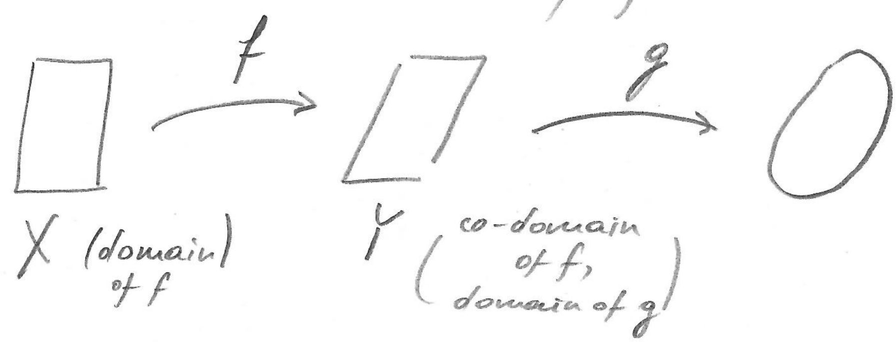
Chain Rule and Implicit Differentiation. (3.3)

The Chain Rule applies to Derivative of Composition.

Let's first review the concept of Composition of functions.



$$g \circ f(x) = g(f(x))$$

↑ composition

f is applied first — "inner function"

g is applied second — "outer function".

Examples: $h(x) = \sin(2x^2+1)$ is a composite function

$g(u) = \sin(u)$ "outer fn", $f(x) = 2x^2+1$ "inner fn".

OR

$g(u) = \sin(2u+1)$ "outer function"; $f(x) = x^2$ "inner function"

Compositions can be "unfolded" in different ways.

Notation: $y = h(x) = g(f(x))$; $g(u)$ – outer fuction

$u = f(x)$ – inner fuction.

$$h'(x) = \frac{dy}{dx}; \quad f'(x) = \frac{du}{dx}$$

Formally: $\left[\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \right]$ (Rule of cancellation of fractions)

This is the simplest form of the Chain Rule.

$$\frac{dy}{du} = g'(u)$$

The Derivative of a Composition is
The derivative of the outer function
times the derivative of the inner function.

$$\left[h'(x) = g'(u)f'(x) = g'(f(x))\, f'(x) \right]$$

substitute $f(x)$ for u!

Example: CO pollution is changing at a rate of 0.02 ppm per person, depending on the number of people in town.

The population of a town is changing at a rate of 1,000 people/year.

Find the rate at which the CO pollution changes with respect to time.

$$\left[0.02 \ \frac{\text{ppm}}{\text{person}}\right] \cdot \left[1{,}000 \ \frac{\text{person}}{\text{year}}\right]$$

$$= 20 \ \frac{\text{ppm}}{\text{year}}$$

Notice that $\left[\frac{\text{ppm}}{\text{person}}\right] \cdot \left[\frac{\text{person}}{\text{year}}\right] = \left[\frac{\text{ppm}}{\text{year}}\right]$!

L - level of CO

P - population; t - time

$$\frac{dL}{dP} = 0.02 \ \frac{\text{ppm}}{\text{person}} \qquad \frac{dP}{dt} = 1000 \ \frac{\text{person}}{\text{year}}$$

So $$\frac{dL}{dt} = \frac{dL}{dP}\frac{dP}{dt}.$$

Examples On using the Chain Rule.

(1) $$y = (1+2x+x^3)^{101}$$

Find $\frac{dy}{dx}$. We use $y = u^{101}$ - outer

$u = 1+2x+x^3$ - inner

outer deriv: $\frac{dy}{du} = 101u^{100}$ (power rule)

$\frac{du}{dx} = 2+3x^2$ (sums and powers)

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 101\,u^{100}\cdot(2+3x^2)$$

$$= 101\left(1+2x+x^3\right)^{100}\cdot\left(2+3x^2\right).$$

must use x in the answer!

$u = 1+2x+x^3$.

(2) $h(x) = e^{-x^2}$ "the bell curve function".

$u = -x^2$ $\qquad h(u) = e^u$

$\frac{du}{dx} = -2x$ $\qquad \frac{dh}{du} = e^u$

$$\frac{dh}{dx} = \frac{dh}{du}\frac{dy}{dx} = e^u(-2x) = e^{-x^2}(-2x)$$

$u = -x^2$

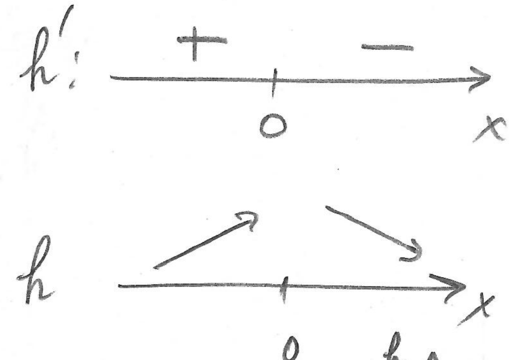
$$= -2x\cdot e^{-x^2}$$

Since $e^{-x^2} > 0$ always

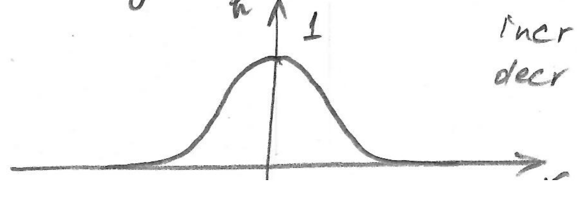
we have $h'(x) > 0$ when $x < 0$

$h'(x) < 0$ when $x > 0$

Diagram:



Graph of h:



Positive,
even,
incr on $(-\infty, 0)$
decr on $(0, \infty)$.

Implicit Functions and Implicit Differentiation

$y = f(x)$ — explicit dependence of y from x.

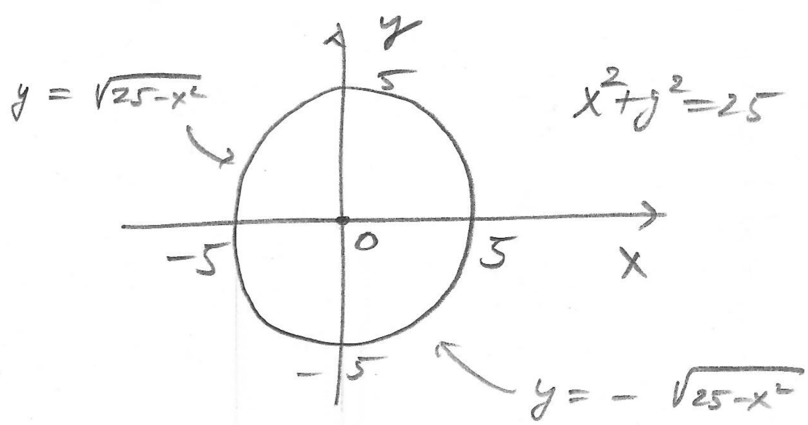
Sometimes, the dependence is more complicated:

$x^2 + y^2 = 25$ — easy when written in the form $F(x,y) = 0$

less easy to express y as explicit function of x:

$$y^2 = 25 - x^2$$

$$y = \pm\sqrt{25 - x^2}$$ — there are 2 fuctions ($\pm$)!



$x^2 + y^2 = 25$ — circle of radius 5 about the origin.

Suppose we want to find the derivative $\frac{dy}{dx}$ from the equation

$$F(x,y) = 0$$

We proceed as follows:

Take $\frac{d}{dx}$ (derivative with resp. to x) of both sides of the equation:

$$\frac{d}{dx} F(x,y) = \frac{d}{dx} 0 = 0 .$$

Treat y as $y(x)$ and apply differentiation rules to $F(x,y)$.

This gives

$$F_1(x,y) + F_2(x,y)\, y' = 0$$

where F_1, F_2 to be computed.

Solve for y':

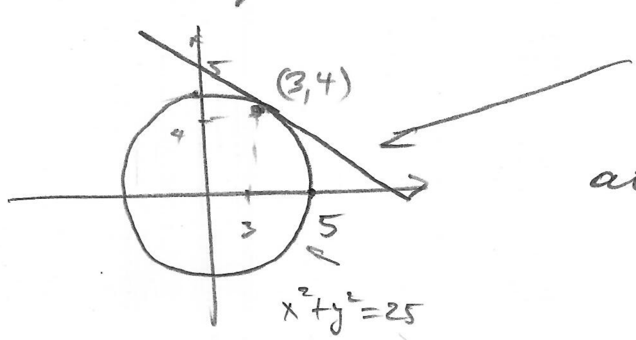
$$y' = - \frac{F_1(x,y)}{F_2(x,y)}$$

— an expression for the derivative.

Example: Find the equation of the tangent line to

$$x^2 + y^2 = 25$$

at the point $(x_0, y_0) = (3,4)$.



Differentiate implicitly:

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(25)$$

$$2x + \frac{d}{dx}(y^2) = 0 \qquad \frac{d}{dx}(x^2) = 2x.$$

Now, $y = y(x)$, so y^2 is a composite function.

Use the Chain Rule:

$$\frac{d}{dx}(y^2) = g'(u)\,u'(x) = 2y(x)\cdot y'(x)$$

$u = y(x)$ – inner $\quad y'(x)$ – inner deriv.

$g(u) = u^2$ – outer $\quad g'(u) = 2u$ – outer deriv.

So

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

When $(x, y) = (3, 4)$

$$y' = -\frac{3}{4} = m$$ [the slope of the tangent.]

Tangent line:

$$y - y_0 = m(x - x_0)$$

$$y - 4 = -\frac{3}{4}(x-3) = -\frac{3}{4}x + \frac{9}{4}$$

Add $4 = \frac{16}{4}$ to both sides

$$y = -\frac{3}{4}x + \frac{25}{4}$$

Derivative of Logarithm

$$y = \ln(x) \quad \Longleftrightarrow \quad e^y = x$$

Differentiate Implicitly:

$$\frac{d}{dx} e^{y(x)} = \frac{d}{dx} x$$

e^u - outer

$u = y(x)$ - inner

$$e^y \cdot y' = 1$$

$$y' = \frac{1}{e^y} = \frac{1}{x}$$

So, for $y = \ln(x)$, $y' = \frac{1}{x}$

Example: log with base 2:

$$\log_2(x) = \frac{\ln(x)}{\ln(2)} \qquad \text{(change of base.)}$$

$$\frac{d}{dx}\log_2(x) = \frac{d}{dx}\underbrace{\frac{\ln(x)}{\ln(2)}}_{\text{constant!}} = \frac{1}{\ln(2)}\frac{d}{dx}\ln(x)$$

$$= \frac{1}{\ln(2)} \cdot \frac{1}{x}$$

More generally, $\frac{d}{dx}\log_b(x) = \frac{1}{\ln(b)}\frac{1}{x}$

The natural log is the simplest!