HOMEWORK

$$U = \langle E \rangle = \sum_{p} \epsilon_{\vec{p}} \langle n_{\vec{p}} \rangle =$$

$$= \frac{4\pi V}{h^3} \int_0^\infty dp\, p^2 \left(\frac{p^2}{2m}\right) \frac{1}{e^{\beta p^2/2m - \beta\mu} + 1}$$

Define $\beta\mu = \nu$ (look in your notes) and set $\beta p^2/2m = x^2$

$$\Rightarrow U = \frac{4\pi V}{h^3 2m} \left(\frac{2m}{\beta}\right)^{5/2} \int_0^\infty \frac{dx\, x^4}{e^{x^2-\nu}+1}$$

Substitute $x^2 = y$

$$U = \frac{4\pi V (2m)^{3/2}}{2h^3 \beta^{5/2}} \int_0^\infty \frac{dy\, y^{3/2}}{e^{y-\nu}+1} =$$

$$= \frac{4\pi V (2m)^{3/2}}{2h^3 \beta^{5/2}} \left(\frac{2}{5}\right) \left\{ \frac{y^{5/2}}{e^{y-\nu}+1}\Bigg|_0^\infty + \int_0^\infty \frac{y^{5/2} e^{y-\nu} dy}{(e^{y-\nu}+1)^2} \right\}$$

$$= \frac{4\pi V (2m)^{3/2}}{5h^3 \beta^{5/2}} \int_0^\infty \frac{y^{5/2} e^{y-\nu} dy}{(e^{y-\nu}+1)^2}$$

Expanding the $y^{5/2}$ in power series around ν

$$y^{5/2} \simeq \nu^{5/2} + \frac{5}{2}\nu^{3/2}(y-\nu) + \frac{15}{8}\nu^{1/2}(y-\nu)^2 + \dots$$

$$U = \frac{4\pi V (2m)^{3/2}}{5h^3 \beta^{5/2}} \int_{-\nu}^{\infty} dt \frac{e^t}{(e^t+1)^2} \left[\nu^{5/2} + \frac{5}{2}\nu^{3/2} t + \frac{15}{8}\nu^{1/2} t^2 \right.$$

$$\simeq \frac{4\pi V (2m)^{3/2}}{5h^3 \beta^{5/2}} \int_{-\infty}^{+\infty} dt \frac{e^t}{(e^t+1)^2} [\qquad\qquad] =$$

$$= \frac{4\pi V (2m)^{3/2}}{5h^3 \beta^{5/2}} \left[I_0 (\beta\mu)^{5/2} + \frac{5}{2} I_1 (\beta\mu)^{3/2} + \frac{15}{8} I_2 (\beta\mu)^{1/2} + \dots \right]$$

$$= \frac{4\pi V (2m)^{3/2}}{5h^3} \mu^{5/2} \left[I_0 + \frac{5}{2} I_1 (\beta\mu)^{-1} + \frac{15}{8} I_2 (\beta\mu)^{-2} + \dots \right]$$

$I_n = 0$ for odd n (look notes)

$I_0 = 1 \qquad I_2 = \frac{\pi^2}{3}$

$$\Rightarrow \quad U = \frac{4\pi V (2m)^{3/2}}{5h^3} \mu^{5/2} \left[1 + \frac{5\pi^2}{8} (\beta\mu)^{-2} + \dots \right]$$

$$\boxed{U = \frac{4\pi V (2m)^{3/2}}{5h^3} \mu^{5/2} \left[1 + \frac{5}{8} \left(\frac{\pi}{\beta\mu}\right)^2 + \dots \right]} \quad (1)$$

From the class notes

$$\frac{\langle N\rangle}{V} = \frac{1}{\lambda^3} f_{3/2}(z) = \frac{1}{\lambda^3}\frac{4}{3\sqrt{\pi}}\left[(\beta\mu)^{3/2} + \frac{\pi^2}{8}(\beta\mu)^{-1/2} + \cdots\right]$$

Since $\lambda = \sqrt{\dfrac{2\pi\hbar^2}{m k_B T}} = \sqrt{\dfrac{h^2}{2\pi m k_B T}}$

$$\Rightarrow \langle N\rangle = V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\frac{4}{3\sqrt{\pi}}(\beta\mu)^{3/2}\left[1 + \frac{\pi^2}{8}(\beta\mu)^{-2} + \cdots\right]$$

$$\boxed{\langle N\rangle = \frac{4V\pi}{3h^3}(2m)^{3/2}\mu^{3/2}\left[1 + \frac{1}{8}\left(\frac{\pi}{\beta\mu}\right)^2 + \cdots\right]} \quad (2)$$

Finally we showed in class that:

$$\mu \simeq E_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2 + \cdots\right] \quad (3)$$

Combiming Eqs. (1) and (2) we obtain:

$$U = \frac{4\pi V(2m)^{3/2}}{5h^3}E_F^{5/2}\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2 + \cdots\right]^{5/2}\left[1 + \frac{5}{8}\left(\frac{\pi}{\beta E_F}\right)^2 + \cdots\right]$$

$$\frac{4\pi V(2m)^{3/2}}{5h^3}E_F^{3/2}E_F\left[1 - \frac{5\pi^2}{24}\left(\frac{k_B T}{E_F}\right)^2 + \frac{5\pi^2}{8}\left(\frac{k_B T}{E_F}\right)^2 + \cdots\right]$$

$$U = \frac{4\pi V (2\pi)^{3/2}}{5 h^3} E_F^{3/2} \, E_F \left[1 + \frac{5}{12} \left(\frac{\pi k_B T}{E_F} \right)^2 + \cdots \right] \qquad (4)$$

Substituting the expression for μ in Eq. (3) in Eq. (2)

$$\langle N \rangle = \frac{4 V \pi \, (2m)^{3/2}}{3 h^3} E_F^{3/2} \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{E_F} \right)^2 + \cdots \right]^{3/2} \times$$

$$\left[1 + \frac{1}{8} \pi^2 \left(\frac{k_B T}{E_F} \right)^2 + \cdots \right]$$

$$= \frac{4 V \pi \, (2m)^{3/2}}{3 h^3} E_F^{3/2} \left[1 \cancel{- \frac{3\pi^2}{24} \left(\frac{k_B T}{E_F} \right)^2 + \frac{\pi^2}{8} \left(\frac{k_B T}{E_F} \right)^2} + \cdots \right]$$

$$\simeq \frac{4 V \pi \, (2m)^{3/2}}{3 h^3} E_F^{3/2}$$

Substituting in Eq. (4) we find:

$$U = \frac{3}{5} \langle N \rangle E_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{E_F} \right)^2 + \cdots \right]$$

Problem 2.

The bulk modulus of the ideal electron gas in three dimensions is

$$B = -V\left(\frac{\partial P}{\partial V}\right)_T \quad \text{and} \quad K_T = \frac{1}{B}.$$

Since: $dU + pdV = TdS \implies$ At $T=0$

$$\implies p = -\left(\frac{\partial U}{\partial V}\right)_T$$

$$B = V\left(\frac{\partial^2 U}{\partial V^2}\right)_T = V\,\frac{d^2U}{dV^2}$$

We have derived in class that:

$$U = \frac{2V}{10\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{5/2} \quad \text{where} \quad E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

Taking the second derivate of U with respect to V and substituting above

$$B = V\left(\frac{10}{9}\right) N \frac{\hbar^2}{2m}\left(3\pi^2 N\right)^{2/3} V^{-8/3}$$

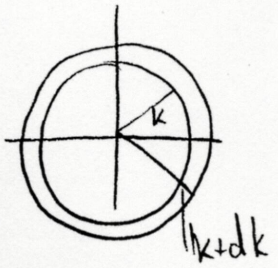
and $K_T = \dfrac{1}{B}$.

1.3 Free electron gas in 2-dimension

Solving the Schrödinger equation and imposing periodic boundary conditions we find (as in the case of 3-dimensions)

$$k_x = \pm n_x \frac{2\pi}{L}, \quad k_y = \pm n_y \frac{2\pi}{L}$$

Thus, the number of states per unit volume in k-space is $\left(\frac{L}{2\pi}\right)^2$.



Thus the number of states between k and k+dk

$$dN = \overset{\text{(spin degeneracy)}}{2} \left(\frac{L}{2\pi}\right)^2 2\pi k dk = \frac{L^2}{\pi} k dk$$

And since $\epsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m} \Longrightarrow d\epsilon = \frac{\hbar^2 k dk}{m}$

$$\Longrightarrow \boxed{D(\epsilon) = \frac{dN}{d\epsilon} = \frac{L^2 m}{\pi \hbar^2} \equiv D_0 = \text{const}} \quad (1)$$

The total number of electrons $N = \int_0^\infty f(\epsilon) D(\epsilon) d\epsilon = D_0 \int_0^\infty f(\epsilon) d\epsilon$

$$= D_0 \int_0^{\epsilon_F} d\epsilon = D_0 \epsilon_F$$

Thus:

$$\epsilon_F = \int_0^\infty f(\epsilon) d\epsilon = \int_0^\infty \frac{d\epsilon}{e^{\epsilon - \mu / k_B T} + 1}$$

Set $x = \frac{\epsilon - \mu}{k_B T}$, $dx = \frac{d\epsilon}{k_B T}$, and obtain

$$\epsilon_F = k_B T \int_{-\mu/k_B T}^{\infty} \frac{dx}{e^x + 1} = k_B T \left[\ln \frac{e^x}{e^x + 1} \right]_{-\mu/k_B T}^{\infty} = -k_B T \left[\ln \left(1 + e^{-x}\right) \right]_{-\mu/k_B T}^{\infty}$$

Thus: $\epsilon_F = k_B T \ln\left(e^{\mu/k_BT} + 1\right)$ or $e^{\epsilon_F/k_BT} = e^{\mu/k_BT} + 1$

$$\Longrightarrow \boxed{\mu = k_B T \ln\left(e^{\epsilon_F/k_BT} - 1\right)}$$

However: $N = 2\left(\frac{L}{2\pi}\right)^2 \pi k_F^2 = \frac{L^2}{2\pi} k_F^2 = \frac{L^2}{2\pi}\frac{2m\epsilon_F}{\hbar^2}$

$$= \frac{L^2 m \epsilon_F}{\pi \hbar^2}$$

Thus $\frac{N}{L^2} = \frac{N}{A} = n = \frac{m\epsilon_F}{\pi\hbar^2}$. Substituting in the above expression:

$$\boxed{\mu = k_B T \ln\left(e^{\frac{\pi n \hbar^2}{m k_B T}} - 1\right)}$$

4.

(a) $$D(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dk}\frac{dk}{d\epsilon} = \frac{L}{2\pi}\left(\frac{d\epsilon}{dk}\right)^{-1} = \frac{L}{2\pi}\frac{1}{\frac{\hbar^2 k}{m}}$$

$$= \frac{L}{2\pi}\left(\frac{m}{\hbar^2}\right)\frac{1}{\sqrt{\frac{2mE}{\hbar^2}}} = \frac{L}{2\pi}\sqrt{\frac{m}{2\hbar^2 E}} = \frac{L}{2\pi}\frac{1}{\hbar}\sqrt{\frac{m}{2E}}$$

(b) $$N = 2\int_0^{E_F} D(E)\,dE = \frac{L}{\pi\hbar}\sqrt{\frac{m}{2}}\int_0^{E_F}\frac{dE}{\sqrt{E}} = \frac{2L}{\pi\hbar}\sqrt{\frac{m}{2}}\,E_F^{1/2}$$

$$E_F = \left(\frac{\pi\hbar}{2}\right)^2\left(\frac{N}{L}\right)^2\left(\sqrt{\frac{2}{m}}\right)^2$$

Problem 11.4 solved in class (lectures)