MATH 255A 10/1/2019

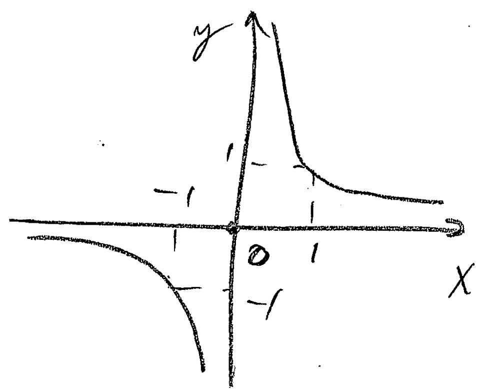
1

Asymptotes and Infinity (2.4)

[Define symbols $+\infty$ and $-\infty$]

Example:

$$y = \frac{1}{x}$$



$$\lim_{x \to 0+} \frac{1}{x} = +\infty$$

means the values $y = \frac{1}{x}$ (positive) become as large as we wish, if x is > 0, and close enough to zero. $(x \to 0+)$

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

means the values of $y = \frac{1}{x}$ become as close to zero as we wish, provided x is large enough, positive or negative.

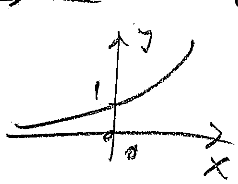
$$\lim_{x \to 0-} \frac{1}{x} = -\infty$$

means the values $y = \frac{1}{x}$ become as large negative as we wish, provided $x < 0$ and close enough to zero $(x \to 0-)$

If $y = f(x) \longrightarrow L$ as $x \to +\infty$ or $x \to -\infty$ we say $y = L$ is a horiz. asymptote.

If $f(x) \to +\infty$ or $f(x) \to -\infty$ as $x \to a+$ or $x \to a-$ we say $x = a$ is a vertical asymptote.

Examples: $y = e^x$ (growing exp)

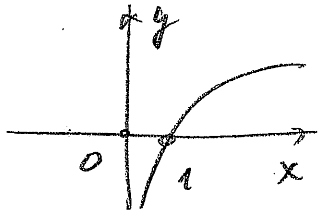


$y = 0$ - horiz. asy as $x \to -\infty$

No asymptote as $x \to +\infty$

$$\lim_{x \to \infty} e^x = +\infty \qquad \text{(no asymptote.)}$$

$$y = \ln x$$



$$\lim_{x \to 0+} \ln x = -\infty$$

$$\lim_{x \to \infty} \ln x = +\infty .$$

<u>Ex</u>:

$$y = \frac{3x+2}{2x-1} = \frac{x\left(3+\frac{2}{x}\right)}{x\left(2-\frac{1}{x}\right)}$$

factor out x – most important as $x \to \infty$

$x \to \infty$

(Horiz. asy) $\longrightarrow$

$$y = \frac{3+\frac{2}{x}}{2-\frac{1}{x}} \longrightarrow \frac{3+0}{2-0} = \frac{3}{2}$$

as $x \to \infty$ $\quad (+\infty$ or $-\infty)$

$$\lim_{x \to \infty} \frac{3x+2}{2x-1} = \lim_{x \to \infty} \frac{3+\frac{2}{x}}{2-\frac{1}{x}} = \frac{\lim 3 + 2 \lim \frac{1}{x}}{\lim 2 - \lim \frac{1}{x}}$$

$$= \frac{3+0}{2-0} = \frac{3}{2}.$$

Technique: "Factor out most important term."

Also:
$$\frac{3x+2}{2x-1} = \frac{(3x+2)/x}{(2x-1)/x} = \frac{3+\frac{2}{x}}{2-\frac{1}{x}}$$

("divide both numer. & denom. by x")

(Vert. Asy)

$$y = \frac{3x+2}{2x-1}$$

Denom : $2x-1 = 0$ when $x = \frac{1}{2}$

but numerator $3x+2 \neq 0$

when $x \approx \frac{1}{2}$.

$$\frac{\text{nonzero}}{\text{approaching zero}} \longrightarrow \pm\infty$$

$$\lim_{x \to \frac{1}{2}+} \frac{3x+2}{2x-1} = \text{"} \frac{\approx \frac{3}{2}+2}{\approx 0+} \text{"} = +\infty$$

$$\lim_{x \to \frac{1}{2}-} \frac{3x+2}{2x-1} = \text{"} \frac{\approx \frac{3}{2}+2}{\approx 0-} \text{"} = -\infty.$$

Example: (1) $\lim_{x \to 2\pm} \frac{x-2}{x^2-4}$ "$\frac{0}{0}$" - meaningless combination

$$\frac{x-2}{x^2-4} = \frac{\cancel{x-2}}{(\cancel{x-2})(x+2)} = \frac{1}{x+2}$$

$$\lim_{x \to 2\pm} \frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}$$

There is no vertical asymptote

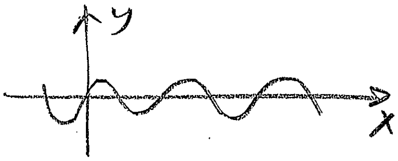
(2) $\lim_{x \to 3+} 3 + \frac{2x}{x-3} = +\infty$.

$$3 + \frac{2x}{x-3} \approx 3 \; \frac{2 \cdot 3}{\approx 0+} = +\infty$$

(3) $\lim_{x \to \infty} \sin x$ DNE

since $\sin x = 0$ for $x = 2\pi n \to \infty$

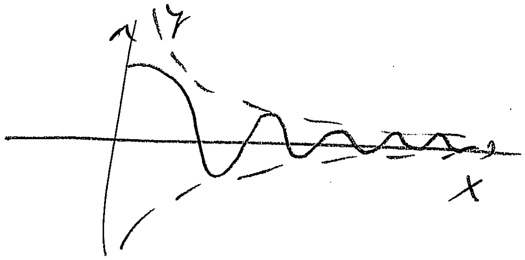
$\sin x = 1$ for $x = \frac{\pi}{2} + 2\pi n \to \infty$



no single value is approached as $x \to \infty$

(4) $\lim_{x \to \infty} \frac{1}{x} \sin x = 0$

Since $-1 \le \sin x \le 1$



$$-\frac{1}{|x|} \le \frac{1}{x} \sin x \le \frac{1}{|x|} |\sin x| \le \frac{1}{|x|}$$

$\downarrow$ as $x \to \infty$ $\qquad$ $\downarrow$ as $x \to \infty$

$0 \qquad\qquad 0$

Therefore $\frac{1}{x} \sin x \longrightarrow 0$ as $x \to \infty$

Example (Ex. 3)

$$\lim_{x \to +\infty} \sqrt{x^2 + x} - x = (+\infty) - (+\infty)$$

undefined!

Does not mean that the limit is undefined, means we need to look more carefully.

(Note: Textbook has $\sqrt{x^2+1} - x$, and contains a few typos.)

$$\sqrt{x^2+x} - x = \frac{\left(\sqrt{x^2+x} - x\right)\left(\sqrt{x^2+x} + x\right)}{\left(\sqrt{x^2+x} + x\right)} = \frac{(x^2+x) - x^2}{\sqrt{x^2+x} + x}$$

$$= \frac{x}{\sqrt{x^2+x} + x}$$

multiply numerator & denominator by conjugate radical

$$\lim_{x \to +\infty} \frac{x}{\sqrt{x^2+x} + x} = \lim_{x \to +\infty} \frac{x \;/x}{(\sqrt{x^2+x} + x)/x}$$

$$= \lim_{x \to +\infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1} = \frac{1}{\sqrt{1+\lim\limits_{x \to +\infty}\frac{1}{x}} + 1}$$

$$= \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

Example
(Ex. 2, 2.4)

"Dose - Response Curve".

$$R(x) = \frac{100\, e^x}{e^x + e^{-5}}$$

– percentage of patients which exhibit abnormal reaction to a drug

$x = \ln$ (Dose)

Dose in mM (millimole)

(a) Find horiz. asymptotes (as $x \to +\infty$ and $x \to -\infty$)

(b) Show that $R(x)$ is increasing (from 0 to 100 (percent))

(c) How large positive does x need to be for $R(x)$ to be within 1% of its asymptotic value?

(a) Horiz. asymptotes:

$$\lim_{x \to +\infty} R(x) = \lim_{x \to +\infty} \frac{100\, e^{x}}{e^{x} + e^{-5}}$$

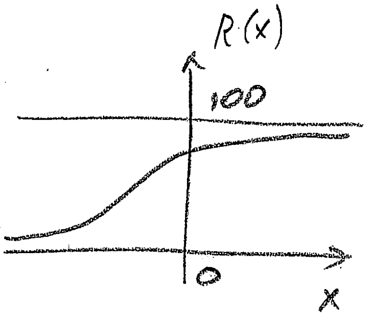
[Factor out most important term: $e^{x} \to \infty$ as $x \to \infty$]

$$\lim_{x \to +\infty} \frac{\cancel{e^{x}} \cdot 100}{\cancel{e^{x}}\left(1 + \frac{e^{-5}}{e^{x}}\right)} = \lim_{x \to +\infty} \frac{100}{1 + e^{-5} e^{-x}}$$

$$= \frac{100}{1 + e^{-5} \underbrace{\lim_{x \to +\infty} e^{-x}}_{=0}} = \frac{100}{1} = 100$$

$$\lim_{x \to -\infty} R(x) = \lim_{x \to -\infty} \frac{100\, e^{x}}{e^{x} + e^{-5}}$$

(Note: e^{x} is not the most important here, since $e^{x} \to 0$ as $x \to -\infty$)



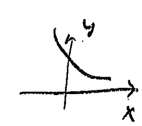
$$= \frac{100 \lim_{x \to -\infty} e^{x}}{\lim_{x \to -\infty} e^{x} + e^{-5}} = \frac{100 \cdot 0}{0 + e^{-5}} = 0$$

$\left(\frac{\text{zero}}{\text{nonzero}}\right)$

(b)

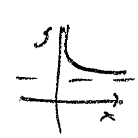
$$R(x) = \frac{100}{1 + e^{-5} e^{-x}}$$

as x increases, e^{-x} decr.



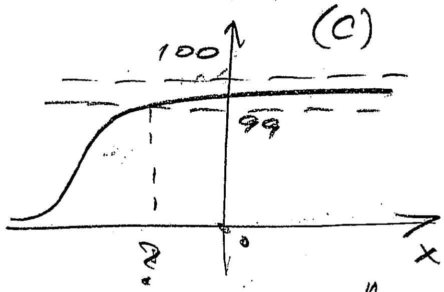
$e^{-5} e^{-x}$ decr.

$1 + e^{-5} e^{-x}$ decr.



$\frac{1}{1 + e^{-5} e^{-x}}$ increases

(c)



$$R(x) \xrightarrow[x \to +\infty]{} 100$$

[asymptotic value for x large and positive.]

"within 1% of 100" means

$$R(x) > 99$$

$$\frac{100}{1+e^{-5}e^{-x}} > 99$$

$$\frac{100}{99} > 1+e^{-5}e^{-x}$$

$$\underbrace{e^{5}\left(\frac{100}{99}-1\right)}_{\doteq 1.5} > e^{-x}$$

$$e^{-x} < 1.5$$

$$e^{x} > \frac{1}{1.5} = \frac{2}{3}$$

$$x > \ln\left(\frac{2}{3}\right) = -0.4055$$

If $\underline{x > -0.4055}$ then

$R(x) > 99$ – within 1% of the limit value 100.