# More on Finite Elements in Two Dimensions

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Mechanical Engineering 501B

**Seminar in Engineering Analysis**

April 27-29, 2009

## Outline

- Review last lecture
  - Quadratic basis functions in two dimensions
- Border integral terms
- Triangular elements
  - Natural coordinates (area coordinates)
  - Linear basis functions
  - Higher order basis functions
  - Boundary terms

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## Review Quadrilateral Equation

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta$$

$$f(\xi,\eta) = \frac{\partial \varphi_i}{\partial \xi}\left[\frac{x_\eta^2 + y_\eta^2}{J}\frac{\partial \varphi_k}{\partial \xi} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial \varphi_k}{\partial \eta}\right] +$$

$$\frac{\partial \varphi_i}{\partial \eta}\left[\frac{x_\xi^2 + y_\xi^2}{J}\frac{\partial \varphi_k}{\partial \eta} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial \varphi_k}{\partial \xi}\right] - \varphi_k a^2 \phi_i J$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \approx \sum_{k=1}^{n}\sum_{j=1}^{n} \gamma_k \gamma_j f(\xi_k,\eta_j)$$

- Select basis functions and n

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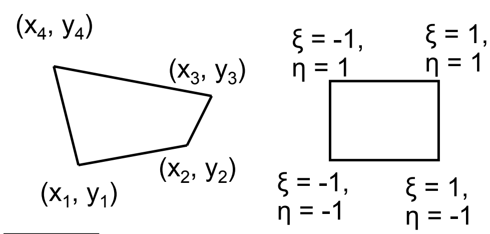
## Review Linear $\phi_i$ Quadrilateral




Note: $\phi_i(\mathbf{x}_{(j)}) = \delta_{ij}$

$$\varphi_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad \varphi_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$\varphi_3 = \frac{(1+\xi)(1+\eta)}{4} \qquad \varphi_4 = \frac{(1-\xi)(1+\eta)}{4}$$

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## Review $\phi$ Derivatives

$$\varphi_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad \frac{\partial \varphi_1}{\partial \xi} = -\frac{(1-\eta)}{4} \qquad \frac{\partial \varphi_1}{\partial \eta} = -\frac{(1-\xi)}{4}$$

$$\varphi_2 = \frac{(1+\xi)(1-\eta)}{4} \qquad \frac{\partial \varphi_2}{\partial \xi} = \frac{(1-\eta)}{4} \qquad \frac{\partial \varphi_2}{\partial \eta} = -\frac{(1+\xi)}{4}$$

$$\varphi_3 = \frac{(1+\xi)(1+\eta)}{4} \qquad \frac{\partial \varphi_3}{\partial \xi} = \frac{(1+\eta)}{4} \qquad \frac{\partial \varphi_3}{\partial \eta} = \frac{(1+\xi)}{4}$$

$$\varphi_4 = \frac{(1-\xi)(1+\eta)}{4} \qquad \frac{\partial \varphi_4}{\partial \xi} = -\frac{(1+\eta)}{4} \qquad \frac{\partial \varphi_4}{\partial \eta} = \frac{(1-\xi)}{4}$$

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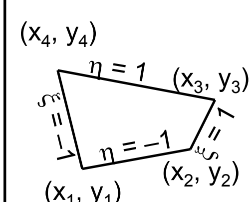
## Review x and y Derivatives




$$x_\xi = \frac{1-\eta}{4}(x_2 - x_1) + \frac{1+\eta}{4}(x_3 - x_4)$$

$$y_\xi = \frac{1-\eta}{4}(y_2 - y_1) + \frac{1+\eta}{4}(y_3 - y_4)$$

$$x_\eta = \frac{1-\xi}{4}(x_4 - x_1) + \frac{1+\xi}{4}(x_3 - x_2)$$

$$y_\eta = \frac{1-\xi}{4}(y_4 - y_1) + \frac{1+\xi}{4}(y_3 - y_2)$$

$$J = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = x_\xi y_\eta - x_\eta y_\xi$$

- Have all information to evaluate $A_{ki}$ for element

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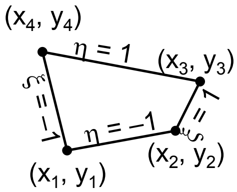
## Boundary Terms




- Element boundaries lie along line of constant $\xi = \pm 1$ or constant $\eta = \pm 1$
- Boundary integral is found along these lines
- Two cases to consider
  - Have gradient (2nd or 3rd kind) boundary condition to include in solution
  - Compute gradients from solution for Dirichlet boundary condition

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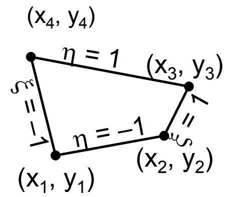
## Boundary Integral




- Find integral below whenever one side of element is on external boundary

$$\int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds \approx \left(\frac{\partial u}{\partial n}\right)_{side} \int_{start}^{end} \varphi_k ds$$

- Differential distance, $ds = S_{\xi=\pm 1} d\eta/2$ or $S_{\eta=\pm 1} d\xi/2$ where $S$ is length of side
  - *E.g.* $S_{\xi=1} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$

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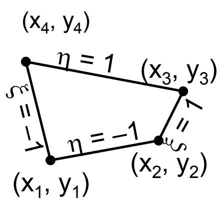
## $\xi$ = 1 Boundary as Example




$$\boxed{\int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds} \quad S_{\xi=1} = \int_2^3 ds = \int_{-1}^{1} S_{\xi=1} \frac{d\eta}{2}$$

- $\phi_1$ and $\phi_4$ are zero along $\xi = 1$ boundary
- Length of side, $S_{\xi=1}$, is $S_{\xi=1} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$
- Differential distance, $ds = S_{\xi=1} d\eta/2$

$$\varphi_2 = \frac{(1+\xi)(1-\eta)}{4} \qquad \varphi_3 = \frac{(1+\xi)(1+\eta)}{4}$$

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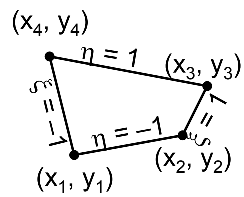
## $\xi$ = 1 Boundary Example II




- Evaluate integral for $\phi_2$ and $\phi_3$ at $\xi = 1$ $\quad \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds$

$$(\varphi_2)_{\xi=1} = \frac{(1+1)(1-\eta)}{4} = \frac{(1-\eta)}{2}$$

$$\int_\Gamma \varphi_2 \frac{\partial \hat{u}}{\partial n} ds = \left(\frac{\partial u}{\partial n}\right)_{\xi=1} S_{\xi=1} \underbrace{\int_{-1}^{1} \frac{1-\eta}{2} \frac{d\eta}{2}}_{= 1/2} = \left(\frac{\partial u}{\partial n}\right)_{\xi=1} \frac{S_{\xi=1}}{2}$$

- Same result for $\phi_3$

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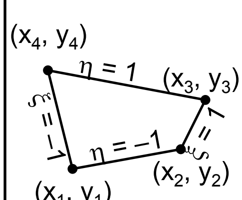
## $\xi$ = 1 Boundary Equations




- Boundary term $\quad B = \left(\frac{\partial u}{\partial n}\right)_{\xi=1} \frac{S_{\xi=1}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = 0$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = B$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = B$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = 0$$

- Equations with B not needed for Dirichlet boundary conditions
  - Used after solution to compute gradients

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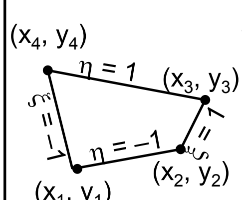
## $\eta$ = 1 Boundary Equations




- Boundary term $\quad B = \left(\frac{\partial u}{\partial n}\right)_{\eta=1} \frac{S_{\eta=1}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = 0$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = 0$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = B$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = B$$

- Equations with B not needed for Dirichlet boundary conditions
  - Used after solution to compute gradients

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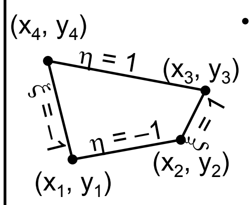
## ξ = –1 Boundary Equations




- Boundary term $B = \left(\frac{\partial u}{\partial n}\right)_{\xi=-1} \frac{S_{\xi=-1}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = B$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = 0$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = 0$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = B$$

- Same sign on B as in ξ = 1 term
  - Outward facing normal derivative is in opposite direction

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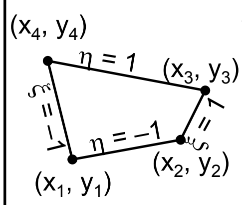
## η = –1 Boundary Equations




- Boundary term $B = \left(\frac{\partial u}{\partial n}\right)_{\eta=-1} \frac{S_{\eta=-1}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = B$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = B$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = 0$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = 0$$

- Same sign on B as in η = +1 term
  - Outward facing normal derivative is in opposite direction

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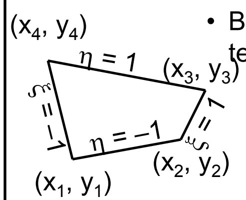
## Review Element Equations




- Boundary term $B = \left(\frac{\partial u}{\partial n}\right)_{bound} \frac{L_{bound}}{2}$

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 + A_{14}u_4 = R_1$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 + A_{24}u_4 = R_2$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 + A_{34}u_4 = R_3$$
$$A_{41}u_1 + A_{42}u_2 + A_{43}u_3 + A_{44}u_4 = R_4$$

- $R_i$ = 0 if node i is not on a boundary
- $R_i = B_{bound\_k}$ node is on boundary k and is $B_{bound\_k} + B_{bound\_m}$ for node on two boundaries

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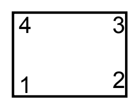
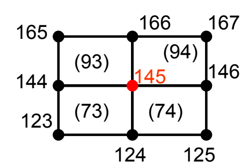
## Review Assembly

- Local *vs.* global numbers




- Assembly looks at all elements that contain a given node
  - Get element equation for local node number
    - Node 145 is node 3 in element 73, node 1 in 94
    - Use equation in 73 where node 3 is multiplied by $A_{33}$, equation in 94 where node 1 is multiplied by $A_{11}$

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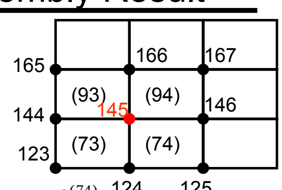
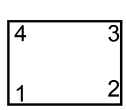
## Review Assembly Result

- Add all element equations with global indices for node 145




$$A_{31}^{(73)}u_{123} + \left[A_{32}^{(73)} + A_{41}^{(74)}\right]u_{124} + A_{42}^{(74)}u_{125} +$$
$$\left[A_{21}^{(93)} + A_{34}^{(73)}\right]u_{144} + \left[A_{11}^{(94)} + A_{22}^{(93)} + A_{33}^{(73)} + A_{44}^{(74)}\right]u_{145}$$
$$\left[A_{12}^{(94)} + A_{43}^{(74)}\right]u_{146} + A_{24}^{(93)}u_{165} +$$
$$\left[A_{14}^{(94)} + A_{23}^{(93)}\right]u_{166} + A_{13}^{(94)}u_{167} = 0$$

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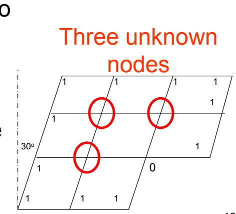
## Review April 27 Homework

- Solve Laplace's Equation for the region and finite element grid shown
  - Details in assignment
  - Boundary values are 1 at all nodes except notch where it is zero
- Elements are rhomboids with sides of h = 0.1 m
  - Use local coordinate system to get space derivatives

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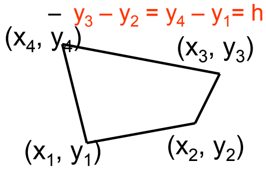
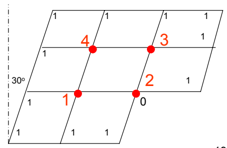
## Review April 27 Homework II

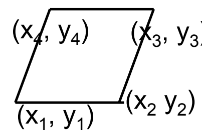
- Local coordinate system (h = 0.1 m)
  - $\xi_1 = -1$, $\eta_1 = -1$ => $x_1 = 0$, $y_1 = 0$ $\theta = 30^o$
  - $\xi_2 = 1$, $\eta_2 = -1$ => $x_2 = h$, $y_2 = 0$ $x_2 - x_1 = h$
  - $\xi_4 = -1$, $\eta_4 = 1$ => $x_4 = h \sin\theta$, $y_4 = h \cos\theta$
  - $\xi_3 = 1$, $\eta_3 = 1$ => $x_3 = x_4 + h$, $y_3 = h \cos\theta$
  - $y_3 - y_2 = y_4 - y_1 = h \cos\theta$






## Review April 27 Homework III




$$x_\xi = \frac{1-\eta}{4}(x_2 - x_1) + \frac{1+\eta}{4}(x_3 - x_4)$$

$$y_\xi = \frac{1-\eta}{4}(y_2 - y_1) + \frac{1+\eta}{4}(y_3 - y_4)$$

$$x_\eta = \frac{1-\xi}{4}(x_4 - x_1) + \frac{1+\xi}{4}(x_3 - x_2)$$

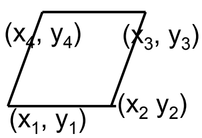
$$x_\xi = \frac{1-\eta}{4}(h) + \frac{1+\eta}{4}(h) = \frac{h}{2}$$

$$y_\xi = \frac{1-\eta}{4}(0) + \frac{1+\eta}{4}(0) = 0$$

$$x_\eta = \frac{1-\xi}{4}(h \sin\theta) + \frac{1+\xi}{4}(h \sin\theta) = \frac{h \sin\theta}{2}$$



## Review April 27 Homework IV




$$y_\eta = \frac{1-\xi}{4}(y_4 - y_1) + \frac{1+\xi}{4}(y_3 - y_2) = \frac{1-\xi}{4}(h\cos\theta) + \frac{1+\xi}{4}(h\cos\theta) = \frac{h\cos\theta}{2}$$

$$J = x_\xi y_\eta - x_\eta y_\xi = \frac{h}{2}\frac{h\cos\theta}{2} - \left(\frac{h\sin\theta}{2}\right)(0) = \frac{h^2\cos\theta}{2}$$

$$\frac{x_\eta^2 + y_\eta^2}{J} = \frac{\left(\frac{h\sin\theta}{2}\right)^2 + \left(\frac{h\cos\theta}{2}\right)^2}{\frac{h^2\cos\theta}{4}} = \frac{\frac{h^2}{4}\left(\sin^2\theta + \cos^2\theta\right)}{\frac{h^2\cos\theta}{4}} = \frac{1}{\cos\theta}$$

$$\frac{x_\eta x_\xi + y_\eta y_\xi}{J} = \frac{\frac{h\sin\theta}{2}\frac{h}{2} + \frac{h\cos\theta}{2}(0)}{\frac{h^2\cos\theta}{4}} = \frac{\sin\theta}{\cos\theta}$$



## Review April 27 Homework V

- Use one-point Gauss quadrature for (symmetric) $A_{ki}$ from April 20-22 lecture

$$f(\xi,\eta) = \frac{\partial\varphi_i}{\partial\xi}\left[\frac{x_\eta^2 + y_\eta^2}{J}\frac{\partial\varphi_k}{\partial\xi} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial\varphi_k}{\partial\eta}\right]$$

$$\left[\frac{x_\xi^2 + y_\xi^2}{J}\frac{\partial\varphi_k}{\partial\eta} - \frac{x_\eta x_\xi + y_\eta y_\xi}{J}\frac{\partial\varphi_k}{\partial\xi}\right] - \varphi_k \cancel{a}^2 \phi_i J$$

a = 0

$$A_{ki} = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \approx 4 f(0,0) \text{ by Gauss quadrature}$$



## Review April 27 Homework VI

$$\frac{x_\xi^2 + y_\xi^2}{J} = \frac{\left(\frac{h}{2}\right)^2 + 0^2}{\frac{h^2\cos\theta}{4}} = \frac{1}{\cos\theta}$$

For q = 30°, sinθ = ½ and cosθ = $3^{1/2}/2$

$$f(\xi,\eta) = \frac{\partial\varphi_i}{\partial\xi}\left[\frac{1}{\cos\theta}\frac{\partial\varphi_k}{\partial\xi} - \frac{\sin\theta}{\cos\theta}\frac{\partial\varphi_k}{\partial\eta}\right] + \frac{\partial\varphi_i}{\partial\eta}\left[\frac{1}{\cos\theta}\frac{\partial\varphi_k}{\partial\eta} - \frac{\sin\theta}{\cos\theta}\frac{\partial\varphi_k}{\partial\xi}\right]$$

$$f(\xi,\eta) = \frac{1}{\cos\theta}\left\{\frac{\partial\varphi_i}{\partial\xi}\left[\frac{\partial\varphi_k}{\partial\xi} - \sin\theta\frac{\partial\varphi_k}{\partial\eta}\right] + \frac{\partial\varphi_i}{\partial\eta}\left[\frac{\partial\varphi_k}{\partial\eta} - \sin\theta\frac{\partial\varphi_k}{\partial\xi}\right]\right\}$$

$$f(\xi,\eta) = \frac{2}{\sqrt{3}}\left\{\frac{\partial\varphi_i}{\partial\xi}\left[\frac{\partial\varphi_k}{\partial\xi} - \frac{1}{2}\frac{\partial\varphi_k}{\partial\eta}\right] + \frac{\partial\varphi_i}{\partial\eta}\left[\frac{\partial\varphi_k}{\partial\eta} - \frac{1}{2}\frac{\partial\varphi_k}{\partial\xi}\right]\right\}$$



## Review April 27 Homework VII

- Compute $A_{ki}$ integrals using Gauss quadrature with one Gauss point
  - See general form of shape function on next chart for computational ease
- Assemble equations for three unknown nodes in diagram
- Substitute boundary values and solve resulting system of three equations for the three unknowns

## Review April 27 Homework VIII

- We can write all shape functions as

$$\varphi_i = \frac{(1+a_i\xi)(1+b_i\eta)}{4}$$

$$\frac{\partial \varphi_i}{\partial \xi} = \frac{a_i(1+b_i\eta)}{4}$$

$$\frac{\partial \varphi_i}{\partial \eta} = \frac{b_i(1+a_i\xi)}{4}$$

| i | $a_i$ | $b_i$ |
|---|---|---|
| 1 | –1 | –1 |
| 2 | 1 | –1 |
| 3 | 1 | 1 |
| 4 | –1 | 1 |

- Solutions use these equations to get integrals for various $A_{ki}$



## Review April 27 Homework IX

$$f(\xi,\eta) = \frac{2}{\sqrt{3}}\left\{\frac{a_i(1+b_i\eta)}{4}\left[\frac{a_k(1+b_k\eta)}{4} - \frac{1}{2}\frac{b_k(1+a_k\xi)}{4}\right] + \frac{b_i(1+a_i\xi)}{4}\left[\frac{b_k(1+a_k\xi)}{4} - \frac{1}{2}\frac{a_k(1+b_k\eta)}{4}\right]\right\}$$

- Set x = h = 0 to get f(0,0) for Gauss quadrature

$$f(0,0) = \frac{2}{\sqrt{3}}\left\{\frac{a_i}{4}\left[\frac{a_k}{4} - \frac{1}{2}\frac{b_k}{4}\right] + \frac{b_i}{4}\left[\frac{b_k}{4} - \frac{1}{2}\frac{a_k}{4}\right]\right\}$$

$$f(0,0) = \frac{2}{\sqrt{3}}\left(\frac{a_ia_k + b_ib_k}{16} - \frac{a_ib_k + b_ia_k}{32}\right) = \frac{2(a_ia_k + b_ib_k) - (a_ib_k + b_ia_k)}{16\sqrt{3}}$$



## Review April 27 Homework X

- $A_{ki}$ = 4f(0,0) for zero-point Gauss

$$A_{ki} = 4f(0,0) = \frac{2(a_ia_k + b_ib_k) - (a_ib_k + b_ia_k)}{4\sqrt{3}}$$

$$\mathbf{A} = \frac{1}{2\sqrt{3}}\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 6 & 0 & -6 \\ -2 & 0 & 2 & 0 \\ 0 & -6 & 0 & 6 \end{bmatrix}$$

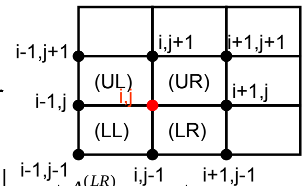
$$\begin{aligned} 2u_{11} \qquad - 2u_{13} \qquad &= 0 \\ 6u_{22} \qquad - 6u_{24} &= 0 \\ -2u_{31} \qquad + 2u_{33} \qquad &= 0 \\ -6u_{12} \qquad + 6u_{44} &= 0 \end{aligned}$$

$A_{11}$ = 2, $A_{13}$ = -2, $A_{22}$ = 6, $A_{24}$ = -6, $A_{31}$ = -2, $A_{33}$ = 2, $A_{42}$ = -6, $A_{44}$ = 6, $A_{12}$ = $A_{14}$ = $A_{21}$ = $A_{23}$ = $A_{32}$ = $A_{34}$ = $A_{41}$ = $A_{43}$ = 0



## Review April 27 Homework XI

- Add all element equations with global indices for node ij

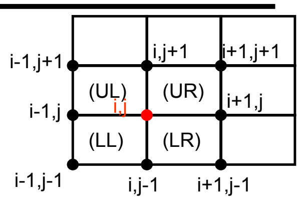



$$A_{31}^{(LL)}u_{i-1,j-1} + \left[A_{32}^{(LL)} + A_{41}^{(LR)}\right]u_{i,j-1} + A_{42}^{(LR)}u_{i+1,j-1} +$$
$$\left[A_{21}^{(UL)} + A_{34}^{(LL)}\right]u_{i-1,j} + \left[A_{11}^{(UR)} + A_{22}^{(UL)} + A_{33}^{(LL)} + A_{44}^{(LR)}\right]u_{i,j}$$
$$\left[A_{12}^{(UR)} + A_{43}^{(LR)}\right]u_{i+1,j} + A_{24}^{(UR)}u_{i-1,j+1} +$$
$$\left[A_{14}^{(UR)} + A_{23}^{(UL)}\right]u_{i,j+1} + A_{13}^{(UR)}u_{i+1,j+1} = 0$$



## Review April 27 Homework XII

- Substitute numerical values (same for all elements)




$$-2u_{i-1,j-1} + [0+0]u_{i,j-1} - 6u_{i+1,j-1} + [0+0]u_{i-1,j} + [2+6+2+6]u_{i,j}$$
$$[0+0]u_{i+1,j} - 6u_{i-1,j+1} + [0+0]u_{i,j+1} - 2u_{i+1,j+1} = 0$$

$$-2u_{i-1,j-1} - 6u_{i+1,j-1} + 16u_{i,j} - 6u_{i-1,j+1} - 2u_{i+1,j+1} = 0$$



## Higher order Shape Functions

- Equation on previous page is valid for any shape functions in a quadrilateral
- Isoparameteric elements use the same order shape functions for both the geometry and the dependent variable
  - Could use linear functions for geometry higher order for dependent variable
  - Higher order functions for geometry would allow elements with curved sides

## Higher Order Shape Functions

- Use notation at right for writing higher order shape functions
- This was used in derivations last time
- Bilinear shape functions using this notation are shown at the right

| i | $a_i$ | $b_i$ |
|---|---|---|
| 1 | –1 | –1 |
| 2 | 1 | –1 |
| 3 | 1 | 1 |
| 4 | –1 | 1 |

$$\varphi_i = \frac{(1+a_i\xi)(1+b_i\eta)}{4}$$

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## Quadratic Lagrangian

- Corner nodes $\varphi_i = \dfrac{a_i\xi(1+a_i\xi)b_i\eta(1+b_i\eta)}{4}$
- Side nodes where $\xi$ = 0 and $\eta$ = ±1 $\varphi_{i+3} = \dfrac{(1-\xi^2)b_i\eta(1+b_i\eta)}{2} \quad i = 2,3$
- Side nodes where $\eta$ = 0 and $\xi$ = ±1 $\varphi_{i+6} = \dfrac{a_i\xi(1+a_i\xi)(1-\eta^2)}{2} \quad i = 1,2$
- Central node $\varphi_9 = (1-\eta^2)(1-\xi^2)$

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## Serendipity Quadratic

- Corner nodes $\varphi_i = \dfrac{(1+a_i\xi)(1+b_i\eta)(a_i\xi+b_i\eta-1)}{4}$
- Side nodes where $\xi$ = 0 and $\eta$ = ±1 $\varphi_{i+3} = \dfrac{(1-\xi^2)(1+b_i\eta)}{2} \quad i = 2,3$
- Side nodes where $\eta$ = 0 and $\xi$ = ±1 $\varphi_{i+6} = \dfrac{(1+a_i\xi)(1-\eta^2)}{2} \quad i = 1,2$

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## Other Basis Functions

- Linear shape functions had four nodes per element and quadratic functions had nine (Lagrangian) or ten (serendipity)
- Cubic shape functions have 16 or 12 nodes for Lagrangian or serendipity
- Hermite cubic shape functions use first derivatives as unknowns
- All involve evaluation of $A_{ki}$ integrals and boundary integrals

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## Return to Basic Result ($N_i = \phi_i$)

- Have general two-dimensional result

$$\sum_{i=0}^{N} u_i \left\{ \int_\Omega \left[ \frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_k}{\partial y} - N_k a^2 N_i \right] dxdy \right\} = \int_\Gamma \varphi_k \frac{\partial \hat{u}}{\partial n} ds$$

$$k = 0, \ldots, N$$

- Apply this equation to any 2D elements
- Limited to example PDE
- Consider triangular elements next
- Use different coordinate system

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## Integral for Triangles

- Result of derivation in terms of triangle area coordinates, $\lambda_i$

$$A_{ki} = 2A \int_0^1 \int_0^{1-\lambda_2} \left[ \frac{a_1^2+b_1^2}{4A^2}\frac{\partial N_i}{\partial \lambda_1}\frac{\partial N_k}{\partial \lambda_1} + \frac{a_2^2+b_2^2}{4A^2}\frac{\partial N_i}{\partial \lambda_2}\frac{\partial N_k}{\partial \lambda_2} + \frac{a_1a_2+b_1b_2}{4A^2}\left( \frac{\partial N_i}{\partial \lambda_1}\frac{\partial N_k}{\partial \lambda_2} + \frac{\partial N_k}{\partial \lambda_1}\frac{\partial N_i}{\partial \lambda_2} \right) - N_k a^2 N_i \right] d\lambda_1 d\lambda_2$$

- In this integral A = area of triangle and $a_i$ and $b_i$ depend on vertex coordinates
  - $a_1 = y_2 - y_3$, $a_2 = y_3 - y_1$, $a_3 = y_1 - y_2$
  - $b_1 = x_3 - x_2$, $b_2 = x_1 - x_3$, $b_3 = x_2 - x_1$

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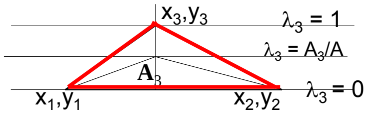
## Triangular Coordinates




- Vertices have coordinates $x_i$, $y_i$
- Natural (area) coordinates, $\lambda_i$ start at zero at side opposite vertex i, are perpendicular to that side, and go to one at vertex i
- Three such coordinates with $\lambda_i = A_i/A$

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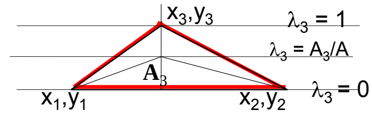
## Triangular Coordinates II




- Triangle area = A = (base)(height)/2 = $(x_2 - x_1)(y_3 - y_1)/2$ (for this triangle)
- Inner area $A_3$ has height which is $\lambda_3$ times total height, *i.e.* $h_3 = \lambda_3(y_3 - y_1)$ so that $A_3 = (x_2 - x_1)[\lambda_3(y_3 - y_1)]/2$
- Dividing $A_3$ by A gives $\lambda_3 = A_3/A$

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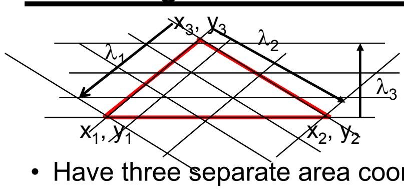
## Triangular Coordinates III




- Have three separate area coordinate variables, $\lambda_1$, $\lambda_2$, and $\lambda_3$
- Each goes from zero to one
- Each $\lambda_i = A_i/A$ so that $\lambda_1 + \lambda_2 + \lambda_3 = A_1/A + A_2/A + A_3/A = A/A = 1$

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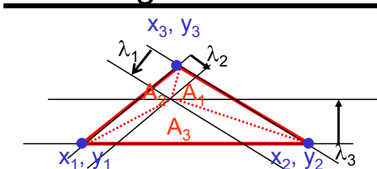
## Triangular Coordinates IV




- Lines from point where all $\lambda_i$ intersect to each vertex defines three subareas
- Previous chart showed that each area, $A_i$ = $(\text{height})_i(\text{base})_i\lambda_i/2 = \lambda_i A$ or $\lambda_i = A_i/A$
- Since $A_1 + A_2 + A_3 = A$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$

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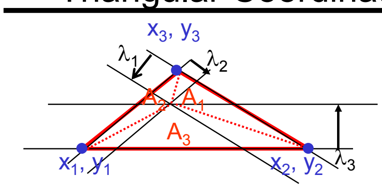
## Triangular Coordinates V




- At any point in the triangle the x, y and $\lambda_i$ coordinates are related as follows

$$x = x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 \qquad y = y_1\lambda_1 + y_2\lambda_2 + y_3\lambda_3$$

- Can write matrix equation to relate x and y to $\lambda_i$ plus constraint that $\lambda_1 + \lambda_2 + \lambda_3 = 1$

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## Coordinate Transformations

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

- Use formula for $b_{ij}$, the components of $\mathbf{B} = \mathbf{A}^{-1}$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$$b_{ij} = \frac{(-1)^{i+j} M_{ji}}{Det\mathbf{A}}$$

- $M_{ji}$ is minor determinant

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## Finding the Inverse

$$Det\begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix} = \begin{matrix}x_2y_3 + x_1y_2 + x_3y_1\\ -x_2y_1 - x_3y_2 - x_1y_3\end{matrix} = 2A$$

- Details for area, A, at end of presentation

$$\begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix}^{-1} = \frac{1}{2A}\begin{bmatrix}x_2y_3 - y_2x_3 & -(y_3 - y_2) & x_3 - x_2\\ -(x_1y_3 - y_1x_3) & y_3 - y_1 & -(x_3 - x_1)\\ x_1y_2 - y_1x_2 & -(y_2 - y_1) & x_2 - x_1\end{bmatrix}$$

$M_{32}$ $b_{23}$

$$b_{ij} = \frac{(-1)^{i+j}M_{ji}}{Det\mathbf{A}}$$

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## Getting $\lambda_i$ from x and y

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = \frac{1}{2A}\begin{bmatrix}x_2y_3 - y_2x_3 & y_2 - y_3 & x_3 - x_2\\ y_1x_3 - x_1y_3 & y_3 - y_1 & x_2 - x_3\\ x_2y_3 - y_1x_2 & y_1 - y_2 & x_2 - x_1\end{bmatrix}\begin{bmatrix}1\\ x\\ y\end{bmatrix}$$

- This equation gives all $\lambda_i$ coordinates for any Cartesian coordinate x, y
- Can express shape functions of any order in terms of $\lambda_i$ coordinates

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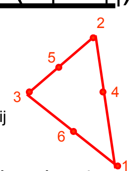
## Triangle Shape Functions ($N_i \equiv \phi_i$)




- Linear shape functions: $N_i = \lambda_i$
  - Three nodes at triangle vertices
  - Satisfy basic result that $N_i(\mathbf{x}_j) = \delta_{ij}$
- Quadratic shape functions
  - Nodes 1, 2, and 3 at vertices and nodes 4, 5, and 6 at midpoints of edge

$$N_1 = \lambda_1(2\lambda_1 - 1) \quad N_4 = 4\lambda_1\lambda_2$$
$$N_2 = \lambda_2(2\lambda_2 - 1) \quad N_5 = 4\lambda_2\lambda_3$$
$$N_3 = \lambda_3(2\lambda_3 - 1) \quad N_6 = 4\lambda_3\lambda_1$$

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## Shape Function Derivatives

- Finite element coefficients have derivatives and integrals of shape functions with respect to x and y
- Have to get derivatives and integrals with respect to the $\lambda_i$
- Only two of the $\lambda_i$ are independent
- Can pick $\lambda_1$ and $\lambda_2$ as independent values of $\lambda_i$
- Have usual equations for transforms

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## Shape Function Derivatives II

- Transform derivative of any function, $\psi$

$$\frac{\partial\psi}{\partial x} = \frac{\partial\lambda_1}{\partial x}\frac{\partial\psi}{\partial\lambda_1} + \frac{\partial\lambda_2}{\partial x}\frac{\partial\psi}{\partial\lambda_2} \qquad \frac{\partial\psi}{\partial y} = \frac{\partial\lambda_1}{\partial y}\frac{\partial\psi}{\partial\lambda_1} + \frac{\partial\lambda_2}{\partial y}\frac{\partial\psi}{\partial\lambda_2}$$

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = \frac{1}{2A}\begin{bmatrix}x_2y_3 - y_2x_3 & y_2 - y_3 & x_3 - x_2\\ y_1x_3 - x_1y_3 & y_3 - y_1 & x_2 - x_3\\ x_2y_3 - y_1x_2 & y_1 - y_2 & x_2 - x_1\end{bmatrix}\begin{bmatrix}1\\ x\\ y\end{bmatrix}$$

$$\lambda_1 = \frac{x_2y_3 - y_2x_3}{2A} + \frac{y_2 - y_3}{2A}x + \frac{x_3 - x_2}{2A}y$$

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## Shape Function Derivatives III

- Summary of partial derivatives required for natural coordinate derivative transformations
  - Definitions of $a_i$ and $b_i$ given here

$$\frac{\partial\lambda_1}{\partial x} = \frac{y_2 - y_3}{2A} = \frac{a_1}{2A} \qquad \frac{\partial\lambda_1}{\partial y} = \frac{x_3 - x_2}{2A} = \frac{b_1}{2A}$$
$$\frac{\partial\lambda_2}{\partial x} = \frac{y_3 - y_1}{2A} = \frac{a_2}{2A} \qquad \frac{\partial\lambda_2}{\partial y} = \frac{x_1 - x_3}{2A} = \frac{b_2}{2A}$$
$$\frac{\partial\lambda_3}{\partial x} = \frac{y_1 - y_2}{2A} = \frac{a_3}{2A} \qquad \frac{\partial\lambda_3}{\partial y} = \frac{x_2 - x_1}{2A} = \frac{b_3}{2A}$$

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## Back to Basics

- Change notation for shape functions from $\phi$ to N and use basic Galerkin equation for Helmholtz equation

$$\sum_{i=0}^{M} u_i \left\{ \underbrace{\int_\Omega \left[ \frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_k}{\partial y} - N_k a^2 N_i \right] dxdy}_{A_{ki}} \right\} = \int_\Gamma N_k \frac{\partial \hat{u}}{\partial n} ds \qquad k = 0, \ldots, M$$

- Triangular element with $N_{(e)}$ nodes
- Transform derivatives and integral from x, y to $\lambda_1$, $\lambda_2$



## Derivative Transformations

- Use previous equations for $\partial\psi/\partial x$, etc.

$$\frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x} = \left[ \frac{\partial \lambda_1}{\partial x}\frac{\partial N_i}{\partial \lambda_1} + \frac{\partial \lambda_2}{\partial x}\frac{\partial N_i}{\partial \lambda_2} \right]\left[ \frac{\partial \lambda_1}{\partial x}\frac{\partial N_k}{\partial \lambda_1} + \frac{\partial \lambda_2}{\partial x}\frac{\partial N_k}{\partial \lambda_2} \right] = \left[ \frac{a_1}{2A}\frac{\partial N_i}{\partial \lambda_1} + \frac{a_2}{2A}\frac{\partial N_i}{\partial \lambda_2} \right]\left[ \frac{a_1}{2A}\frac{\partial N_k}{\partial \lambda_1} + \frac{a_2}{2A}\frac{\partial N_k}{\partial \lambda_2} \right]$$

$$\frac{\partial N_i}{\partial y}\frac{\partial N_k}{\partial y} = \left[ \frac{\partial \lambda_1}{\partial y}\frac{\partial N_i}{\partial \lambda_1} + \frac{\partial \lambda_2}{\partial y}\frac{\partial N_i}{\partial \lambda_2} \right]\left[ \frac{\partial \lambda_1}{\partial y}\frac{\partial N_k}{\partial \lambda_1} + \frac{\partial \lambda_2}{\partial y}\frac{\partial N_k}{\partial \lambda_2} \right] = \left[ \frac{b_1}{2A}\frac{\partial N_i}{\partial \lambda_1} + \frac{b_2}{2A}\frac{\partial N_i}{\partial \lambda_2} \right]\left[ \frac{b_1}{2A}\frac{\partial N_k}{\partial \lambda_1} + \frac{b_2}{2A}\frac{\partial N_k}{\partial \lambda_2} \right]$$



## Derivative Transformations II

- Combine equations from last chart

$$\frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_k}{\partial y} = \left[ \frac{a_1}{2A}\frac{\partial N_i}{\partial \lambda_1} + \frac{a_2}{2A}\frac{\partial N_i}{\partial \lambda_2} \right]\left[ \frac{a_1}{2A}\frac{\partial N_k}{\partial \lambda_1} + \frac{a_2}{2A}\frac{\partial N_k}{\partial \lambda_2} \right] + \left[ \frac{b_1}{2A}\frac{\partial N_i}{\partial \lambda_1} + \frac{b_2}{2A}\frac{\partial N_i}{\partial \lambda_2} \right]\left[ \frac{b_1}{2A}\frac{\partial N_k}{\partial \lambda_1} + \frac{b_2}{2A}\frac{\partial N_k}{\partial \lambda_2} \right]$$

$$= \frac{a_1^2 + b_1^2}{4A^2}\frac{\partial N_i}{\partial \lambda_1}\frac{\partial N_k}{\partial \lambda_1} + \frac{a_2^2 + b_2^2}{4A^2}\frac{\partial N_i}{\partial \lambda_2}\frac{\partial N_k}{\partial \lambda_2} + \frac{a_1 a_2 + b_1 b_2}{4A^2}\left( \frac{\partial N_i}{\partial \lambda_1}\frac{\partial N_k}{\partial \lambda_2} + \frac{\partial N_k}{\partial \lambda_1}\frac{\partial N_i}{\partial \lambda_2} \right)$$



## Transform dxdy Integral to $d\lambda_1 d\lambda_2$

- Use Jacobian determinant, J

$$d\lambda_1 d\lambda_2 = J dxdy = \begin{vmatrix} \frac{\partial \lambda_1}{\partial x} & \frac{\partial \lambda_1}{\partial y} \\ \frac{\partial \lambda_2}{\partial x} & \frac{\partial \lambda_2}{\partial y} \end{vmatrix} dxdy$$

$$J = \begin{vmatrix} \frac{\partial \lambda_1}{\partial x} & \frac{\partial \lambda_1}{\partial y} \\ \frac{\partial \lambda_2}{\partial x} & \frac{\partial \lambda_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y_3 - y_2}{2A} & \frac{x_3 - x_2}{2A} \\ \frac{y_3 - y_1}{2A} & -\frac{x_3 - x_1}{2A} \end{vmatrix} = \frac{(y_3 - y_2)(x_3 - x_1)}{4A^2} - \frac{(y_3 - y_1)(x_3 - x_2)}{4A^2}$$

- J = 1/2A (details at end of presentation)



## Integral for Triangles

- Original $A_{ki}$ equation

$$A_{ki} = \int_\Omega \left[ \frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_k}{\partial y} - N_k a^2 N_i \right] dxdy$$

- After transformations and Jacobian
  - Derivation of integral limits on charts at end of presentation

$$A_{ki} = 2A \int_0^1 \int_0^{1-\lambda_2} \left[ \frac{a_1^2 + b_1^2}{4A^2}\frac{\partial N_i}{\partial \lambda_1}\frac{\partial N_k}{\partial \lambda_1} + \frac{a_2^2 + b_2^2}{4A^2}\frac{\partial N_i}{\partial \lambda_2}\frac{\partial N_k}{\partial \lambda_2} + \frac{a_1 a_2 + b_1 b_2}{4A^2}\left( \frac{\partial N_i}{\partial \lambda_1}\frac{\partial N_k}{\partial \lambda_2} + \frac{\partial N_k}{\partial \lambda_1}\frac{\partial N_i}{\partial \lambda_2} \right) - N_k a^2 N_i \right] d\lambda_1 d\lambda_2$$



## Triangular Element Equations

- For linear shape functions, $N_i = \lambda_i$
  - Integrals at end of presentation

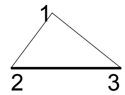
$$A_{ki} = \int_\Omega \left[ \frac{\partial N_i}{\partial x}\frac{\partial N_k}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_k}{\partial y} - N_k a^2 N_i \right] dxdy = \frac{a_i a_k + b_i b_k}{4A} + \frac{Aa^2}{12}(1 + \delta_{ik})$$

- Right-hand sides $B_k$ = 0 except for boundary nodes

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 = B_1$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 = B_2$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 = B_3$$

## Border Terms

- Consider side 2-3, ds = $L_{23} d\lambda_3$
- Others will be similar




$$\sum_{i=0}^{M} A_{ki}u_i = \int_{\Gamma^{(e)}} N_k \frac{\partial \hat{u}}{\partial n} ds = \left.\frac{\partial \hat{u}}{\partial n}\right|_{23} \int_2^3 N_k ds = L_{23} \left.\frac{\partial \hat{u}}{\partial n}\right|_{23} \int_0^1 N_k d\lambda_3 \qquad k = 1,2,3$$

- Along side 2-3, $\lambda_1 = 0$ we only have to evaluate this integral for k = 2 and k = 3

$$\sum_{i=0}^{M} A_{3i}u_i = B_3^{(23)} \equiv L_{23}\left.\frac{\partial \hat{u}}{\partial n}\right|_{23}\int_0^1 \lambda_3 d\lambda_3 = L_{23}\left.\frac{\partial \hat{u}}{\partial n}\right|_{23}\left[\frac{\lambda_3^2}{2}\right]_0^1 = \frac{L_{23}}{2}\left.\frac{\partial \hat{u}}{\partial n}\right|_{23}$$



## Border Terms II

- To get border integral for k = 2 with $N_2 = \lambda_2$, use $\lambda_1 + \lambda_2 + \lambda_3 = 1$ so $d\lambda_3 = -d\lambda_1 - d\lambda_2 = -d\lambda_2$ since $\lambda_1$ is constant along 23
  - On side 23, $\lambda_2$ goes from 1 to zero as we integrate in a counterclockwise direction

$$\sum_{i=0}^{M} A_{2i}u_i = B_2^{(23)} = L_{23}\left.\frac{\partial \hat{u}}{\partial n}\right|_{23}\int_1^0 \lambda_2(-d\lambda_2) = -L_{23}\left.\frac{\partial \hat{u}}{\partial n}\right|_{23}\left[\frac{\lambda_3^2}{2}\right]_1^0 = \frac{L_{23}}{2}\left.\frac{\partial \hat{u}}{\partial n}\right|_{23}$$

- Similar results for other borders
  - Terms appear in two of three element equations



## Border Terms III

- Element equations for border on side 23

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 = 0$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 = B_2^{(23)}$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 = B_3^{(23)}$$
$$B_2^{(23)} = B_3^{(23)} = \frac{L_{23}}{2}\left.\frac{\partial \hat{u}}{\partial n}\right|_{23}$$

- Element equations for border on side 31

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 = B_1^{(31)}$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 = 0$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 = B_3^{(31)}$$
$$B_1^{(31)} = B_3^{(31)} = \frac{L_{31}}{2}\left.\frac{\partial \hat{u}}{\partial n}\right|_{31}$$



## Border Terms III

- Element equations for border on side 12

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 = B_1^{(12)}$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 = B_2^{(12)}$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 = 0$$
$$B_1^{(12)} = B_2^{(12)} = \frac{L_{12}}{2}\left.\frac{\partial \hat{u}}{\partial n}\right|_{12}$$

- Define $R_i = 0$ for nodes not on external boundary and $R_i = B_i^{(jk)}$ otherwise

$$A_{11}u_1 + A_{12}u_2 + A_{13}u_3 = R_1$$
$$A_{21}u_1 + A_{22}u_2 + A_{23}u_3 = R_2$$
$$A_{31}u_1 + A_{32}u_2 + A_{33}u_3 = R_3$$



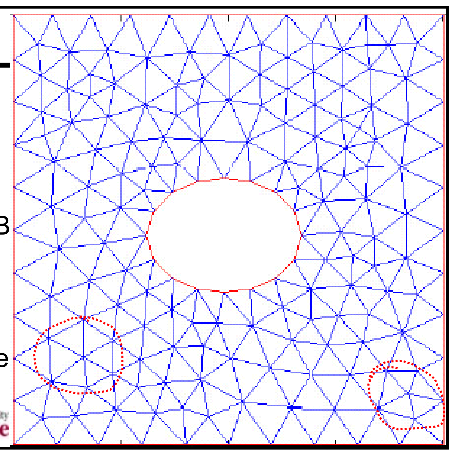
## Assembly

- Away from boundary typical node is part of five or six triangles
- Must consider element equations for that node from all triangles
- Each element has three equations
  - To assemble equations for one node, pick the one equation (from three) from each element where node appears
    - pick equation where the nodal value is multiplied by $A_{kk}$ (coefficient whose subscripts are the same)



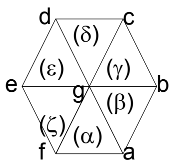
## Mesh

- Triangular mesh from MATLAB PDE toolbox
- 5 or 6 per node



## Assembly for Six Triangles

- Central node g is part of six elements labeled from (α) to (ζ)
  - Assume node g in global system is node 3 for all triangles in local system




$$A_{31}^{(\alpha)}u_f + A_{32}^{(\alpha)}u_a + A_{33}^{(\alpha)}u_g = R_3^{(\alpha)} \qquad A_{31}^{(\delta)}u_c + A_{32}^{(\delta)}u_d + A_{33}^{(\delta)}u_g = R_3^{(\delta)}$$

$$A_{31}^{(\beta)}u_a + A_{32}^{(\beta)}u_b + A_{33}^{(\beta)}u_g = R_3^{(\beta)} \qquad A_{31}^{(\varepsilon)}u_d + A_{32}^{(\varepsilon)}u_e + A_{33}^{(\varepsilon)}u_g = R_3^{(\varepsilon)}$$

$$A_{31}^{(\gamma)}u_b + A_{32}^{(\gamma)}u_c + A_{33}^{(\gamma)}u_g = R_3^{(\gamma)} \qquad A_{31}^{(\varsigma)}u_e + A_{32}^{(\varsigma)}u_f + A_{33}^{(\varsigma)}u_g = R_3^{(\varsigma)}$$

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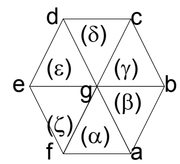
## Assembly for Six Triangles II

- Add all element equations for node g
  - Other systems will not have all node g coefficients as $A_{33}$




$$\left(A_{32}^{(\alpha)} + A_{31}^{(\beta)}\right)u_a + \left(A_{31}^{(\gamma)} + A_{32}^{(\beta)}\right)u_b + \left(A_{32}^{(\gamma)} + A_{31}^{(\delta)}\right)u_c$$
$$+\left(A_{32}^{(\delta)} + A_{31}^{(\varepsilon)}\right)u_d + \left(A_{32}^{(\varepsilon)} + A_{31}^{(\varsigma)}\right)u_e + \left(A_{32}^{(\varsigma)} + A_{31}^{(\alpha)}\right)u_f$$
$$+\left(A_{33}^{(\alpha)} + A_{33}^{(\beta)} + A_{33}^{(\gamma)} + A_{33}^{(\delta)} + A_{33}^{(\varepsilon)} + A_{33}^{(\varsigma)}\right)u_g$$
$$= R_3^{(\alpha)} + R_3^{(\beta)} + R_3^{(\gamma)} + R_3^{(\delta)} + R_3^{(\varepsilon)} + R_3^{(\varsigma)}$$

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## Triangle Quality

- For accuracy want all triangles to be as close to equilateral triangles as possible
- Area of equilateral triangle with side b

$$A = \frac{bh}{2} = \frac{b}{2}\frac{b\sqrt{3}}{2} = \frac{b^2\sqrt{3}}{4}\frac{3}{3} = \frac{3b^2}{4\sqrt{3}} \quad \Rightarrow \quad \frac{4\sqrt{3}A}{3b^2} = 1$$

- Quality of triangle with sides $b_1$, $b_2$, and $b_3$

$$Q = \frac{4\sqrt{3}A}{b_1^2 + b_2^2 + b_3^2}$$

- Keep Q > 0.6

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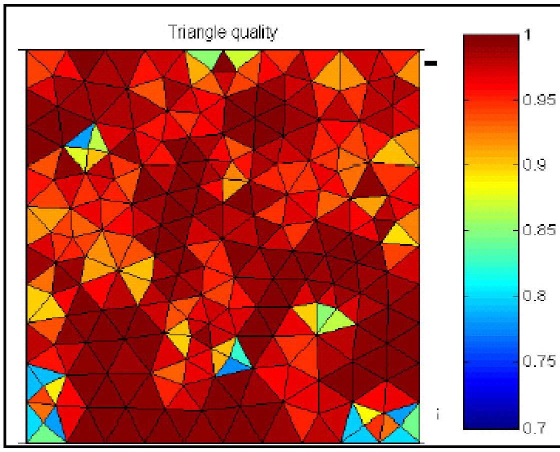
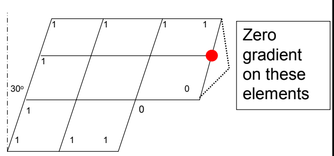



## May 4 Homework

- Two problems
  - Repeat finite element problem from last week using gradient boundaries
  - Triangular finite elements
- Get element equations from homework solution
  - One gradient boundary node

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## May 4 Homework II

- Recall quadrilateral shape function form

$$\varphi_i = \frac{(1 + a_i\xi)(1 + b_i\eta)}{4}$$

| i | $a_i$ | $b_i$ |
|---|---|---|
| 1 | –1 | –1 |
| 2 | 1 | –1 |
| 3 | 1 | 1 |
| 4 | –1 | 1 |

- Use exact result for $A_{ki}$ integrals

$$A_{ki} = \frac{a_i a_k + b_i b_k + \dfrac{2a_i a_k b_i b_k}{3} - (a_i b_k + b_i a_k)\sin\theta}{4\cos\theta}$$

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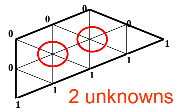
## May 4 Homework III

- Triangular element problem
  - All triangles equilateral
  - Each side is h = 0.1 m
  - Dirichlet boundary conditions
  - Definitions of $a_i$ and $b_i$ shown below
  - Coordinates for one triangle: $x_1 = y_1 = 0$; $x_2 = h$, $y_2 = 0$; $x_3 = h/2$, $y_3 = 3^{1/2}h/2$
  - Area, $A = 3^{1/2}h^2/4$




$$a_1 = y_2 - y_3 \quad b_1 = -x_2 + x_3$$
$$a_2 = y_3 - y_1 \quad b_2 = -x_3 + x_1$$
$$a_3 = y_1 - y_2 \quad b_3 = -x_1 + x_2$$

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## May 4 Homework IV

- Integrals for $A_{ki}$ from April 27-29 lectures
  - a = 0 for Laplace's equation

$$A_{ki} = \frac{a_i a_k}{4A} + \frac{b_i b_k}{4A} - \cancelto{0}{\frac{Aa^2}{12}}(1+\delta_{ik})$$

- Have three equations for each element
  - Nine (symmetric) $A_{ki}$ values
- Assemble equations from six adjoining elements to get two equations for two unknowns in overall grid

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## Additional Charts

- Derivations not covered in class
  - Charts 70 and 71: the area of a triangle in terms of the coordinates of its vertices
  - Charts 72 and 73: Jacobian for converting area dxdy to area $d\lambda_1 d\lambda_2$s
  - Charts 74 and 75: getting and verifying limits for triangle integration limits
  - Charts 76 to 78: details of $A_{ki}$ integration for linear shape functions

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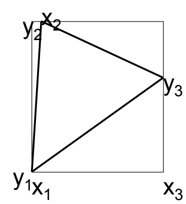
## Area Calculation Details

- Create rectangle around triangle with arbitrary orientation
- Triangle area is rectangle area minus area of three triangles




- $A = (y_2 - y_1)(x_3 - x_1) - (y_2 - y_1)(x_2 - x_1)/2 - (y_3 - y_1)(x_3 - x_1)/2 - (y_2 - y_3)(x_3 - x_2)/2$
- $2A = 2(y_2 - y_1)(x_3 - x_1) - (y_2 - y_1)(x_2 - x_1) - (y_3 - y_1)(x_3 - x_1) - (y_2 - y_3)(x_3 - x_2)$

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## Area Calculation Details II

- Multiply out and cancel terms with capital letters for labels

$$2A = \underbrace{2y_3x_2}_{2G} \underbrace{-2y_3x_1}_{-2B} \underbrace{-2y_1x_2}_{-2C} + \underbrace{2y_1x_1}_{2D} \underbrace{-y_3x_3}_{-E} + \underbrace{y_3x_1}_{B} + y_1x_3 \underbrace{-y_1x_1}_{-D}$$
$$\underbrace{-y_2x_2}_{-F} + y_2x_1 + \underbrace{y_1x_2}_{C} \underbrace{-y_1x_1}_{-D} \underbrace{-y_3x_2}_{-G} + \underbrace{y_3x_3}_{E} + \underbrace{y_2x_2}_{F} - y_2x_3$$

- Final result

$$2A = y_3x_2 - y_3x_1 - y_1x_2 + y_1x_3 + y_2x_1 - y_2x_3$$

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## Jacobian from dxdy to $d\lambda_1 d\lambda_2$

- Use Jacobian determinant, J

$$d\lambda_1 d\lambda_2 = J dxdy = \begin{vmatrix} \frac{\partial \lambda_1}{\partial x} & \frac{\partial \lambda_1}{\partial y} \\ \frac{\partial \lambda_2}{\partial x} & \frac{\partial \lambda_2}{\partial y} \end{vmatrix} dxdy$$

$$J = \begin{vmatrix} \frac{\partial \lambda_1}{\partial x} & \frac{\partial \lambda_1}{\partial y} \\ \frac{\partial \lambda_2}{\partial x} & \frac{\partial \lambda_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y_3 - y_2}{2A} & \frac{x_3 - x_2}{2A} \\ \frac{y_3 - y_1}{2A} & -\frac{x_3 - x_1}{2A} \end{vmatrix} = \frac{(y_3 - y_2)(x_3 - x_1)}{4A^2} - \frac{(y_3 - y_1)(x_3 - x_2)}{4A^2}$$

$$4A^2 J = (y_3 - y_2)(x_3 - x_1) - (y_3 - y_1)(x_3 - x_2)$$

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## Jacobian Details

- Multiply by $4A^2$ and show that numerator gives previous expression for 2A in terms of triangle vertices' coordinates

$$4A^2 J = (y_3 - y_2)(x_3 - x_1) - (y_3 - y_1)(x_3 - x_2)$$
$$= \cancel{y_3 x_3} - y_3 x_1 - y_2 x_3 + y_2 x_1$$
$$- \cancel{y_3 x_3} + y_3 x_2 + y_1 x_3 - y_1 x_2 = 2A$$

$$d\lambda_1 d\lambda_2 = J dxdy = \frac{1}{2A} dxdy \qquad dxdy = 2A d\lambda_1 d\lambda_2$$

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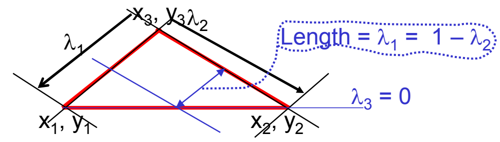
## Integrating Triangle Area




- Use following limits for integration over entire area of triangle

$$\int_\Omega f(\lambda_1, \lambda_2, \lambda_3) dxdy = \int_\Omega f(\lambda_1, \lambda_2, \lambda_3) 2A d\lambda_1 d\lambda_2$$
$$= 2A \int_0^1 \int_0^{1-\lambda_2} f(\lambda_1, \lambda_2, \lambda_3) d\lambda_1 d\lambda_2$$

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## Confirm Integration Limits

- If f = 1 we should get area = 1/2

$$A = \int_\Omega dxdy = \int_\Omega 2A d\lambda_1 d\lambda_2 = 2A \int_0^1 \int_0^{1-\lambda_2} d\lambda_1 d\lambda_2 = 2A \int_0^1 [\lambda_1]_0^{1-\lambda_2} d\lambda_2$$
$$= 2A \int_0^1 (1 - \lambda_2) d\lambda_2 = 2A \left[ \lambda_2 - \frac{\lambda_2^2}{2} \right]_0^1 = 2A \left[ 1 - \frac{1}{2} \right] = A$$

- Confirms correct limits for integrating complete area of a triangle using the natural (area) coordinates, $\lambda_i$
  - Also shows integral value = 1/2

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## Linear Shape Functions

- Start with general two-dimensional integral for Helmholtz equation example

$$A_{ki} = \int_\Omega \left[ \frac{\partial N_i}{\partial x} \frac{\partial N_k}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_k}{\partial y} - N_k a^2 N_i \right] dxdy$$

- For linear shape functions $N_i = \lambda_i$

$$A_{ki} = 2A \int_0^1 \int_0^{1-\lambda_2} \left[ \frac{\partial \lambda_i}{\partial x} \frac{\partial \lambda_k}{\partial x} + \frac{\partial \lambda_i}{\partial y} \frac{\partial \lambda_k}{\partial y} \right] d\lambda_1 d\lambda_2 - a^2 2A \int_0^1 \int_0^{1-\lambda_2} \lambda_k \lambda_i d\lambda_1 d\lambda_2$$
$$= 2A \int_0^1 \int_0^{1-\lambda_2} \left[ \frac{a_i}{2A} \frac{a_k}{2A} + \frac{b_i}{2A} \frac{b_k}{2A} \right] d\lambda_1 d\lambda_2 - a^2 2A \int_0^1 \int_0^{1-\lambda_2} \lambda_k \lambda_i d\lambda_1 d\lambda_2$$

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## Linear Shape Functions II

$$2A \int_0^1 \int_0^{1-\lambda_2} \left[ \frac{a_i}{2A} \frac{a_k}{2A} + \frac{b_i}{2A} \frac{b_k}{2A} \right] d\lambda_1 d\lambda_2 =$$
$$= \frac{a_i a_k + b_i b_k}{2A} \int_0^1 \int_0^{1-\lambda_2} d\lambda_1 d\lambda_2 = \frac{a_i a_k + b_i b_k}{2A} \frac{1}{2} = \frac{a_i a_k + b_i b_k}{4A}$$

- Use following general result (we have shown it is correct for $m_1 = m_2 = m_3 = 0$)

$$\int_\Omega \lambda_1^{m_1} \lambda_2^{m_2} \lambda_3^{m_3} d\Omega = 2A \frac{m_1! m_2! m_3!}{(m_1 + m_2 + m_3 + 2)!}$$

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## Linear Shape Functions III

- Use general form just stated to get remainder of $A_{ki}$ integral for $N_i = \lambda_i$

$$\int_\Omega N_k a^2 N_i dxdy = a^2 \int_\Omega \lambda_k \lambda_i dxdy =$$
$$\left. \begin{array}{ll} a^2 2A \dfrac{1!1!0!}{(1+1+0+2)!} = \dfrac{Aa^2}{12} & i \neq k \\ a^2 2A \dfrac{2!0!0!}{(2+0+0+2)!} = \dfrac{Aa^2}{6} & i = k \end{array} \right\} = \frac{Aa^2}{12} (1 + \delta_{ik})$$

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