Midterm 1: Study Guide

Textbook coverage:

1.1: Real Numbers and Functions. Interval notation, domains, ranges, intervals of increase and decrease. Examples 2, 3, 7.

1.2: Data Fitting with Linear and Periodic Functions; Examples 2-6.

1.3: Power functions and scaling laws. Equations of lines, linear regression, trigonometric functions, transformations. Proportionality notation. Geometric similarity. Examples 1-5.

1.4: Exponential functions. Exponential growth and decay. Doubling time and half-life. Compound interest. Solving exponential equations with logarithms. Examples 1, 2, 5, 6.

1.5: Function building. Transformations of graphs, compositions. Examples 1, 2, 4, 5, 7, 8.

1.6: Inverse functions and logarithms; Examples 3, 4, 6-12.

Review Questions

1. (see also 1.1: 13-18) Find the domains of the functions and compute the indicated values (or state that the corresponding values are not in the domain)

   (a) $f(x) = (1-3x)^{-1/3}$, $f(0)$, $f(\frac{1}{3})$, $f(3)$.

   (b) $f(x) = \begin{cases} 3x+5, & x<2 \\ 3x^2-1, & x>2 \end{cases}$, $f(1)$, $f(2)$, $f(3)$.

   Sketch the graphs and determine the ranges of the functions.

2. (see also 1.1: 44-47) *Friend's rule* is a method for calculating pediatric drug dosages in terms of child's age (up to $12\frac{1}{2}$ years). If A is the adult dose (in mg) and n is the age of the child (in years) then the child's dose is given by

   $$D(n) = \frac{2}{25}nA.$$

   (a) What is the domain of the function defined by (n, D)?

   (b) Graph this function for $A = 600$ mg.

   (c) If a 3-year-old child receives 150 mg of a certain drug, what is the corresponding dose for a 7 year old? What is the adult dose?

3. (see also 1.2: 1-6, 11-18)

(a) Find a linear function ($y = mx + b$) that passes through the points $(-1, 4)$ and $(1, 3)$.

(b) Find an exponential function ($y = ab^x$) that passes through the same points.

(c) Find an equation for a straight line through the points $(-1, 4)$ and $(-1, 3)$. Does this equation represent a function $y = f(x)$? Does it represent a function $x = g(y)$?

4. (see also 1.2: 47) The following Table presents the data on the average carbon dioxide (CO_2) level in the atmosphere, measured in parts per million (ppm), at Mauna Loa Observatory from 2004 to 2012.

Year	CO_2 level (in ppm)
2004	377.5
2006	381.9
2008	385.6
2010	389.9
2012	393.8

(a) Use the linear regression method to find the best-fitting data line. Plot the line and the points.

(b) Use the linear equation obtained in part (a) to obtain a prediction for the CO_2 level at Mauna Loa Observatory in 2016.

5. (see also 1.3: 11-16, 34, 35) If $x \propto y^6$ and $y \propto z^{1/3}$ then how does z change as x decreases from 16 to 4?

6. (see also 1.3: 25-27) Consider a cylinder of radius r and height $5r$. Express the volume V and the surface area S of this cylinder as a function of r. If r is doubled, what happens to V? If V is decreased by a factor of 64, what happens to S?

7. (see also 1.3: 36-38) A sample based on nineteen mountain ash trees of different sizes yielded a relationship between the leaf area, A [m^2], of the tree, and the stem diameter at breast height (DBH), d, [cm]. The relationship obtained was $A \propto d^{2.89}$. If one of the points that this relationship passed through was $(d, A) = (30, 78)$, find the equation and sketch the graph.

8. (see also 1.4: 31, 32) Assume $\$1,000$ is invested for $t = 10$ years at a rate $r = 0.08$ compounded (a) annually (b) daily (c) continuously. Calculate the future value. (Round your answers to two significant digits.)

9. (see also 1.4: 35, 36) The height of beer froth is modeled by the function $H(t) = 17(0.94)^{t/15}$ where t is measured in seconds. Estimate (accurate to 1 second) at what time the froth is (a) one half of its original height (b) one tenth of its original height.

10. (see also 1.4: 33, 39-42, 1.5: 37, 38) If a bacterial population initially has 20 individuals and doubles every 9.3 hours then how many individuals will it have after three days?

11. (see also 1.5: 34, 35; 1.2: 31-36, 47, 48) The tides at La Selva Beach, California, on Saturday September 17, 2016 are given by the following table (actual data have been approximated for simplicity of calculation):

Time	Height (ft)	Tide
11:30am	5.41	High
5:30pm	0.69	Low
11:30pm	5.41	High

Let

$$T(t) = A\cos(B(t - C)) + D$$

denote the height of the tide t hours after 12:00 noon.

(a) Find values of A, B, C, and D such that the function fits the La Selva Beach data.

(b) Sketch the function over the interval $[-12, 12]$.

12. (see also 1.6: 15-30; 1.3: 1-10) Write the expressions in terms of base e and simplify where possible:

(a) 5^{2x}

(b) 3^{x^2}

(c) $\log_2(x^2 - 2)$

(d) $\log_{10}(ex + e)$

Note: 3^{x^2} is usually interpreted as $3^{(x^2)}$, while $\left(3^x\right)^2$ is the same as 3^{2x}.

13. Which one is greater: $\log_5 127$ or $\log_{10} 999$? Answer without using calculator!

14. Solve the equations:

(a) $\log_5 x + \log_5(x - 2) = 1$.

(b) $e^{x^2-4x+5} = e$.

(c) $3e^{-3x} = e^{-x}$.

(d) $e^{-x} = \frac{x}{2} + 1$. [Solve graphically, do not use calculator!]

(e) $e^{-x} = \frac{x}{2}$. [Use equation solving function on a graphing calculator, show work!]

15. (see also 1.6: 9-14) Find the inverses of the functions. State the domain and the range of the inverse.

    (a) $y = x/(x+1)$;

    (b) $y = 0.1\, e^{3x-5}$;

    (c) $y = \exp(-x^2)$ on $[0, \infty)$.

    Note: $\exp(x)$ is another notation for e^x.

16. (see also 1.6: 31-34)

    (a) Sketch the indicated points on the logarithmic scale (use base 10): 0.01, 0.2, 10, 25,000.

    (b) Sketch a log-log graph of the function $y = \frac{10}{x^{1/2}}$ over the interval $[0.01, 100]$. On the vertical axis, mark the values that correspond to $x = 10^{-2}, 10^0, 10^2$.

17. (1.6: 44, 46) From one individual to another within the same species of mammal it is found that the brain volume V varies with the body weight W according to the power law $V = aW^b$. The following data are obtained from measurements of a number of adult chimpanzees. Show, using a log-log plot that the power law fits these data quite well, and obtain the values of a and b:

W, kg	31	36	38	42	47	48	53
V, cm^3	365	380	382	397	410	415	427

Answers:

1. (a) Domain: $(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$. Values: 1, undefined, $-\frac{1}{2}$;

   (b) Domain $(-\infty, 2) \cup (2, \infty)$. Values: 8, undefined, 26.

2. (a) Domain: $[0, 12.5]$. Note: algebraically, the domain is $\mathbb{R}$ (i.e. $(-\infty, \infty)$). However, it's meaningless to apply the formula for n outside the interval $[0, 12.5]$, therefore the answer above is correct.

   (b) The graph is a straight line that raises from 0 to 600 [mg] in the interval from 0 to 12.5 years.

   (c) 350 mg; 625 mg.

3. (a) $y = -0.5x + 3.5$.

(b) $y = \sqrt{12}\left(\frac{3}{4}\right)^{t/2}$ $\left(a = \sqrt{12},\, b = \sqrt{\frac{3}{4}}\right)$.

(c) $x = -1$. Not a function $y = f(x)$ (the vertical line test fails). Works OK as a function $x = g(y)$ ($g(y)$ is a constant -1).

4. (a) $y = 2.03x - 3690.5$.

   (b) 401.98.

5. $x \propto z^2$, x is decreased by a factor of 4, therefore z is decreased by a factor of 2.

6. If r is doubled, V increases by a factor of 8. If V is decreased by a factor of 64 then S is decreased by a factor of 16.

7. $A = 0.0042\, d^{2.89}$; the graph has appearance similar to the cubic parabola $y = x^3$, passes through $(30, 78)$.

8. $2,158.92, $2,225.35, $2,225.54.

9. 168 seconds; 558 seconds.

10. 4,281.

11. $A = 2.36$; $B = \frac{\pi}{6}$, $C = -0.5$, $D = 3.05$. (Hint for the graph: plot the points of the data and match with a graph of cosine with a horizontal shift, vertical stretching and a vertical shift.)

12. (a) $e^{2x \ln 5}$; (b) $e^{x^2 \ln 3}$; (c) $\ln(x^2 - 2)/\ln(2)$; (d) $(1 + \ln(x+1))/\ln(10)$.

13. $\log_5 127 > \log_{10} 999$.

14. (a) $1 + \sqrt{6}$; (b) 2; (c) $\log(3)/2$; (d) 0; (e) $x \approx 0.852606$.

15. (a) $y = \frac{x}{1-x}$ (note that the inverse is expressed in the form $y = f(x)$); (b) $y = \frac{1}{3}(5 + \ln(10x))$; (c) $y = \sqrt{-\ln x}$ on (0,1].

16. (b) The graph is a straight line $Y = -\frac{1}{2}X + 1$ that passes through the points $(-2, 2)$, $(0, 1)$, $(2, 0)$. The values marked on the y axis are 100, 10, 1.

17. $a = 130.62$, $b = 0.298$.