Final Exam Study Guide

Types of questions

1. Exponential growth/decay. Doubling time and half-life. (Sections 1.4, 1.6, online homework)
2. Limits of functions; limits at infinity, asymptotes. Applying limit rules to determine limits.
3. Continuity; piecewise functions.
4. Calculate a derivative; use derivative rules. Use derivatives of basic functions: power, exponential, logarithmic, trigonometric. Derivatives of implicit functions. Derivatives in the Fundamental Theorem of Calculus. Interpret the derivative (state units for dimensional quantities, discuss interpretation as rate of growth/decay).
5. Linear and quadratic approximations of functions. Tangent lines. Sensitivity and elasticity. When is the linear approximation an underestimate or an overestimate?
6. Graphing: find asymptotes, critical points, intervals of increase/decrease, inflection points and intervals of concavity up/down. Graph a function given a list of properties.
7. Optimization: local and global extrema; First and Second Derivative Tests. Closed and Open Interval Methods. Applications.
8. Compute an integral using one of the techniques: use properties of integral and interpretation as area; use Riemann sum to approximate; use the Fundamental Theorem of Calculus to evaluate.
9. Find indefinite integral of a function. Find antiderivative that satisfies certain condition.
10. Applications of integration: accumulated change, population growth, growing grapes (degree-days) examples.

The following list provides examples of questions for each type. Some of these questions appeared on past final exams, others were taken from the textbook, homework problems, or midterm reviews. You can find more examples of questions in worked textbook examples, online homework, past midterms and practice problems from the course schedule.

Review Problems

1. If a bacterial population initially has twenty individuals and grows exponentially with doubling time 9.3 hours, then how many individuals will it have after three days?

2. The size of the population of humpback whales is modeled by the function $N(t) = 420(1.05)^t$, where t is measured in years since the beginning of 1980. (a) How long does it take for the population to double in size? (b) Find $N'(10)$, state the units, and write a sentence to interpret its value.

3. Carbon-14 has a half-life of 5,730 years. How much is left of 150 g Carbon-14 after t years?

4. Find $\frac{dy}{dx}$ for each of the following:

   (a) $y = x\tan(3x^2 - 2x)$

   (b) $y = \dfrac{0.21\,x}{x^2 + x + 1}$

   (c) $2xy^3 = 6y - 3x^2$

   (d) $y = \displaystyle\int_0^x \frac{e^{t^3}}{t^2 + t + 1}\,dt.$

5. Determine the limits. Show work using the rules of limits.

   (a) $\lim\limits_{x\to 2} \frac{(x-2)^2}{|x-2|}$

   (b) $\lim\limits_{x\to 2} \frac{(x-2)}{|x-2|}$

   (c) $\lim\limits_{t\to\infty} 0.05 + 1500(0.9998)^t.$

   (d) $\lim\limits_{t\to\infty} \frac{3n^2-2n+5}{4n-8n^2}.$

6. Given $f(x) = \frac{6e^x}{e^x+2.1}$.

   (a) Find the limit $L = \lim\limits_{x\to\infty} f(x)$. Show work using the rules of limits.

   (b) Determine how large x needs to be so that $f(x)$ is within 0.01 from the limit value. *[Hint: To solve $|f(x) - L| \le 0.01$ notice that $f(x) < L$, so $f(x) - L < 0$ and $|f(x) - L| = L - f(x)$.]*

7. Find the value that needs to be assigned to d, if any, to guarantee that f will be continuous for all $x > 0$:

$$f(x) = \begin{cases} 3x + \dfrac{12}{x}, & x \le 3 \\ -5x + d, & x > 3. \end{cases}$$

8. Find the values that need to be assigned to parameters m and d, if any, to guarantee that f will be *continuous* and *differentiable* at $x = 1$:

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ d + m(x-1), & x > 1. \end{cases}$$

9. (a) Find the linear approximation of the function $y = \sqrt{4+x}$ around $x = 0$. (b) Sketch a graph of the function and the linear approximation near $x = 0$. (c) If $\Delta x = 0.03$ use the linear approximation to estimate Δy. (b) Using second-order derivative, determine whether the linear approximation overestimates or underestimates the values of the function near $x = 0$.

10. A certain cell is modeled as a sphere. If the formulas $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$ are used to compute the surface area and volume of the sphere, respectively, estimate the effect on S and V produced by a 1% increase in the radius r.

11. A drug is given to a patient in order to lower the blood pressure level. The blood pressure in mm Hg is a function of elapsed time (in minutes) since the drug was administered, that is, $B = f(t)$. Interpret the meaning of the statement $f'(30) = -0.01$.

12. Given $f(x) = x^3 - x + 1$.

    (a) Find all critical points of f and classify them using the first or second derivative tests.

    (b) Find all inflection points and determine intervals of concavity up or down.

    (c) Sketch a graph of $f(x)$ showing vertical intercepts, asymptotes (if applicable) and all maxima and minima and inflection points.

13. (a) Sketch a graph of a function $y = f(x)$ with all of the following properties:

    - $f(x)$ is continuous on $(-\infty, \infty)$
    - $y = 2$, $y = 0$ are horizontal asymptotes.
    - $f'(x) > 0$ for all x;
    - $f''(x) < 0$ for $x < -5$; $f''(x) > 0$ for $x > -5$;
    - $f(-5) = 1$.

    (b) Which of the following are correct (circle all that apply):

A. The value $x = -5$ is a critical point for $f(x)$.

B. The value $f(1)$ is a local extremum.

C. The point $(-5, 1)$ is an inflection point for $f(x)$.

D. The maximum of $f(x)$ is 2.

14. Find the global maximum and the global minimum of $f(x) = x^2 - \ln x$ on $[0.2, 2]$. Is there a global maximum and minimum on $(0, \infty)$?

15. The function

$$C(t) = 100(e^{-0.25t} - e^{-0.5t}) \quad [\text{mg/l}]$$

is used to describe the concentration of a drug in the bloodstream t hours after the drug is administered.

(a) Find $C'(t)$. Compute $C'(2)$ and $C'(4)$ and interpret.

(b) Use $C'(t)$ to determine the intervals of increase and decrease of $C(t)$.

(c) Find the maximal concentration in [mg/l] and the time it occurs.

16. Compute definite integrals:

(a) $\displaystyle\int_1^2 \frac{17x^2}{(2x^3+1)^2}\,dx.$

(b) $\displaystyle\int_0^5 \frac{5,000e^{0.2t}}{1+e^{0.2t}}\,dt.$

(c) $\displaystyle\int_0^2 (e^x - e^{-x})^2\,dx.$

(d) $\displaystyle\int_0^{\pi/4} \tan x\,dx.$

17. Let $f(x) = 1 - \frac{4}{\sqrt{x}}$.

(a) Find $\int f(x)\,dx$.

(b) Find the antiderivative $F(x)$ such that $F(0) = 1$.

(c) Find the antiderivative $F(x)$ such that the least value of $F(x)$ is zero.

18. Using the Riemann sum for a uniform subdivision with $n = 8$ and left endpoints, approximate the area under the curve $y = \sqrt{3+x}$ over the interval $[1, 5]$. Geometrically, explain whether this approximation underestimates or overestimates the true area.

19. Use properties of definite integrals and appropriate formulas from geometry to calculate the integral *exactly*:

$$\int_0^5 \left(7 - \frac{x}{5} - 2\sqrt{25 - x^2}\right) dx.$$

20. The population of Mariposa, CA is changing at the rate $P'(t) = 1 + 3t^{1/2}$ people per month. The current population is 2,200 people. Obtain a prediction for the size of the population sixteen months from now.

21. Sweet corn in Western Oregon has lower developmental threshold $T_0 = 50°\text{F}$ and requires approximately 1,597 degree-days to reach maturity. Suppose the temperature in the fields is given by

$$T(t) = 68 + 17\sin(2\pi t),$$

where t is time in days.

Given the formula for the number of growing degree-days (GDD) that accumulate from $t = 0$ to $t = x$,

$$\text{GDD} = \int_0^x (T(t) - T_0)\, dt.$$

(a) Calculate the number of degree-days that accumulate from $t = 0$ to $t = 10$.

(b) Estimate (rounding to the nearest integer) the time x required for the corn to reach maturity.

Answers:

1. 4,281.

2. (a) About 14.2 years. (b) $N'(10) = 33.38$ [whales/year]. At the beginning of 1990 the whale population was growing at a rate of 33.38 whales per year.

3. $150\, 2^{-t/5730}$, or $150e^{-t\ln(2)/5730}$

4. (a) $\tan(3x^2 - 2x) + (6x^2 - 2x)\sec^2(3x^2 - 2x)$; (b) $\frac{0.21(1-x^2)}{(1+x+x^2)^2}$; (c) $y' = \frac{\frac{1}{3}y^3+x}{1-xy^2}$; (d) $\frac{e^{x^3}}{1+x+x^2}$.

5. (a) 0; (b) the limit does not exist; (c) 0.05; (d) -0.375.

6. (a) $L = 6$; (b) $x \geq 7.1372$.

7. $d = 28$.

8. $d = 0$, $m = 2$.

9. (a) $y = \frac{1}{4}x + 2$; (c) $\Delta y \approx 0.0075$ (d) $f''(x) = -\frac{1}{4(4+x)^{3/2}} < 0$, the function is concave down, so the graph lies below the tangent line. The linear approximation is an overestimate.

10. S increases by $\approx 2\%$; V increases by $\approx 3\%$. [The elasticity of S to r is 2, and the elasticity of V to r is 3, no matter what the value r is.]

11. After 30 minutes the blood pressure decreases at a rate ≈ 0.01 mm Hg per minute.

12. (a) $x = -\frac{1}{\sqrt{3}}$ is a local maximum, $x = \frac{1}{\sqrt{3}}$ is a local minimum (b) $x = 0$ is an inflection point, f is concave down on $(0, \infty)$, concave up on $(-\infty, 0)$.

13. (a) One possible graph in an increasing sigmoidal curve as described in Example 2 of Section 2.4. (b) Only C is correct.

14. Global minimum $\frac{1}{2}(1 + \ln 2) \approx 0.84657$ at $x = \frac{1}{\sqrt{2}}$. Global maximum $4 - \ln 2 \approx 3.3069$ at $x = 2$. Same global minumum on $(0, \infty)$, but no global maximum since $f(x) \to \infty$ as $x \to 0+$ or $x \to \infty$.

15. (a) $C'(t) = -25e^{-0.25t} + 50e^{-0.5t}$. $C'(2) = 3.23$, $C'(4) = -2.43$. After 2 hours the concentration is increasing at a rate of 3.23 [mg/l/hr]; after 4 hours the concentration is decreasing at a rate of 2.43 [mg/l/hr]. (b) $C(t)$ is increasing on $(0, 4\ln(2))$, decreasing on $(4\ln(2), \infty)$. (c) The maximum concentration of 25 [mg/l] is achieved when $t = 4\ln(2) \approx 2.77$ hours.

16. (a) 7/9; (b) $25,000 \ln \frac{1+e}{2}$; (c) $\frac{1}{2}(e^4 - e^{-4}) - 4$; (d) $\frac{1}{2}\ln 2$.

17. (a) $x - 8\sqrt{x} + C$. (b) $F(x) = x - 8\sqrt{x} + 1$. (c) $F(x) = x - 8\sqrt{x} + 16$.

18. 9.5430. This is an underestimate, since $y = \sqrt{3 + x}$ is increasing on $[1, 5]$ and the left end point of each interval I_i produces the least value of the function on this interval.

19. $\frac{5}{2}(13 - 5\pi)$.

20. 2344.

21. (a) 180 degree-days; (b) 88.43 days.