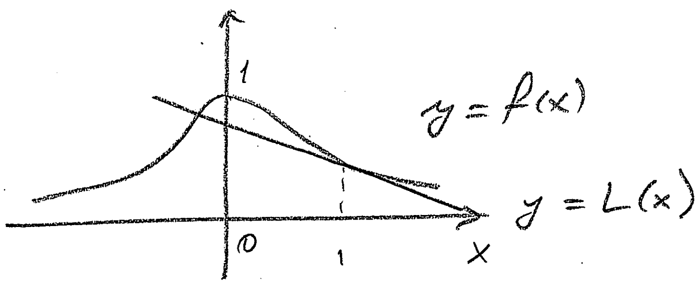
HOMEWORK SECTION 3.5

#5

$$f = \frac{1}{1+x^2}\ ; \quad x_0 = 2\ ; \qquad f(x_0) = \frac{1}{1+4} = \frac{1}{5} = 0.2$$

$$f' = \frac{-2x}{(1+x^2)^2}\ ; \quad f'(x_0) = \frac{-4}{(1+4)^2} = -\frac{4}{25} = -0.16$$

$$L(x) = 0.2 + (-0.16)(x-2)$$
$$= 0.52 - 0.16X$$



Based on graph, $y = L(x)$ is an underestimate.

#12. Estimate $e^{-0.2}$ using $x_0 = 0$

$$f(x) = e^x\ ;\ f'(x) = e^x$$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$= 1 + (x - 0) = 1 + X$$

$$X = -0.2 \Rightarrow L(-0.2) = 1 - 0.2 = 0.8$$

Calculator: $e^{-0.2} = 0.81873$

Approximate value is correct to one decimal place.

#18. $y = \frac{1}{x+1}$, $x_0 = 0$, $\Delta x = -0.05$

$$f = \frac{1}{x+1} \; ; \; f' = \frac{-1}{(x+1)^2} \; ; \; f'(0) = -1 = S$$

$$\Delta y \approx S \cdot \Delta x = (-1)(-0.05) = 0.05$$

#24. $y = \frac{1}{x+1}$, $x_0 = 0$, $\delta x = 0.12$ (12%)

$$f(x) = \frac{1}{x+1} \; ; \; f' = \frac{-1}{(x+1)^2} \; ; \; f'(0) = -1$$

$$E = \frac{f'(0)}{f(0)} \cdot 0 = 0$$

$$\delta y = E \cdot 0.12 = 0 \quad (0\%).$$

Actually "12% error on x" is not well-defined when $x_0 = 0$, so that part of the problem is simply poorly constructed.

#30. $S = 4\pi r^2$; $V = \frac{4\pi}{3} r^3$

Suppose r increases by 1%. How do S and V change?

$$E_S = \frac{S'(r)\, r}{S(r)} = \frac{4\pi \cdot 2r \cdot r}{4\pi r^2} = 2$$

$$\delta S \cong 2 \cdot \delta r = 2\%$$

$$E_V = \frac{V'(r)\,r}{V(r)} = \frac{\frac{4\pi}{3}\cdot 3r^2\cdot r}{\frac{4\pi}{3}\cdot r^3} = 3$$ (3)

$$\delta S \cong 3\cdot\delta r = 3\%$$

S increases by $\simeq 2\%$

V increases by $\simeq 3\%$.

#10.

$$f(x) = \frac{4}{\sqrt{x}}$$

Find $f^{(4)}(x)$.

$$f(x) = 4x^{-\frac{1}{2}}$$

$$f'(x) = 4\cdot\left(-\tfrac{1}{2}\right)x^{-\frac{3}{2}}$$

$$f''(x) = 4\cdot\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)x^{-\frac{5}{2}}$$

$$f'''(x) = 4\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)x^{-\frac{7}{2}}$$

$$f^{(4)}(x) = 4\cdot\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)\left(-\tfrac{7}{2}\right)x^{-\frac{9}{2}}$$

$$= \frac{4\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 2\cdot 2}\, x^{-\frac{9}{2}} = \frac{105}{4}x^{-\frac{9}{2}}.$$

#26.

$$y = 1 + 2x + \frac{18}{x};$$

Determine intervals of increase, decrease
concave up/down
find points of inflection.

$$f(x) = 1 + 2x + \frac{18}{x}$$

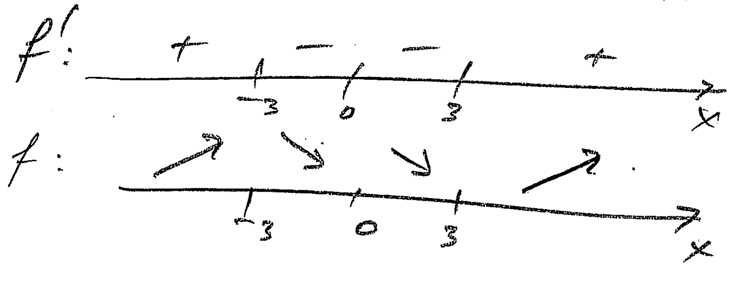
$$f'(x) = 2 + \left(\frac{18}{x}\right)' = 2 - \frac{18}{x^2} = 2\left(1 - \frac{9}{x^2}\right)$$

$$f'(x) = 0: \quad 1 - \frac{9}{x^2} = 0$$

$$1 = \frac{9}{x^2}$$

$$x^2 = 9$$

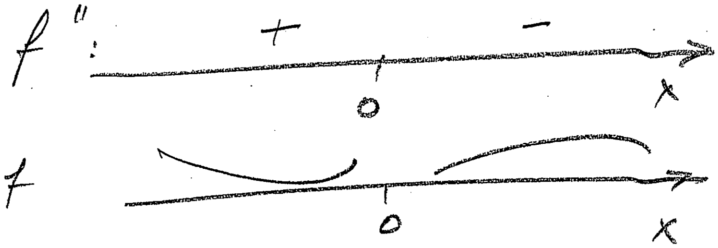
$x = \pm 3$ – critical points.



$$f''(x) = \left(2 - \frac{18}{x^2}\right)' = \frac{(-18)(-2)}{x^3} = \frac{36}{x^3}$$

$f''(x) > 0$ when $x > 0$

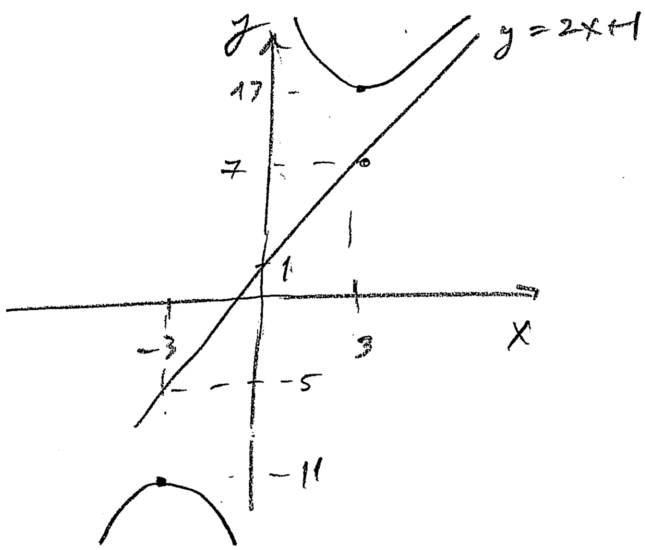
$f''(x) < 0$ when $x < 0$



$x = 0$ – vertical asymptote (not an inflection point)

$x = -3, +3$ – critical points

$f(-3) = 1 - 6 - 6 = -11$; $f(3) = 1 + 6 + 6 = 13$



f incr on

$(-\infty, -3)$ and

$(3, \infty)$

f decr on

$(-3, 0)$ and

$(0, 3)$.

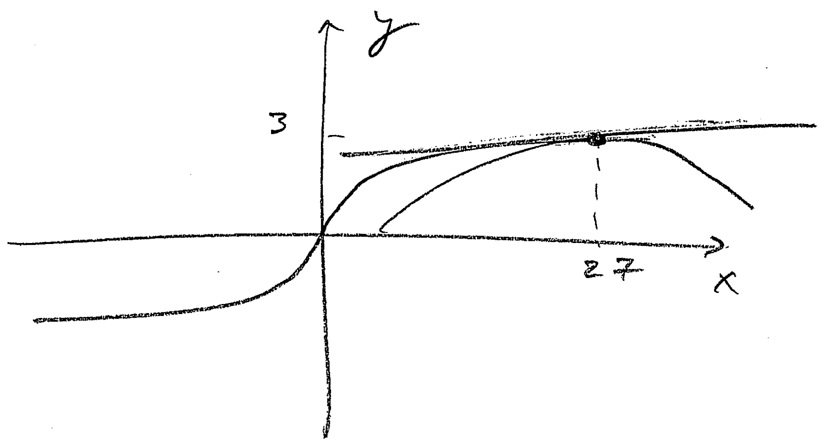
#40. $f(x) = \sqrt[3]{x}$; $x_0 = 27$

Find the first and the second order approximations.
Sketch a graph.

$f'(x) = \left(x^{\frac{1}{3}}\right)' = \frac{1}{3}x^{-\frac{2}{3}}$; $f'(x_0) = \frac{1}{3}\frac{1}{3^2} = \frac{1}{27}$

$f''(x) = \frac{1}{3}\left(x^{-\frac{2}{3}}\right)' = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-\frac{5}{3}}$; $f''(x_0) = -\frac{2}{9}\frac{1}{3^5}$

$= -\frac{2}{2187} \approx -0.001$



linear: $L(x) = 3 + \frac{1}{27}(x-27) = 2 + \frac{1}{27}x$

quadr: $Q(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2$

#46.

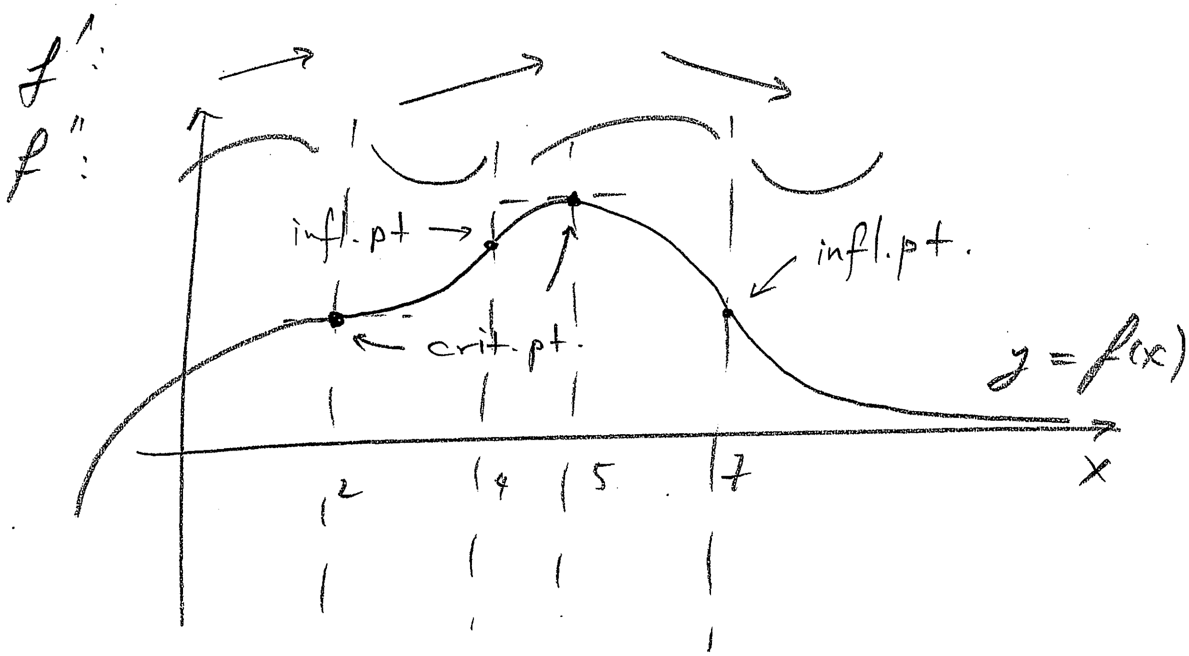
$f'(x) > 0$ when $x < 2$ and $2 < x < 5$

$f'(x) < 0$ when $x > 5$

$f'(2) = 0$

$f''(x) < 0$ when $x < 2$ and when $4 < x < 7$

$f''(x) > 0$ when $2 < x < 4$ and when $x > 7$



#8 $y = \frac{1}{3}x^3 - 9x + 2$

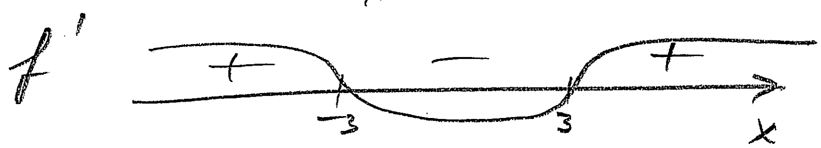
$f(x) = \frac{1}{3}x^3 - 9x + 2$; Continuous everywhere, no asymptotes;

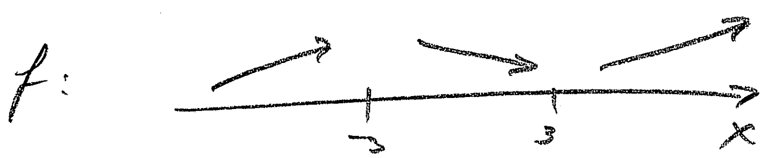
$f'(x) = x^2 - 9$

$f'(x) = 0$: $x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$ – critical points.





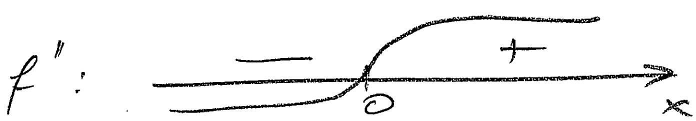
⇒ $x = -3$ → local max

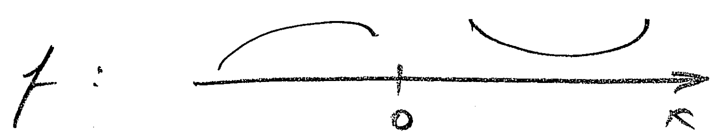
$x = 3$ → local min.

$f''(x) = 2x$

$f''(x) = 0$: $2x = 0$

$x = 0$





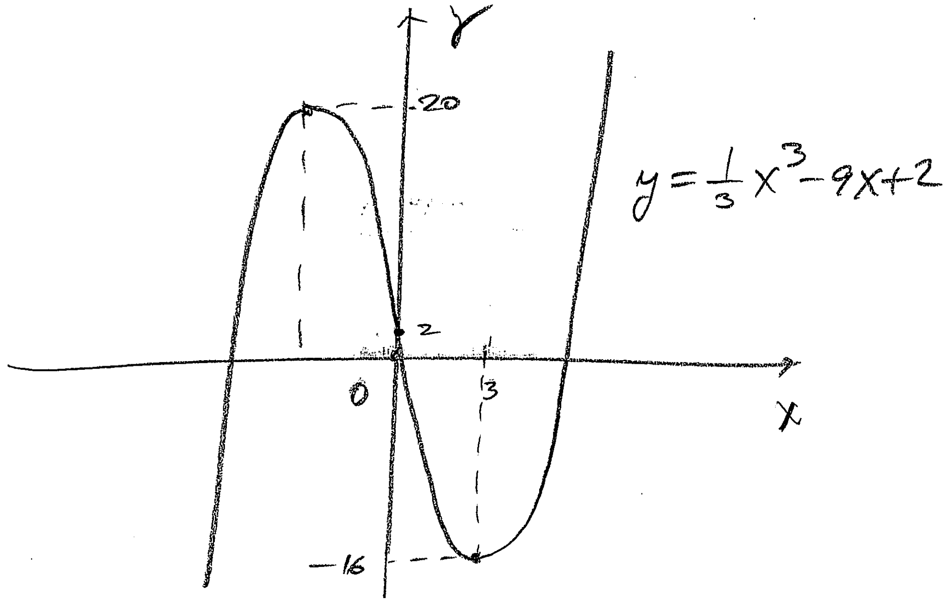
$x = 0$ – inflection pt.

Values: $f(3) = 9 - 9 \cdot 3 + 2 = -16$

$f(-3) = -9 + 9 \cdot 3 + 2 = 20$

$f(0) = 2$

(x-intercepts are difficult to find.)



#10.

$$y = 2e^x + e^{-x}$$

$$f(x) = 2e^x + e^{-x}$$

$$f'(x) = 2e^x - e^{-x}$$

$$f'(x) = 0 \iff 2e^x - e^{-x} = 0$$

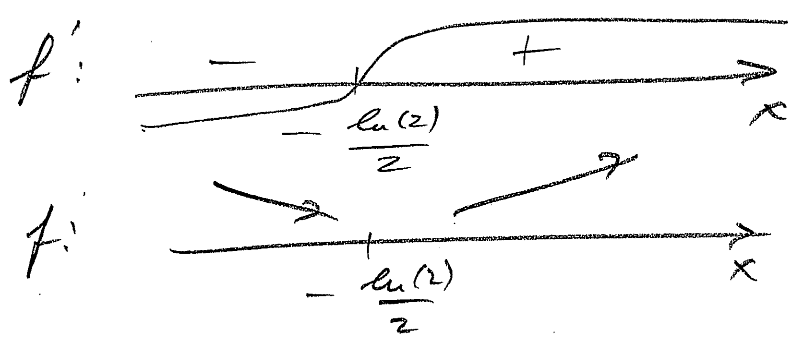
$$2e^x = e^{-x} \qquad (* e^x)$$

$$2e^{2x} = 1$$

$$e^{2x} = \frac{1}{2}$$

$$2x = \ln\frac{1}{2} = -\ln 2$$

$$x = -\frac{\ln 2}{2} = \frac{-0.69}{2} \simeq -0.34$$



$x = -\frac{\ln(2)}{2}$ — point of local min.

$$f''(x) = 2e^x + e^{-x} > 0$$

since $e^x > 0$, $e^{-x} > 0$.

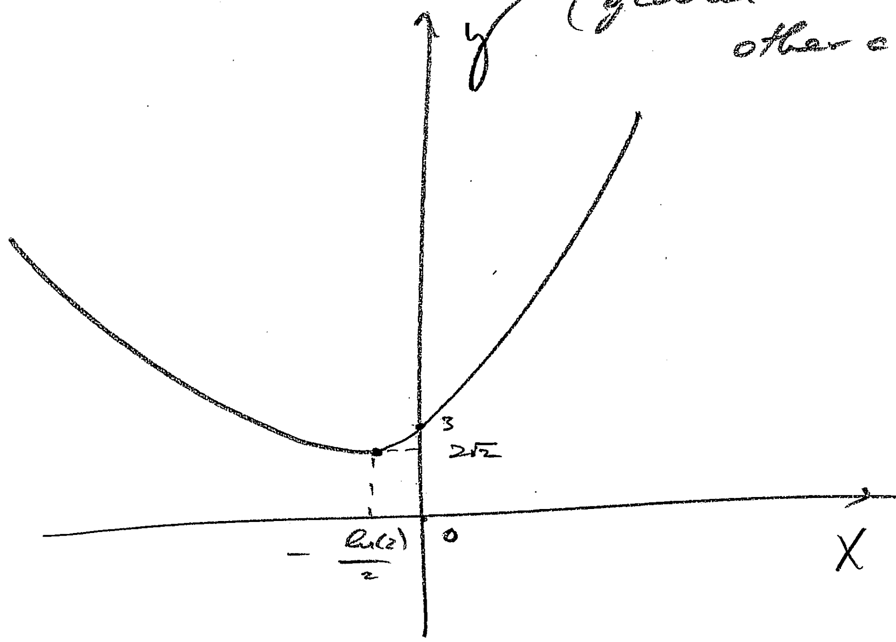
no infl. pts; graph is always concave up.

Values: $f(0) = 2+1 = 3$ (y-intercept)

$$f\left(-\frac{\ln(2)}{2}\right) = 2e^{-\frac{\ln(2)}{2}} + e^{\frac{\ln(2)}{2}}$$

$$= \frac{2}{\sqrt{2}} + \sqrt{2} = 2\sqrt{2}$$
$$\approx 2.82$$

(global min, since no other critical pts.)



decr. on $\left(-\infty, -\frac{\ln(2)}{2}\right)$

incr. on $\left(-\frac{\ln(2)}{2}, \infty\right)$

always concave up.

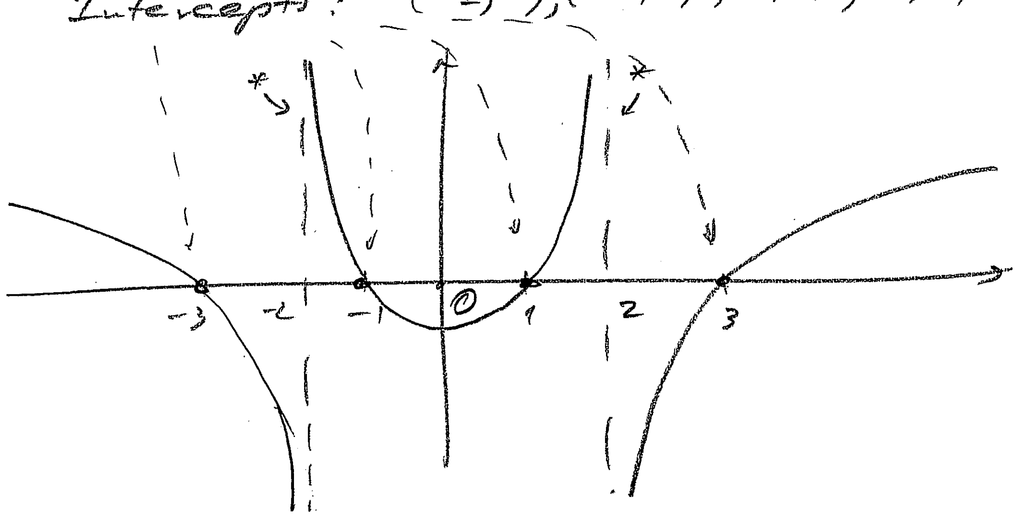
#21: (*) $x=2$, $x=-2$ - vert. asy.

(**) $f(x)$ incr for $0<x<2$ and $x>2$

(***) $f(x)$ decr for $x<2$ and $-2<x<0$.

(iv) concave down on $(-\infty,-2)$ and $(2,\infty)$

(v) Intercepts: $(-1,0)$, $(-3,0)$, $(3,0)$, $(1,0)$

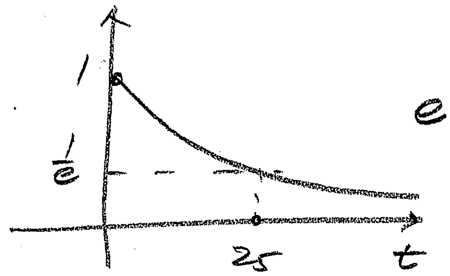


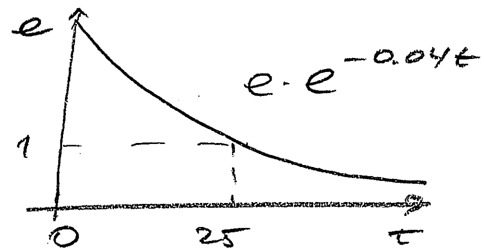
(**)
(***) f:

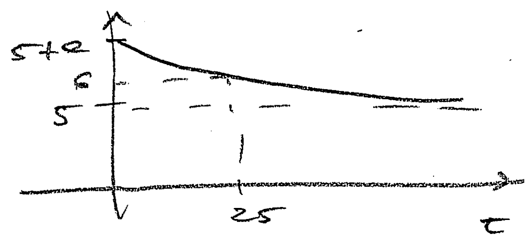
#32:

$$A(t) = 5 + e^{-0.04t+1}$$

$$= 5 + e \cdot \underbrace{e^{-0.04t}}_{\text{decaying exp.}}$$







$A(0) = 5 + e \approx 7.718$

$A(25) = 6$

$A(t) > 5$ (horiz. asymptote)

$A(t)$ - decreasing (decaying exp.)

$A(t)$ concave up (exp. fn.).