

$$= \lim_{h\to 0} \frac{(-h)}{h(6+h)\cdot 6} = \lim_{h\to 0} \frac{-1}{(6+h)\cdot 6} = -\frac{1}{36}$$

(plug in $h=0$) simplified form is continuous at $h=0$.



Tangent line:

$$y - y_0 = m(x - x_0)$$

$$x_0 = 5,\ y_0 = f(5) = \frac{1}{1+5} = \frac{1}{6}$$

$$m = f'(5) = -\frac{1}{36}$$

$$y - \frac{1}{6} = -\frac{1}{36}(x-5)$$

$$y - \frac{1}{6} = -\frac{1}{36}x + \frac{5}{36}$$

$$y = -\frac{1}{36}x + \frac{5}{36} + \frac{1}{6}$$

$$y = -\frac{1}{36}x + \frac{11}{36}$$

The derivative characterizes the behavior of a function near $x=a$: the graph is close to its tangent line; the slope of the tangent is $-\frac{1}{36}$. $f(x)$ changes at a rate $\simeq -\frac{1}{36}$ when x is close to 5.

Example

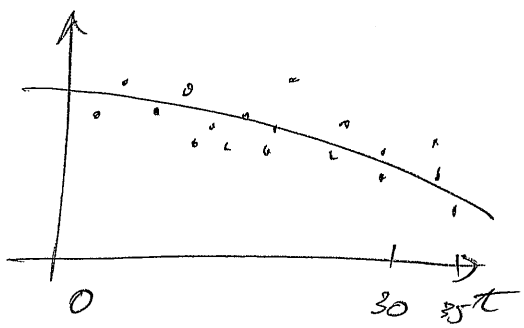
(Ex. 7 in 2.6)

Melting Arctic Sea Ice

$$S(t) = 7.292 + 0.023t - 0.004t^2$$

$$[\times 10^6 \text{ km}^2]$$

Size of the ice cap in millions (area) of sq. km. t years after 1980.



$S(t)$ comes from a quadratic regression of observational data

(a) Find $S'(32)$.

$t = 30 \to$ year $= 2010$

$t = 32 \to$ year $= 2012$

(b) Determine the units and discuss the meaning of the obtained value.

$$S'(32) = \lim_{h \to 0} \frac{S(32+h) - S(32)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{7.292} + 0.023(\cancel{32}+h) - 0.004(32+h)^2 - \ldots \\ \ldots - (\cancel{7.292} + 0.023(\cancel{32}) - 0.004(32)^2)}{h}$$

Constant terms : 7.292 cancel

Linear terms : $0.023 \cdot 32$ cancels with $-0.023 \cdot 32$

Constant multiples can be factored out.

Group similar terms together:

$$= (\cancel{7.292} - \cancel{7.292}) + 0.023 \lim_{h\to 0} \frac{(\cancel{32}+h) - \cancel{32}}{h} + 0.004 \lim_{h\to 0} \frac{(32+h)^2 - 32^2}{h}$$

$$= 0.023 \lim_{h\to 0} 1 - 0.004 \lim_{h\to 0} \frac{\cancel{32^2} + 2\cdot 32 \cdot h + h^2 - \cancel{32^2}}{h}$$

$$= 0.023 - 0.004 \lim_{h\to 0} (64 + h)$$

$$= 0.023 - 0.256 = -0.233.$$

(c) $S'(32) = \lim_{\Delta t\to 0} \frac{\Delta S}{\Delta t}$, so the units for $S'(32)$ are $\frac{[\text{units of } S]}{[\text{units of } t]}$.

In this case

$$\left[\frac{10^6 \cdot km^2}{\text{year}}\right] = \left[\begin{array}{c}\text{millions of} \\ \text{square km} \\ \underline{\text{per}}\ \text{year}\end{array}\right]$$

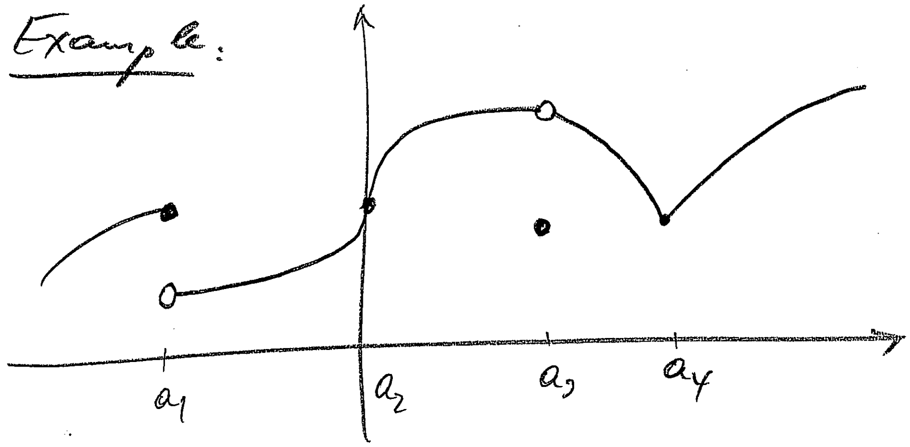
On the average, the size of the ice cap is decreasing at a rate of 233,000 km^2/year.

<u>Differentiability and Continuity</u>

<u>Def</u>: f is differentiable at 'a' if $f'(a)$ exists (and is finite)

f is differentiable on an interval (A,B) if $f'(a)$ exists for every a in (A,B).

Example:



f is discontinuous at a_1
not differentiable (no tangent line)

f is continuous at a_2
not differentiable (vertical tangent line)

f is discontinuous at a_3
not differentiable
(tangent line has to pass through $(a, f(a))$
— does not work)

f is continuous at a_4
not differentiable
graph has a "corner"
no tangent line.

Generally Differentiable $\Rightarrow$ Continuous
Continuous $\not\Rightarrow$ Differentiable

Indeed, if f is differentiable,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{b\to a} \frac{f(b)-f(a)}{b-a} \quad \Bigg\} \text{ exist.}$$

The denominator $h = b-a \longrightarrow 0$

So for the limit to remain finite

the numerator must approach zero:

$$\lim_{b\to a} f(b) - f(a) = 0$$

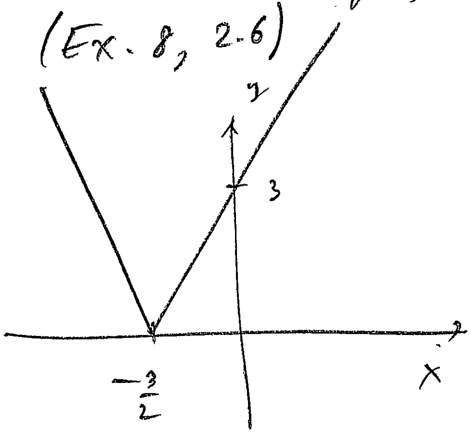
$$\lim_{b\to a} f(b) = f(a)$$

(Definition of continuity!)

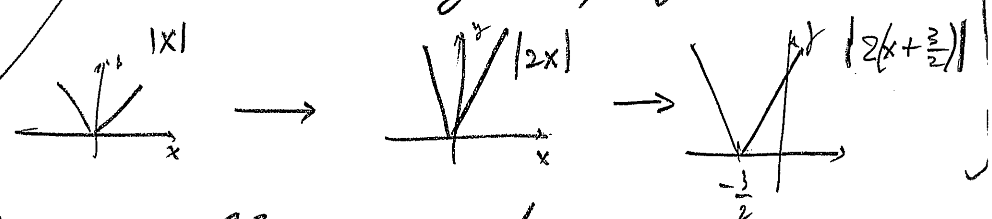
<u>Example</u>: $f(x) = |2x+3| = \begin{cases} 2x+3, & 2x+3 \geq 0 \\ -(2x+3), & 2x+3 < 0 \end{cases}$

(Ex. 8, 2.6)

$$= \begin{cases} 2x+3, & x \geq -\frac{3}{2} \\ -2x-3, & x < -\frac{3}{2} \end{cases}$$



Graph obtained from $|x|$ by shifting/stretching:



$f(x)$ is differentiable everywhere

(graph is a straight line!) except at $x = -\frac{3}{2}$

$$\frac{f\left(-\frac{3}{2}+h\right)-f\left(-\frac{3}{2}\right)}{h} = \frac{\left|2\overbrace{\left(-\frac{3}{2}+h\right)+3}^{=h}\right| - \left|\overbrace{2\left(-\frac{3}{2}\right)+3}^{=0}\right|}{h}$$

$$= \frac{|2h|}{h} = 2\,\frac{|h|}{h} = \begin{cases} 2, & h>0 \\ -2, & h<0 \end{cases}$$

So $\displaystyle\lim_{h\to 0+} \frac{f(a+h)-f(a)}{h} = 2$

$\displaystyle\lim_{h\to 0-} \frac{f(a+h)-f(a)}{h} = -2$

the two-sided limit does not exist.

<u>Note.</u>

$$\lim_{h\to 0+} \frac{f(a+h)-f(a)}{h}$$

$$\lim_{h\to 0-} \frac{f(a+h)-f(a)}{h}$$

are sometimes referred to as one-sided derivatives.

To be differentiable at a the function must have <u>both</u> one-sided derivatives defined, and they must be equal.