California State University Northridge

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Engineering Analysis Notes

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Vectors and Matrices

Introduction

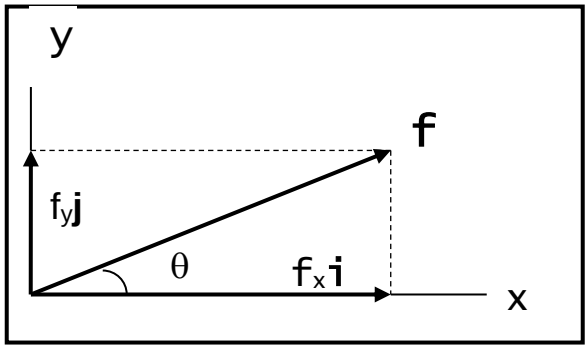
These notes provide an introduction to the use of vectors and matrices in engineering analysis. In addition, they provide a discussion of how the simple concept of a vector in mechanics leads to the concept of vector spaces for engineering analysis.

Matrix notation is used to simplify the representation of linear algebraic equations. In addition, the matrix representation of systems of equations provides important properties regarding the system of equations. The discussion here presents many results without proof. You can refer to a general advanced engineering math text, like the one by Kreyszig or a text on linear algebra for such proofs.

Parts of these notes have been prepared for use in a variety of courses to provide background information on the use of matrices in engineering problems. Consequently, some of the material may not be used in this course and different sections from these notes may be assigned at different times in the course.

Vectors, Linear Independence and Basis Sets

A vector is a common concept in engineering mechanics that most students first saw in their high-school physics courses. Vectors are usually described in introductory courses as a quantity that has a magnitude and a direction. Force and velocity are common examples of vectors used in a basic mechanics course.



In addition to representing a vector in terms of its magnitude and direction, we can also represent a vector in terms of its components. This is illustrated in the figure at the right. Here we have a force vector, **f**, with a magnitude, |**f**|, and a direction, θ, relative to the x axis. (Note that the notation of the vector, **f**, and its magnitude, |**f**|, are different. The vector is the full specification of a magnitude and direction; e.g., 2000 pounds force at an angle of 30° from the x axis. The magnitude |**f**| is 2000 pounds in this example.) The components of the vector in the x and y directions are called f_x and f_y, respectively. These are not vectors, but are scalars that are multiplied by the unit vectors in the x and y direction to give the vector forces in the coordinate directions. The unit vectors in the x and y direction are usually given the symbols **i** and **j**, respectively. In this case we would write the vector in terms of its components as $\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j}$. The vector components are called scalars to distinguish them from vectors. (Formally a scalar is defined as a quantity which is invariant under a coordinate transformation.)

The concept of writing a vector in terms of its components is an important one in engineering analysis. Instead of writing $\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j}$, we can write $\mathbf{f} = [f_x \ f_y]$, with the understanding that the first number is the x component of the vector and the second number is the y component of the vector. Using this notation we can write the unit vectors in the x and y directions as **i** = [1 0] and **j** = [0 1]. This notation for unit vectors provides a link between representing a vector as a row or column matrix, as we will do below, and the conventional vector notation: $\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j}$ and $\mathbf{f} = [f_x \ f_y]$. If we substitute **i** = [1 0] and **j** = [0 1] in the equation $\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j}$, we get the result that $\mathbf{f} = f_x[1, 0] + f_y[0, 1] = [f_x \ f_y]$. In place of the notation f_x and f_y for the x and y components, we can use numerical subscripts for the coordinate directions and components. In this scheme we would call the x and y coordinate directions the x_1 and x_2 directions and the vector components

would be labeled as f_1 and f_2. The numerical notation allows a generalization to systems with an arbitrary number of dimensions.

From the diagram of the vector, **f**, and its components, we see that the magnitude of the vector, |**f**|, is given by Pythagoras's theorem: $|\mathbf{f}| = \sqrt{f_x^2 + f_y^2} = \sqrt{f_1^2 + f_2^2}$. We know that we can extend the two-dimensional vector shown on the previous page to three dimensions. In this case our vectors have three components, one in each coordinate direction. We can write the unit vectors in the three coordinate directions as **i** = [1 0 0], **j** = [0 1 0], and **k** = [0 0 1]. We would then write our three-dimensional vector, using numerical subscripts in place of x, y, and z subscripts, as $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ or $\mathbf{f} = [f_1 \;\; f_2 \;\; f_3]$. If we substitute **i** = [1 0 0], **j** = [0 1 0], and **k** = [0, 0, 1] in the equation $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$, we get the result that $\mathbf{f} = f_1[1\;0\;0] + f_2[0\;1\;0] + f_3[0\;0\;1] = [f_1, f_2, f_3]$.

The dot product of two vectors, **a** and **b** is written as **a•b**. The dot product is a scalar and its value is $|\mathbf{a}||\mathbf{b}|\cos(\theta)$, where θ is the angle between the two vectors. The magnitude of the unit vectors, **i**, **j**, and **k**, is one. Each unit vector is parallel to itself so if we evaluate **i•i**, **j•j**, or **k•k**, we get |1||1|cos(0) = 1 for the dot product. Any two different unit vectors are perpendicular to each other so the angle between them is 90°; thus the dot product of any two different unit vectors is |1||1| cos(90°) = 0. The dot product of two vectors, expressed in terms of their components can be written as follows. $\mathbf{a}\bullet\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\bullet(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = a_1b_1\mathbf{i}\bullet\mathbf{i} + a_1b_2\mathbf{i}\bullet\mathbf{j} + a_1b_3\mathbf{i}\bullet\mathbf{k} + a_2b_1\mathbf{j}\bullet\mathbf{i} + a_2b_2\mathbf{j}\bullet\mathbf{j} + a_2b_3\mathbf{j}\bullet\mathbf{k} + a_3b_1\mathbf{k}\bullet\mathbf{i} + a_3b_2\mathbf{k}\bullet\mathbf{j} + a_3b_3\mathbf{k}\bullet\mathbf{k} = a_1b_1 + a_2b_2 + a_3b_3$. This result – the dot product of two vectors is the sum of the products of the individual components – is the basis for the generalization of the dot product into the inner product as discussed below.

The dot product represents the magnitude of the first component along the direction of the second component times the magnitude of the second component. The most familiar application of the dot product is engineering mechanics is in the definition of work as dW = **f•dx**; this gives the product of the magnitude of the force component in the direction of the displacement times the magnitude of the displacement.

The fact that the unit vectors are perpendicular to each other gives a particularly simple relationship for the dot product. This is an important tool in later application of vectors. We use the word **orthogonal** to define a set of vectors that are mutually perpendicular. In addition, when we have a set of mutually perpendicular vectors, each of which has a magnitude of one, we call this set of vectors an **orthonormal** set.

We can represent any three-dimensional vector in terms of the three unit vectors, **i**, **j**, and **k**. Because of this we say that these three vectors are a **basis set** for representing any three real, three-dimensional vector. In fact, we could use any three vectors in place of **i**, **j**, and **k**, to represent any three-dimensional vector *so long as the set of three vectors is linearly independent*.

For example, we could use a new set, **m** = **i** + **j** + **k, n** = **i** + **j** – **k** and **o** = **i + k**. This would be an inconvenient set to use, since the unit vectors are not orthogonal and the dot products would be hard to compute. Nevertheless, we could represent any vector, $\mathbf{a} = a_1\mathbf{m} + a_2\mathbf{n} + a_3\mathbf{o}$, instead of the equivalent vector $(a_1 + a_2 + a_3)\mathbf{i} + (a_1 + a_2)\mathbf{j}.+ (a_1 - a_2 + a_3)\mathbf{k}$. We can convert the vector $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ components into the **m**,**n**,**o** basis by solving the following set of equations:

$$\begin{aligned} a_1 + a_2 + a_3 &= b_1 \\ a_1 + a_2 \qquad &= b_2 \\ a_1 - a_2 + a_3 &= b_3 \end{aligned} \qquad [1]$$

You can verify that the general solution to this set of equations is the one shown below.

$$\begin{aligned} a_1 &= -0.5b_1 + b_2 + 0.5b_3 \\ a_2 &= 0.5b_1 \quad\quad - 0.5b_3 \\ a_3 &= \quad b_1 \quad - b_2 \end{aligned} \qquad [2]$$

The two sets of equations above allow us to convert between the two different representations. However, consider the following set of vectors, **m** = **i** + **j** + **k, n** = **i** + **j** – **k** and **o** = **i + j**, where we have made only a slight change in **o** from its previous definition as **i** + **k**. In this case we see that vector, **A** = a_1**m** + a_2**n** + a_3**o**, is equal to (a_1 + a_2 + a_3)**i** + (a_1 + a_2 + a_3)**j**.+ (a_1 - a_2)**k**. When can try to convert the vector **B** = b_1**i** + b_2**j** + b_3**k** components into the **m**,**n**,**o** basis by solving the following set of equations:

$$\begin{aligned} a_1 + a_2 + a_3 &= b_1 \\ a_1 + a_2 + a_3 &= b_2 \\ a_1 - a_2 \quad\quad &= b_3 \end{aligned} \qquad [3]$$

However, we find that subtracting the first two equations gives the result that 0 = b_1 – b_2, instead of an equation that we can solve for a_1 or a_2. Thus, we conclude that the set of equations has no solution and we cannot use the proposed set of vectors to represent any three-dimensional vector. The reason for this is that the new proposed set does not have vectors that are **linearly independent**. Instead, the three proposed vectors satisfy the following linear equation: **m** + **n** + 2**o** = 0. That is, we can solve for one of these vectors in terms of the other two. We will later see that any set of vectors that we want to use to represent any other vector in the space (such a set of vectors is called a basis set) must be linearly independent.

We will extend these basic concepts of vectors, particularly the resolution of a vector into a set of components, the use of a linearly independent basis set to represent any vector in the particular analysis of interest, and the dot product of two vectors. These ideas will be later used to define a generalized vector space that applies to sets of numbers or functions whose behavior is similar to the familiar physical vectors from engineering mechanics. First, we will develop the general notation of matrices, which includes a representation of vectors in terms of their components.

Matrices and their Operations

A matrix (plural matrices) is represented as a two-dimensional array of elements, a_{ij}, where i is the row index and j is the column index. The entire matrix is represented by the single symbol **A**. In general, we speak of a matrix as having n rows and m columns. Such a matrix is called an (n by m) or (n x m) matrix. Equation [4] shows the representation of a typical (n x m) matrix.

In general, the number of rows may be different from the number of columns. Sometimes the matrix is written as $\mathbf{A}_{(n \times m)}$ to show its size. (Size is defined as the number of rows and the number of columns.) A matrix that has the number of rows equal to the number of columns is called a **square matrix**.

Matrices are used to represent physical quantities that have more than one number. These are usually used for engineering systems such as structures or networks in which we represent a collection of numbers, such as the individual stiffness of the members of a structure, as a single symbol known as a stiffness matrix. Networks of pipes, circuits, traffic streets, and the like may be represented by a connectivity matrix which indicates which pair of nodes in the matrix are directly joined to each other. The use of matrix notation and formulae for matrices leads to important analytical results. We will see that a matrix property knows as its eigenvalues represents the fundamental vibration frequencies in a mechanical system. The structure of an (n x m) matrix is shown in the equation below.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} \end{bmatrix} \qquad [4]$$

Two matrices can be added or subtracted if both matrices have the same size. If we define a matrix, **C**, as the sum (or difference) of two matrices, **A** and **B**, we can write this sum (or difference) in terms of the matrices as follows.

$$\mathbf{C} = \mathbf{A} \pm \mathbf{B} \quad (\textit{possible only if } \mathbf{A} \textit{ and } \mathbf{B} \textit{ have the same size}) \qquad [5]$$

The components of the **C** matrix are simply the sum (or difference) of the components of the two matrices being added (or subtracted). Thus for the matrix sum (or difference) shown in equation [5], the components of **C** are given by the following equation.

$$\mathbf{C} = \mathbf{A} \pm \mathbf{B} \qquad \Rightarrow \qquad c_{ij} = a_{ij} \pm b_{ij} \quad (i = 1, n;\ j = 1, m) \qquad [6]$$

The product of a matrix, **A**, with a single number, x, yields a second matrix whose size is the same as that of matrix **A**. Each component of the new matrix is the component of the original matrix, a_{ij}, multiplied by the number x. The number x in this case is usually called a scalar to distinguish it from a matrix or a matrix component.

$$\mathbf{B} = x\mathbf{A} \qquad \textit{if } b_{ij} = x a_{ij} \quad (i = 1, n;\ j = 1, m) \qquad [7]$$

We define two special matrices, the null matrix, **0**, and the identity matrix, **I**. The null matrix is an arbitrary shape (may or may not be square) matrix in which all the elements are zero. The identity matrix is a square matrix in which all the diagonal terms are 1 and the off-diagonal terms are zero. These matrices are sometimes written as $\mathbf{0}_{(m \times n)}$ or $\mathbf{I}_n$ to specify a particular size for the null or identity matrix. The null matrix and the identity matrix are shown below.

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & & & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \qquad [8]$$

A matrix that has the same pattern as the identity matrix, but has terms other than ones on its principal diagonal is called a **diagonal matrix**. The general term for such a matrix is $d_i\delta_{ij}$, where d_i is the diagonal term for row i and δ_{ij} is the Kronecker delta; the latter is defined such that $\delta_{ij} = 0$ unless i = j, in which case $\delta_{ij} = 1$. A diagonal matrix is sometimes represented in the following form: **D** = **diag**(d_1, d_2, d_3,…,d_n); this says that **D** is a diagonal matrix whose diagonal components are given by d_i.

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & d_n \end{bmatrix} \qquad [9]$$

We call the diagonal for which the row index is the same as the column index, the main or principal diagonal. Algorithms in the numerical analysis of differential equations lead to matrices whose nonzero terms lie along diagonals. For such a matrix, all the nonzero terms in a particular diagonal may be represented by symbols like $a_{i,i-k}$ or $a_{i,i+k}$. Diagonals with subscripts $a_{i,i-k}$ or $a_{i,i+k}$ are said to lie, respectively, below or above the main diagonal.

If the n rows and m columns in a matrix, A, are interchanged, we will have a new matrix, B, with m rows and n columns. The matrix B is said to be the **transpose** of A, written as A^T.

$$\mathbf{B} = \mathbf{A}^T \qquad if\ b_{ij} = a_{ji} \quad [i = 1,n;\ j = 1,m;\ \mathbf{A}\ is\ (n\ x\ m);\ \mathbf{B}\ is\ (m\ x\ n).] \qquad [10]$$

An example of an original **A** matrix and its transpose is shown below.

$$\mathbf{A} = \begin{bmatrix} 3 & 12 & -6 \\ 14 & -2 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 3 & 14 \\ 12 & -2 \\ -6 & 0 \end{bmatrix} \qquad [11]$$

The transpose of a product of matrices equals the product of the transposes of individual matrices, with the order reversed. Two examples are shown below. Can you complete the third?

$$(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} \qquad (\mathbf{ABC})^{\mathbf{T}} = \mathbf{C}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} \qquad (\mathbf{ABCD})^{\mathbf{T}} = \ldots \qquad [12]$$

Matrices with only one row are called row matrices; matrices with only one column are called column matrices.[1] Although we can write the elements of such matrices with two subscripts, the subscript of one for the single row or the single column is usually not included. The examples below for the row matrix, **r,** and the column matrix, **c**, show two possible forms for the subscripts. In each case, the first row or column matrix has a double subscript, such as 1j for a component in the single-row of a row matrix or i1 for single column of a column matrix; the second form has the commonly used single subscript. When row and column matrices are used in formulas, such as the formula for the multiplication of two matrices shown below in equation [19], that have two matrix subscripts, the first form of the matrices shown below are implicitly used to give the missing subscript (with a value of 1) for the equation.

[1] Row and column matrices are called row vectors or column vectors when they are used to represent the components of a vector. In these notes, we will use upper case boldface letters such as **A** and **B** to represent matrices with more than one row or more than one column; we will use lower case boldface letters such as **a** or **b** to represent matrices with only one row or only one column. We will generally refer to these matrices as vectors.

$$\mathbf{r} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & \cdots & r_{1m} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 & \cdots & \cdots & r_m \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ \vdots \\ \vdots \\ c_{n1} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} \qquad [13]$$

The transpose of a column matrix is a row matrix; the transpose of a row matrix is a column matrix. This is sometimes used to write a column matrix in the middle of text by saying, for example, that **c** = [1 3 -4 5]^T.

Matrix Multiplication

The definition of matrix multiplication seems unusual when encountered for the first time. However, it has its origins in the treatment of linear equation systems. For a simple example, we consider three two-dimensional coordinate systems. The coordinates in the first system are x_1 and x_2. The coordinates for the second system are y_1 and y_2. The third system has coordinates z_1 and z_2. Each coordinate system is related by a coordinate transformation given by the following relations.

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 & \qquad z_1 &= b_{11}y_1 + b_{12}y_2 \\ y_2 &= a_{21}x_1 + a_{22}x_2 & \qquad z_2 &= b_{21}y_1 + b_{22}y_2 \end{aligned} \qquad [14]$$

We can obtain a relationship between the z-coordinate system and the x-coordinate system by combining the various components of equation [14] to eliminate the y_i coordinates as follows.

$$\begin{aligned} z_1 &= b_{11}[a_{11}x_1 + a_{12}x_2] + b_{12}[a_{21}x_1 + a_{22}x_2] \\ z_2 &= b_{21}[a_{11}x_1 + a_{12}x_2] + b_{22}[a_{21}x_1 + a_{22}x_2] \end{aligned} \qquad [15]$$

We can rearrange these terms to obtain a set of equations similar to those in equation [14] that relates the z coordinate system to the x-coordinate system.

$$\begin{aligned} z_1 &= [b_{11}a_{11} + b_{12}a_{21}]x_1 + [b_{11}a_{12} + b_{12}a_{22}]x_2 = c_{11}x_1 + c_{12}x_2 \\ z_2 &= [b_{21}a_{11} + b_{22}a_{21}]x_1 + [b_{21}a_{12} + b_{22}a_{22}]x_2 = c_{21}x_1 + c_{22}x_2 \end{aligned} \qquad [16]$$

We see that the coefficients c_{ij}, for the new transformation are related to the coefficients for the previous transformations as follows.

$$\begin{aligned} c_{11} &= [b_{11}a_{11} + b_{12}a_{21}] & \qquad c_{12} &= [b_{11}a_{12} + b_{12}a_{22}] \\ c_{21} &= [b_{21}a_{11} + b_{22}a_{21}] & \qquad c_{22} &= [b_{21}a_{12} + b_{22}a_{22}] \end{aligned} \qquad [17]$$

There is a general form for each c_{ij} coefficient in equation [17]. Each is a sum of products of two terms. The first term from each product is a b_{ik} value whose first subscript (i) is the same as the first subscript of the c_{ij} coefficient being computed. The second term in each product is an a_{kj} value whose second subscript (j) is the same as the second subscript of the c term being computed. In each $b_{ik}a_{kj}$ product, the second b subscript (k) is the same as the first a subscript. From these observations, we can write a general equation for each of the four coefficients in equation [17] as follows.

$$c_{ij} = \sum_{k=1}^{2} b_{ik} a_{kj} \qquad (i = 1,2;\ j = 1,2) \qquad [18]$$

The definition of matrix multiplication is a generalization of the simple example in equation [18] to any general sizes of matrices. In this general case, we define the product, **C** = **AB**, of two matrices, **A** with n rows and p columns, and **B** with p rows and m columns by the following equation.

$$\mathbf{C}_{(n\,x\,m)} = \mathbf{A}_{(n\,x\,p)}\mathbf{B}_{(p\,x\,m)} \quad \Rightarrow \quad c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \quad (i = 1,\ldots,n;\ j = 1,\ldots,m) \qquad [19]$$

There are two important items to consider in the formula for matrix multiplication. The first is that order is important. The product **AB** is different from the product **BA**. In fact, one of the products may not be possible. The second item is the need for compatibility between the first and second matrix in the **AB** product.[2] In order to obtain the product **AB** the number of columns in **A** must equal the number of rows in **B**. A simple example of matrix multiplication is shown below.

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -6 \\ 4 & -2 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 6 & 1 \end{bmatrix} \qquad [20]$$

$$\mathbf{AB} = \begin{bmatrix} 3(3)+0(1)-6(6) & 3(4)+0(2)-6(1) \\ 4(3)-2(1)+0(6) & 4(4)-2(2)+0(1) \end{bmatrix} = \begin{bmatrix} -27 & 6 \\ 10 & 12 \end{bmatrix}$$

Matrix multiplication is simple to program. The C++ code for multiplying two matrices is shown below.[3] This code assumes that all variables have been properly declared and initialized. The code uses the obvious notation to implement equation [19]. The array components are denoted as a[i][k]. b[k][j] and c[i][j]. The product matrix, **C**, has the same number of rows, n, as in matrix **A** and the same number of columns, m, as in matrix **B**. The number of columns in **A** is equal to p, which must also equal the number of rows in **B**.

```
for (i = 1; i <= n; i++ )
   for ( j = 1; j <= m; j++ )
   {
      c[i][j] = 0.0;
      for ( k = 1; k <= p; k++ )
         c[i][j] += a[i][k] * b[k][j];
   }
```

[2] The terms **premultiply** and **postmultiply** are commonly used to indicate the order of the matrices involved in matrix multiplication. In the matrix product **AB**, we say that **B** is **premultiplied** by **A** or that **A** is **postmultiplied** by **B**. Alternatively, the terms **left multiplied** and **right multiplied** are used. In the **AB** product, **A** is right multiplied by **B** and **B** is left multiplied by **A**.

[3] The basic code structure is the same in any language. There are three nested loops. The two outer loops cover all possible combinations of i and j to ensure that all the c_{ij} components are computed. The inner loop code is the typical code for summing a number of items. C++ programmers will note that the loop indices used in this code ignore the fact that the minimum index for a C++ array is zero. This was done deliberately for all code examples in these notes to provide similar numbering for array indices in the notes and those in the code.

We now examine how the coordinate transformations that we used above to introduce matrix multiplication can be represented as matrix equations. We can define matrices, **A**, **B**, and **C** to represent the coefficients that we used in our coordinate transformation equations.

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad \mathbf{C}=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \qquad [21]$$

The various coordinate pairs can be represented as column matrices as shown below.

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \qquad \mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \qquad \mathbf{z}=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} z_{11} \\ z_{21} \end{bmatrix} \qquad [22]$$

With these matrix definitions, the two sets of simultaneous linear equations shown in equation [14] can be represented by the following pair of matrix equations:

$$\mathbf{y}=\mathbf{Ax} \qquad \textit{and} \qquad \mathbf{z}=\mathbf{By} \qquad [23]$$

You can verify that the equations above are correct by applying the general formula for matrix multiplication in equation [19] to the matrix equations in [23]. To do this, you should use the definitions of **A**, **B**, **x**, **y**, and **z**, provided in equations [21] and [22]. If we combine the matrix equations in [23] to eliminate the **y** matrix, we get the following result.

$$\mathbf{z}=\mathbf{By}=\mathbf{BAx} \qquad \textit{or} \qquad \mathbf{z}=\mathbf{Cx} \quad \textit{with} \quad \mathbf{C}=\mathbf{BA} \qquad [24]$$

Note the importance of the order of multiplication. In general, **BA**; it is not equal to **AB**.

There are two cases where the order is not important. These are multiplication by a null matrix, which produces a null matrix, and multiplication by an identity matrix, which produces the original matrix.

$$\mathbf{0A}=\mathbf{A0}=\mathbf{0} \qquad \textit{and} \qquad \mathbf{AI}=\mathbf{IA}=\mathbf{A} \qquad [25]$$

Although the order is not important here, the actual identity and null matrices used may be different. We can rewrite equations [25] to explicitly show the rows and columns in each matrix.

$$\mathbf{0}_{(p\,x\,n)}\mathbf{A}_{(n\,x\,m)}=\mathbf{0}_{(p\,x\,m)} \qquad \mathbf{A}_{(n\,x\,m)}\mathbf{0}_{(m\,x\,q)}=\mathbf{0}_{(n\,x\,q)}$$
$$\mathbf{A}_{(n\,x\,m)}\mathbf{I}_{(m\,x\,m)}=\mathbf{I}_{(n\,x\,n)}\mathbf{A}_{(n\,x\,m)}=\mathbf{A}_{(n\,x\,m)} \qquad [26]$$

.

By definition the identity matrix is a square matrix. One size specification for the identity matrix, the number of rows or the number of columns, is set by the compatibility condition for matrix multiplication. Once this is done, the other size is set by the requirement that **I** is square. For the null matrices in equation [26], the size specifications, n or m, must match the sizes for the **A** matrix. Although the size specifications p and q, for the null matrices in equation [26] are arbitrary, they are usually taken as p = m and q = n to give a square null matrix as the **0A** product.

Simultaneous Linear Algebraic Equations

The coordinate transformation equations are simple examples of a more general case for simultaneous linear algebraic equations. In the general case, we can have a set of simultaneous equations that is written as follows

$$\sum_{j=1}^{m} a_{ij} x_j = b_i \qquad i = 1, \ldots, n \qquad [27]$$

We expect that a well-determined problem will have the number of equations, n, equal to the number of unknowns, m, but in the general case, shown above, these n may be different from m. We see that this system of equations in equation [27] can be represented by matrices $A_{(n \times m)}$, $\mathbf{x}_{(m \times 1)}$, and $\mathbf{b}_{(n \times 1)}$, where **A** has the form shown in equation [4], **x** is the column matrix $[x_1, x_2, x_3, \ldots x_m]^T$ and **b** is the column matrix $[b_1, b_2, b_3, \ldots b_n]^T$. With these definitions, equation [27] is the same as the general equation for matrix multiplication shown in equation [19]. (Recall that we have omitted the second subscript, which is one, on the components of **x** and **b**.) The system of equations shown in equation [27] is written, in matrix form, in equation [28], below.

$$\mathbf{Ax} = \mathbf{b} \qquad [28]$$

These matrices are written out in detail below. Here the column matrix, **x**, appears to have more rows than the coefficient matrix, **A**. This is done to emphasize the notion that m may be different from n in general. Of course, m may be equal to or less than n rather than greater than n as implied in the matrices shown in the equation below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix} \qquad [29]$$

In order to solve a set of simultaneous linear equations we use a process that replaces equations in the set by equivalent equations. We can replace any equation in the set by a linear combination of other equations without changing the solution of the system of equations. For example, consider the simple set of two equations with two unknowns

$$\begin{aligned} 3x_1 + 5x_2 &= 13 \\ 7x_1 - 4x_2 &= -1 \end{aligned} \qquad [30]$$

You can confirm that $x_1 = 1$ and $x_2 = 2$ is a solution to this set of equations. To solve this set of equations we can replace the second equation by a new equation, which is a linear combination of the two equations without changing the solution. The particular combination we seek is one that will eliminate x_1. We can do this by subtracting the first equation, multiplied by 7/3, from the second equation to obtain the following pair of equations, which is equivalent to the original set in equation [30].

$$\begin{aligned} 3x_1 + 5x_2 &= 13 \\ -\frac{47}{3} x_2 &= -\frac{94}{3} \end{aligned} \qquad [31]$$

We can readily solve the second equation to find $x_2 = 2$, and substitute this value of x_2 into the first equation to find that $x_1 = [13 – 5(2)]/3 = 1$. The general process for solving the system of equations represented by equations [27], [28], or [29], known as Gauss elimination, is similar to the one just shown. It requires a series of operations on the coefficients a_{ij} and b_i to produce a set of equations with the form shown in equation [32], below, without changing the solution of the initial problem.

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \cdots & \alpha_{1n-1} & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \cdots & \alpha_{2n-1} & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & \cdots & \cdots & \alpha_{3n-1} & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & \cdots & \cdots & \alpha_{n-1n-1} & \alpha_{n-1n} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \qquad [32]$$

The basic rule in the Gauss elimination process is that we can use a linear combination of two equations to replace one of those equations, without changing the solution to the problem. This is the process that we used above in going from the set of equations in [30] to the set of equations in [31]. Both sets of equations are equivalent in the sense that both sets of equations give the same answers for x_1 and x_2. However, the second set of equations can be directly solved for all the unknowns.

The revised coefficient matrix in equation [32] is called an upper triangular matrix. The only nonzero terms are on or above the principal diagonal. The same operations that are used to obtain the revised coefficient matrix are used to obtain the revised right-hand-side matrix.

The revised **A** and **b** matrices are obtained in a series of steps. In the first step, the x_1 coefficients are eliminated from all equations except the first one. This is done by the following replacement operations on the coefficients in equations 2 to n. The replacement notation (←) from computer programming is used here to indicate that an old value of a_{ij} is being replaced by the results of a calculation. This avoids the need to use mathematical notation that would require separate symbols for the old value and the new value of a_{ij}.

$$\left(a_{ij} \leftarrow a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \quad j = 1,\ldots n \quad \textit{and} \quad b_i \leftarrow b_i - \frac{a_{i1}}{a_{11}} b_1 \right) \quad i = 2,\ldots n \qquad [33]$$

After equation [33] is applied to all rows below the first row, the only nonzero x_1 coefficient is in the first equation (represented by the first row of the matrix.) You can confirm that this will set $a_{i1} = 0$ for $i > 1$. You can also apply the formulae in [33] to equation [30] to see that the result is equation [31]. The elimination process is next applied to make the x_2 coefficients on all equations below the second equation zero.

$$\left(a_{ij} \leftarrow a_{ij} - \frac{a_{i2}}{a_{22}} a_{2j} \quad j = 2,\ldots n \quad \textit{and} \quad b_i \leftarrow b_i - \frac{a_{i2}}{a_{22}} b_2 \right) \quad i = 3,\ldots n \qquad [34]$$

Equation [34] has the same form as equation [33]; only the starting points for the row and column operations are different. The process described by equations [33] and [34] continues until the form shown in equation [32] is obtained. From equation [32], the various values of x can be found by back substitution. We can simply find x_n as β_n/α_{nn}. The remaining values of x are found in reverse order by the following equation.

$$x_i = \frac{\beta_i - \sum_{j=i+1}^{n} \alpha_{ij} x_j}{\alpha_{ii}} \qquad i = n-1, n-2, \ldots, 1 \qquad [35]$$

When we are solving for x_i, all previous values of x_j required in the summation are known.

The C++ code below shows a simplified version[4] of how the Gauss elimination method is applied to the solution of equations. As in previous code examples, all data values are assumed to be properly declared and initialized. The number of equations is equal to the number of unknowns, n. The row that is subtracted from all rows below it is called the pivot row. The main outer loop in the first part of the code uses the variable, pivot, to represent this row. The code execution is simplified by augmenting the a matrix so that $a_{i,n+1} = b_i$. This allows the code to proceed without separate consideration of similar operations on the **A** and **b** matrix components.

```
// augment a matrix with b values
for ( row = 1; row <=n; row++) a[row][n+1] = b[row];
// get upper triangular array
for (pivot = 1; pivot < n; pivot++ )
   for ( row = pivot+1; row <= n; row++ )
      for ( column = row+1; column <= n+1; column++)
         a[row][column] -= a[row][pivot] * a[pivot] [column]
                         / a[pivot][pivot];
// Upper triangular matrix complete; get x values
for (row = n; row <= 1; row--)
{
   x[row] = a[row][n+1];
   for ( column = n; column < row; column-- )
         x[row] -= a[row][column] * x[column];
   x[row] /= a[row][row];
}
```

The process outlined above for the solution of a set of simultaneous equations is known as the Gaussian elimination procedure. Alternative procedures such as the Gauss-Jordan method and LU decomposition, work in a similar manner. They produce an upper triangular matrix or diagonal matrix that is then used to solve for the values of x_i in reverse order.

Matrix Rank Determines Existence and Uniqueness of Solutions

If the solution process outlined above is used on certain matrices, it may not be possible to obtain a solution. Consider the two sets of equations shown below.

$$\begin{aligned} 3x_1 + 5x_2 &= 13 \\ 6x_1 + 10x_2 &= 26 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} 3x_1 + 5x_2 &= 13 \\ 6x_1 + 10x_2 &= 27 \end{aligned} \qquad [36]$$

In the set of equations on the left, the second equation is simply twice the first equation. If we multiply the first equation by two and subtract it from the second equation, we get the result that 0 = 0. Thus, the

[4] Actual code would have to account for the possibility that the system of equations might not have a solution. It would also use different operations to reduce round-off error. This example continues the practice used previously of starting the array subscripts at 1 and ending them at n to be consistent with the notation in equations in these notes. Typical C++ code starts the array subscripts at 0.

second equation gives us no new information on the relationship between x_1 and x_2. We say that this system of equations has an infinite number of solutions. Any value of $x_2 = 2.6 – 0.6x_1$ will satisfy both equations. The second set of equations has no solutions. If we multiply the first equation by two and subtract it from the second equation, we have the result that 0 = 1! Thus, this second set of equations is incompatible and does not have a solution.[5]

This simple example can be generalized to discuss the existence and uniqueness of solutions for the general set of equations. If we carry out the solution process outlined above to form an upper triangular matrix, we may have the result that one (or more) of the final rows in the coefficient matrix is all zero. This means that we cannot obtain a unique solution. Such a case is called a singular matrix. The **rank** of a matrix is formally defined as the number of linearly independent rows in a matrix. (This can be shown to be equal to the number of linearly independent columns.) The practical determination of rank is based on the Gauss elimination process outlined above. If in the final matrix in the elimination process is a matrix with n rows of which n_{zero} rows contain all zeros, the rank of the matrix is $n – n_{zero}$. (This rank is the same for both the original matrix and the upper-triangular matrix because the Gauss elimination operations do not change the matrix rank.) The **A** matrix for both sets of equations in equation [36] has only one linearly independent row, thus its rank is one. The upper triangular form that results when a matrix is tested for rank is sometimes called the ***row-echelon*** form. Sometimes in this form each row is divided by the diagonal element on that row so that all the diagonal elements are one.

The two matrices in equation [37] below have been placed in row-echelon form by using Gauss elimination on the original matrices. Can you determine the rank of the original matrices before looking at the answers below?

The matrix on the left of equation [37] has four rows that are not all zero; thus, its rank is four. The one on the right has six rows that are not all zero, thus its rank is six. This rank-six matrix has eight columns. Because the number of linearly independent columns and the number of linearly independent rows are both the same as the rank of six, we know that these eight columns will be related by two different linear equations.

$$\begin{bmatrix} 6 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & 6 & 2 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 9 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 2 & 0 & 0 & 0 & 1 & 6 \\ 0 & 1 & 7 & 8 & 6 & 2 & 8 & 4 \\ 0 & 0 & 2 & 0 & 3 & 5 & 8 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 & 6 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix} \qquad [37]$$

The existence and uniqueness of solutions are defined in terms of the rank of the augmented matrix, [**A**,**b**]. This is the matrix in which the right-hand side column matrix, **b**, is added as an extra column in the **A** matrix. This augmented matrix is shown below for the general case of n equations and m unknowns. The n equations mean that there are n rows in the matrix. The m unknowns give m + 1 columns to the augmented matrix.

[5] This result has a geometric interpretation. When we have two simultaneous linear algebraic equations, we can plot each equation in x_1–x_2 space. The solution to the pair of equations is located at the point where both equations intersect. If we did this for the left set of equations in [36], we would only have a single line.

$$[\mathbf{A},\mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} & b_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} & b_n \end{bmatrix} \qquad [38]$$

The existence and uniqueness of solutions to **Ax** = **b** is stated below without proof.

> If the rank of the original matrix, **A**, equals the rank of the augmented matrix, [**A**,**b**], and equals the number of unknowns, m, there is a **unique solution** to the matrix equation, **Ax** = **b**.
>
> If the rank of the original matrix, **A**, equals the rank of the augmented matrix, [**A**,**b**], but is less than the number of unknowns, m, there are an **infinite number of solutions** to the matrix equation, **Ax** = **b**.
>
> If the rank of the original matrix, **A**, is not equal the rank of the augmented matrix, [**A**,**b**], there is **no solution** to the matrix equation, **Ax** = **b**.

We can see that these statements are consistent with the examples in equation [36]. A formal proof of these statements is given in linear algebra texts.

These guidelines for the existence and uniqueness of solutions to simultaneous linear equations are illustrated in the three sets of equations shown below. Each equation set has three equations in three unknowns. The original equation set, shown in the first column, is converted to an upper triangular form in the second column. We see that the first set has a unique solution. The second and third sets do not have a unique solution; however, there is a difference between these two. The second set has an infinite number of solutions. For any value, α that we pick for x_3 we can determine a value of x_1 and x_2 that is consistent with the original set of equations. However, for the third set of equations, the upper triangular form gives an inconsistent third equation. Thus, this set of equations has no solution.

	Original Equation Set	Upper Triangular Form	Solutions
Set I	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $7x_1 + 3x_2 + 8x_3 = -13$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $50.5x_3 = 50.5$	$x_1 = 0$ $x_2 = -7$ $x_3 = 1$
Set II	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $-2x_1 + 10x_2 + 61x_3 = -9$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $0 = 0$	$x_1 = 12 - 8\alpha$ $x_2 = -2.5 - 4.5\alpha$ $x_3 = \alpha$
Set III	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $-2x_1 + 10x_2 + 61x_3 = -8$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $0 = 1$	No Solution

These three sets of equations are shown in terms of their **A** and augmented [**A** **b**] matrices in the table below. We see that the set of equations in the table above corresponds to the data in the augmented matrix. The first set of equations has rank **A** = rank [**A** **b**] = 3, the number of unknowns. We have already seen that this provides the unique solution above. The second set of equations has rank **A** = rank [**A** **b**] = 2, less than the number of unknowns. This means that we have an infinite number of solutions. Again, this

corresponds to the result above. Finally, the third case below has rank **A** = 2, but rank [**A** **b**] = 3. This difference in rank shows that there are no solutions.

	Original Matrices		Row-Echelon Form		Rank	
	A	[**A** **b**]	**A**	[**A** **b**]	**A**	[**A** **b**]
Set I	$\begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 7 & 3 & 8 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 7 & 3 & 8 & -13 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 50.5 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 50.5 & 50.5 \end{bmatrix}$	3	3
Set II	$\begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ -2 & 10 & 61 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ -2 & 10 & 61 & -9 \end{bmatrix}$		$\begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	2	2
Set III	$\begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ -2 & 10 & 61 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ -2 & 10 & 61 & -8 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	2	3

There is one final case to consider; that is the case of ***homogenous equations***, where the **b** matrix is all zeros. If there are n equations and the rank of the coefficient matrix is n then the only solution to the set of equations is that all $x_i = 0$. (This is called the trivial solution.) However, if the rank is less than n, it is possible to have a solution in which all the x_i are not zero. However, such a solution is not unique.

Consider the two sets of homogenous equations shown below. Each set of equations has a right-hand side that is all zeros. (The two equation sets are identical except for the coefficient of the x_1 term in the first equation.)

$$\begin{aligned} -x_1 - 4x_2 + 3x_3 &= 0 \\ -4x_1 + 11x_2 - 6x_3 &= 0 \\ x_1 - 8x_2 + 5x_3 &= 0 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} x_1 - 4x_2 + 3x_3 &= 0 \\ -4x_1 + 11x_2 - 6x_3 &= 0 \\ x_1 - 8x_2 + 5x_3 &= 0 \end{aligned} \qquad [39]$$

If we carry out the usual solution process to create an upper triangular matrix for these two sets of equations, we obtain the following results.

$$\begin{aligned} -x_1 - 4x_2 + 3x_3 &= 0 \\ 27x_2 - 18x_3 &= 0 \\ 0 &= 0 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} x_1 - 4x_2 + 3x_3 &= 0 \\ -6x_2 + 6x_3 &= 0 \\ -2.8x_3 &= 0 \end{aligned} \qquad [40]$$

For the set of equations on the right, the rank of the coefficient matrix is the same as the number of equations. Here we have a unique solution in which all of the $x_i = 0$. The rank of the coefficient matrix for the equations on the left is less than the number of equations. In this case, we have an infinite number of solutions. If we pick $x_3 = \alpha$, an arbitrary constant, we can satisfy all three equations if $x_2 = 2\alpha/3$ and $x_1 = 3\alpha - 4(2\alpha/3) = \alpha/3$. One of the infinite solutions, with $\alpha = 0$, is the trivial solution where all $x_i = 0$.[6]

[6] You should make sure that you can place the original sets of equations in [39] and the upper triangular forms in [40] into an **A** and an augmented [**A** **b**] matrix and show that both sets of equations have rank **A** = rank [**A** **b**]. Do both

Determinants

A determinant is a single numerical value that can be computed for a square array. The values of determinants play a theoretical role in matrix analysis and can be used for calculations on small matrices. For matrices whose rank is greater than 3x3 or 4x4, alternative calculation methods are used in place of determinants.

Various notations are available for a determinant. If **A** is a matrix, then Det **A** is the determinant for the coefficients in a matrix. The determinant for an array of numbers that looks like a matrix can be written using the absolute value sign, | <array>|, instead of the brackets, [<array>], that we have been using for matrix coefficients. The various notations are shown below for a 2x2 array. For this array, the formula for the determinant, which is shown as the final part of equation [41], is particularly simple.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad Det\ \mathbf{A} = Det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \quad [41]$$

For a 3x3 array, the determinant is a bit more complex.

$$Det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{matrix} \quad [42]$$

The general equation for computing a determinant is given in terms of minors (or cofactors) of a determinant. The minor, M_{ij} of a determinant is the smaller determinant that results if row i and column j are eliminated from the original determinant. The cofactor, C_{ij}, equals $(-1)^{i+j}M_{ij}$. For example, if we start with a 3x3 determinant, such as the one shown in equation [42] we can define nine possible minors (and cofactors). Four of these are shown below:

$$\begin{aligned} C_{33} &= (-1)^{3+3} M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad C_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \\ C_{32} &= (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad C_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \end{aligned} \quad [43]$$

The determinant of a matrix can be written in terms of its minors or cofactors as follows.

$$\begin{aligned} Det\ \mathbf{A}_{(n\ x\ n)} &= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij} \\ &= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} \end{aligned} \quad [44]$$

Note that the sum is taken over any one row or over any one column. In applying this formula, one seeks rows or columns with a large number of zeros to simplify the calculation of the determinant. We can show

---

sets of equations produce **A** matrices with the same rank? What are the ranks of the **A** and [**A** **b**] matrices for the two sets of equations?

that this equation is consistent with the results given previously for the determinants of 2x2 and 3x3 arrays. Applying equation [44] to the third row of a 3x3 array gives the following result.

$$Det\, C_{(3\,x\,3)} = \sum_{j=1}^{3} a_{3j} C_{3j} = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \qquad [45]$$

We could have applied equation [44] to any of the three rows or any of the three columns to compute the determinant. I chose to use the third row since the necessary cofactors can be found in equation [43]. If we use equation [41] to expand the (2 x 2) cofactors in [43] and apply those results to equation [45], we obtain the following result.

$$\begin{aligned} Det\, A_{(3\,x\,3)} &= a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33} \\ &= a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{31}(a_{12}a_{23} - a_{22}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}) + a_{33}(a_{11}a_{22} - a_{12}a_{21}) \qquad [46] \\ &= a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{32}a_{11}a_{23} + a_{32}a_{21}a_{13} + a_{33}a_{11}a_{22} - a_{33}a_{12}a_{21} \end{aligned}$$

The final result, after some rearrangement, is the same as the one in equation [42].

Two rules about determinants are apparent from equation [45]:

- A determinant is zero if any row or any column contains all zeros.
- If one row or one column of a determinant is multiplied by a constant, k, the value of the determinant is multiplied by the same constant. Note the implication for matrices: if a matrix is multiplied by a constant, k, then each matrix element is multiplied by k. If **A** is an n x n matrix, Det(k**A**) = k^nDet(**A**).

Additional rules for and properties of determinants are stated below without proof.

- If one row (or one column) of a determinant is replaced by a linear combination of that row (or column) with another row (or column), the value of the determinant is not changed. This means that the operations of the Gauss elimination process do not change the determinant of a matrix.
- If two rows (or two columns) of a determinant are linearly dependent the value of the determinant is zero.
- The determinant of the product of two matrices, **A** and **B** is the product of the determinants of the individual matrices: Det(**AB**) = Det(**A**) Det(**B**).
- The determinant of transposed matrix is the same as the determinant of the original matrix: Det($\mathbf{A}^T$) = Det(**A**).

If we apply the column expansion of equation [45] to an upper triangular matrix, **A**, we find that Det **A** = $a_{11}A_{11}$, since the a_{11} term is the only term in the first column. We can apply equation [45] repeatedly to the cofactors. Each application shows that the determinant is simply the new term in the upper left of the array times its cofactor. Continuing in this fashion we see that the determinant of an upper triangular matrix is simply the product of the diagonal terms. We can combine this result with the fact noted above that the operations of the Gauss elimination process do not change the determinant of a matrix to develop a practical for computing determinants of any matrix. Apply Gauss elimination to get the matrix in upper triangular form then the determinant (of both the original matrix and the one in upper triangular form) is simply the product of the diagonal elements.

As an example, consider the matrices from "Set 1" in the table on page 14. The original matrix was $\begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 7 & 3 & 8 \end{bmatrix}$; its upper triangular form was $\begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 50.5 \end{bmatrix}$. We can readily compute the determinant as the product (1)(2)(50.5) = 101. You can show that the same value is obtained by the conventional formula for the evaluation of the original 3 x 3 determinant.

Determinants are not used in normal numerical calculations. However, if you need to find the numerical value for a large determinant, the process outlined above is the most direct numerical approach.

Cramer's rule gives the solution to a system of linear equations in terms of determinants. This approach is never used except for very small numbers of equations, typically two or three. According to Cramer's rule the solution for a particular unknown x_i is the ratio of two determinants. The determinant in the denominator uses all the usual matrix coefficients, a_{ij}. The determinant in the numerator consists of the a_{ij} coefficients except in one column. When we are solving for x_j we replace column i in the a_{ij} coefficients by the right-hand-side matrix coefficients, b_i. For a set of three equations in three unknowns, Cramer's rule would give the solutions shown in equation [47].

Cramer's rule allows us to find an analytical expression for the solution of a set of equations, and it is sometimes used to solve small sets of equations (2 x 2 or 3 x 3). However, it is never used for numerical calculations of larger systems because it is extremely time consuming.

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \quad [47]$$

Determinants are also related to rank. An array in which the rows are not linearly independent will have a zero determinant. As an example of this consider the left-hand set of equations from equation [39]. Recall that the coefficients for that set of three equations had a rank of two because the equations were not linearly independent. When we evaluate the determinant for this array below, using equation [42] for the determinant of a (3x3) array, we find that the determinant is zero.

$$\begin{vmatrix} -1 & -4 & 3 \\ -4 & 11 & -6 \\ 1 & -8 & 5 \end{vmatrix} \begin{array}{l} = (-1)(11)(5) + (-4)(-8)(3) + (1)(-4)(-6) \\ -(-1)(-8)(-6) - (-4)(-4)(5) - (1)(11)(3) \\ = -55 + 96 + 24 - (-48) - 80 - 33 = 0 \end{array} \quad [48]$$

This gives us another approach to determining when a set of equations with all zeros on the right-hand side has a solution other than the simple one that all x_i are zero. This condition is that the determinant of the coefficient matrix is zero. If Det(**A**) = 0 then a solution to **Ax** = **b**, where **b** contains all zeros, that does not have all $x_i = 0$ is possible. There are actually an infinite number of such solutions. These solutions differ by an arbitrary multiplier. We will use this idea below when considering matrix eigenvalues.

Inverse of a Matrix

We have defined operations for adding, subtracting and multiplying matrices. Matrix inversion is the matrix analog of division. For a square matrix, **A**, we define a matrix inverse, $\mathbf{A}^{-1}$, by the following equation.

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \qquad [49]$$

If we have a matrix equation, **Ax** = **b**, we can, in principle, solve this equation by premultiplying both sides of the equation by $\mathbf{A}^{-1}$. This gives the following result.

$$If\ \mathbf{A}\mathbf{x} = \mathbf{b}, \qquad \mathbf{A}^{-1}\mathbf{A}x = \mathbf{A}^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{I}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \qquad [50]$$

The various steps in equation [50] use the definition, in equation [49], that the product of a matrix and its inverse is the identity matrix and the definition, in equation [25] that the product of any matrix with the identity matrix is the original matrix. Although the result that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ may be written as the solution to the original equation, the actual solution of matrix equations like **Ax** = **b** is done by methods other than the direct calculation of the inverse. It is not always possible to find the inverse. A square matrix that has no inverse is called a **singular matrix**.

It is usually not necessary to find the inverse of a matrix. If necessary, you can find a numerical value of the inverse by the same process used to solve simultaneous linear algebraic equations. To understand how this is done, we define a second matrix, **B**, as $\mathbf{A}^{-1}$. Then, by the definition of inverse we have the following equation.

$$If\ \mathbf{B} = \mathbf{A}^{-1}, \qquad \mathbf{A}\mathbf{B} = \mathbf{I} \qquad [51]$$

Equation [52] shows the matrices involved in this equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \qquad [52]$$

We have a form similar to the usual problem of solving a set of equations. The coefficient matrix, **A**, is the same, but we have n right-hand side columns of known values. Each of these columns of known values corresponds to one column of unknowns in the **B** matrix that is $\mathbf{A}^{-1}$. If we use our usual process for solving **Ax** = **b**, with, for example, **b =** $[1\ 0\ 0\ 0\ \ldots 0]^T$, we will obtain the first column of **B** = $\mathbf{A}^{-1}$. Repeating the process for similar **b** columns, which are all zeros except for a 1 in row k gives us column k of the inverse. For example, equation [53] shows the solution for the second column of **B** = $\mathbf{A}^{-1}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \\ \vdots \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \qquad [53]$$

Because the operations for solving a set of simultaneous linear equations are based on the **A** matrix only, the solution for the inverse is actually done simultaneously for all columns.

An analytical expression for the inverse can be obtained in terms of the cofactors discussed in the section on determinants. We continue to define $\mathbf{B} = \mathbf{A}^{-1}$; the components of the inverse, b_{ij}, are then given in terms of the minors or cofactors, C_{ij}, of the original A matrix and its determinant.

$$If\ \mathbf{B} = \mathbf{A}^{-1}, \qquad b_{ij} = \frac{C_{ji}}{Det(\mathbf{A})} = (-1)^{i+j}\frac{M_{ji}}{Det(\mathbf{A})} \qquad [54]$$

The simplest application of this equation is to a 2x2 matrix. For such a matrix, the coefficients of $\mathbf{B} = \mathbf{A}^{-1}$ are given by the following equations.

$$b_{11} = (-1)^{1+1}\frac{M_{11}}{Det(\mathbf{A})} = \frac{a_{22}}{Det(\mathbf{A})} \qquad b_{12} = (-1)^{1+2}\frac{M_{21}}{Det(\mathbf{A})} = -\frac{a_{12}}{Det(\mathbf{A})}$$
$$b_{21} = (-1)^{2+1}\frac{M_{12}}{Det(\mathbf{A})} = -\frac{a_{21}}{Det(\mathbf{A})} \qquad b_{22} = (-1)^{2+2}\frac{M_{22}}{Det(\mathbf{A})} = \frac{a_{11}}{Det(\mathbf{A})} \qquad [55]$$

Combining the results of equation [55] with equation [41] for a 2x2 determinant, gives the following result for the inverse of a 2x2 matrix.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad [56]$$

You can easily show that this is correct by multiplying the original matrix by its inverse. You will obtain a unit matrix by either multiplication: $\mathbf{AA}^{-1}$ or $\mathbf{A}^{-1}\mathbf{A}$. The same process can be used to find the inverse of a (3x3) matrix; the result is shown below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} (a_{22}a_{33} - a_{32}a_{23}) & (a_{32}a_{13} - a_{33}a_{12}) & (a_{12}a_{23} - a_{22}a_{13}) \\ (a_{31}a_{23} - a_{33}a_{21}) & (a_{11}a_{33} - a_{31}a_{13}) & (a_{21}a_{13} - a_{11}a_{23}) \\ (a_{21}a_{32} - a_{31}a_{22}) & (a_{31}a_{12} - a_{11}a_{32}) & (a_{11}a_{22} - a_{21}a_{12}) \end{bmatrix}}{\begin{pmatrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ -a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{pmatrix}} \qquad [57]$$

Equations [56] and [57] show the value of determinants in providing analytical solutions to inverses. Although determinants are valuable in such cases any use of determinants should be avoided in numerical work.

The general rule for the inverse of a matrix product and the inverses of the individual matrices is similar to the same equation for the transpose of a matrix product and the product of the transposes of the individual matrices. This relation is shown below.

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad (\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \qquad (\mathbf{ABCD})^{-1} = \ldots \quad [58]$$

Matrix Eigenvalues and Eigenvectors

If a square matrix can premultiply a column vector and return the original column vector multiplied by a scalar, the scalar is said to be an eigenvalue of the matrix and the column vector is called an eigenvector. In the following equation, the scalar, λ, is an eigenvalue of the matrix **A**, and **x** is an eigenvector.

$$\mathbf{A}_{(n\,x\,n)}\mathbf{x}_{(n\,x\,1)} = \lambda\mathbf{x}_{(n\,x\,1)} \quad [59]$$

We can use the identity matrix to rewrite this equation as follows.

$$[\,\mathbf{A}_{(n\,x\,n)} - \mathbf{I}_{(n\,x\,n)}\lambda]\mathbf{x}_{(n\,x\,1)} = \mathbf{0}_{(n\,x\,1)} \quad [60]$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \quad [61]$$

As discussed above in the sections on simultaneous linear equations and determinants, equation [61] has the solution that all values of x_i are zero. It may have a nonzero solution if the determinant of the coefficient matrix is zero. That is,

$$Det[\mathbf{A} - \mathbf{I}\lambda] = Det\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn}-\lambda \end{bmatrix} = 0 \quad [62]$$

From the general expression for a determinant, we see that one component of the final expression for a determinant of any size is the product of all elements on the principal diagonal. In equation [62] this term will give an n^{th} order polynomial in λ (for our n x n matrix). This n^{th} order polynomial is known as the **characteristic equation** of the matrix. This characteristic equation can be solved for n values of λ, not all

of which may be distinct. For a two-by-two matrix, setting Det[**A** – **I**λ]=0 gives the following quadratic equation.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} \quad \begin{matrix} =(a_{11}-\lambda)(a_{22}-\lambda)-a_{21}a_{12}= \\ \lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{21}a_{12}=0 \end{matrix} \qquad [63]$$

We can solve the quadratic equation in [63] to get two roots that give us the two possible eigenvalues:

$$\lambda = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2+4\left(a_{21}a_{12}-a_{11}a_{22}\right)}}{2} \qquad [64]$$

Each eigenvalue will have its own eigenvector. Each eigenvector is found by the solution of equation [57]. If we denote the eigenvectors as $\mathbf{x}_{(1)}$ and $\mathbf{x}_{(2)}$, the components of eigenvector j may be written as $x_{(j)1}$ and $x_{(j)2}$. Accordingly, we have to solve the set of equations shown below two times once for λ_1 and once for λ_2.

$$\begin{aligned} (a_{11}-\lambda_j)x_{(j)1}+a_{12}x_{(j)2}&=0 \\ a_{21}x_{(j)1}+(a_{22}-\lambda_j)x_{(j)2}&=0 \end{aligned} \qquad [65]$$

Again, the solution is not unique. Any set of **x** values, multiplied by an arbitrary constant, will satisfy this set of equations. For simplicity we pick $x_{(j)1} = \alpha$. We have two possible results for the eigenvector component, $x_{(j)2}$ depending on which equation we use.

$$x_{(j)1}=\alpha \qquad x_{(j)2}=\frac{(\lambda_j-a_{11})}{a_{12}}\alpha=\frac{a_{21}}{(\lambda_j-a_{22})}\alpha \qquad [66]$$

There appear to be two different solutions for $x_{(j)2}$, depending on the use of the first or second equation to get this eigenvector component. However, equating these two values for $x_{(j)2}$, will eliminate the arbitrary constant, α, and obtain equation [63] that we solved for λ. Thus the two possible expressions for $x_{(j)2}$ in equation [66] will result in the same value.

As a numerical example, consider the determination of the eigenvalues and eigenvectors for the matrix, $\mathbf{A}=\begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$. You can find the answer using equations [65] and [66]. However, we will outline the entire solution process as an example of finding eigenvalues and eigenvectors for larger systems. Solving the equation Det[**A** – **I**λ] for this matrix gives the following result.

$Det[\mathbf{A}-\mathbf{I}\lambda]=\begin{vmatrix} 1-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix}=(1-\lambda)(2-\lambda)-(0)(5)=0$. The roots to this equation are $\lambda_1 = 2$ and $\lambda_2 = 1$. (Here we have used the numbering convention that the highest eigenvalues has the lowest index.) We now substitute each eigenvalues into the equation $(\mathbf{A}-\mathbf{I}\lambda_1)\mathbf{x}_{(1)}=\mathbf{0}$, and solve for the components of each eigenvector. For the first eigenvector we obtain.

$$\begin{aligned} (1-2)x_{(1)1}+5x_{(1)2}&=0 \\ 0x_{(1)1}+(2-2)x_{(j)2}&=0 \end{aligned} \qquad [67]$$

We see that the last equation results in 0 = 0, which gives us no useful information. Since we know that the homogenous equation set has an infinite number of solutions, we pick an arbitrary value, α, for $x_{(1)2}$. With

this value, the first equation gives us the result that $x_{(1)1} = 5\alpha$. Thus our first eigenvector, $\mathbf{x}_{(1)} = [5\alpha \quad \alpha]^T$. We can apply the same procedure to find the second eigenvector.

$$\begin{aligned}(1-1)x_{(2)1} + 5x_{(2)2} &= 0 \\ 0x_{(2)1} + (2-1)x_{(2)2} &= 0\end{aligned} \qquad [68]$$

Here both equations tell us that $x_{(2)2}$ must be zero. However, there is no information about $x_{(2)1}$. We conclude that this must be an arbitrary quantity that we will call β. This gives our second eigenvector, $\mathbf{x}_{(2)} = [\beta \quad 0]^T$. We can verify our solution for eigenvalues and eigenvectors by showing that they satisfy the defining equation, $\mathbf{Ax} = \lambda\mathbf{x}$.

$$\mathbf{Ax}_{(1)} = \begin{bmatrix}1 & 5\\ 0 & 2\end{bmatrix}\begin{bmatrix}5\alpha\\ \alpha\end{bmatrix} = \begin{bmatrix}(1)(5\alpha)+(5)(\alpha)\\ (0)(5\alpha)+(2)(\alpha)\end{bmatrix} = \begin{bmatrix}10\alpha\\ 2\alpha\end{bmatrix} = 2\begin{bmatrix}5\alpha\\ \alpha\end{bmatrix} = \lambda_1\mathbf{x}_{(1)} \qquad [69]$$

$$\mathbf{Ax}_{(2)} = \begin{bmatrix}1 & 5\\ 0 & 2\end{bmatrix}\begin{bmatrix}\beta\\ 0\end{bmatrix} = \begin{bmatrix}(1)(\beta)+(5)(0)\\ (0)(\beta)+(2)(0)\end{bmatrix} = \begin{bmatrix}\beta\\ 0\end{bmatrix} = 1\begin{bmatrix}\beta\\ 0\end{bmatrix} = \lambda_2\mathbf{x}_{(2)} \qquad [70]$$

The calculations above show that the definition of eigenvalues and eigenvectors is satisfied regardless of our choices for α and β. This is a general result. We are always free to choose one component of the eigenvector. However, the remaining components will be set. Typically the eigenvector components are chosen to give a simple expression for the eigenvector (*i.e*, one in which all the components are integers or simple fractions) or a unit vector.[7]

In the example of the two-by-two matrix used above, we could express the eigenvectors we found in any of the ways shown immediately below. The last expression shown for each eigenvector is a unit vector. Note that the two eigenvectors are not orthogonal in this example.

$$\mathbf{x}_{(1)} = \begin{bmatrix}5\alpha\\ \alpha\end{bmatrix} \quad \mathbf{x}_{(1)} = \begin{bmatrix}5\\ 1\end{bmatrix} \quad \mathbf{x}_{(1)} = \begin{bmatrix}5/\sqrt{26}\\ 1/\sqrt{26}\end{bmatrix} \qquad \mathbf{x}_{(2)} = \begin{bmatrix}\beta\\ 0\end{bmatrix} \quad \mathbf{x}_{(2)} = \begin{bmatrix}1\\ 0\end{bmatrix} \qquad [71]$$

Matrix Transformations Using Eigenvectors

It is possible to use the eigenvalues and eigenvectors to transform the original **A** matrix into a diagonal matrix. To do this we define a matrix, **X**, whose columns are the different eigenvectors. That is $\mathbf{X} = [\mathbf{x}_{(j)}, \mathbf{x}_{(2)}, \mathbf{x}_{(3)},\ldots \mathbf{x}_{(n)}]$. We also define a diagonal matrix, **Λ**, whose elements are the eigenvectors; i.e., $\mathbf{\Lambda} = [\lambda_i\delta_{ij}]$. The matrix product, **AX** can be viewed as the product of **A** with each eigenvector. That is $\mathbf{AX} = [\mathbf{Ax}_{(1)}, \mathbf{Ax}_{(2)}, \mathbf{Ax}_{(3)},\ldots \mathbf{Ax}_{(n)}] = [\lambda_1\mathbf{x}_{(1)}, \lambda_2\mathbf{x}_{(2)}, \lambda_3\mathbf{x}_{(3)},\ldots \lambda_n\mathbf{x}_{(n)}]$. We can use this result to show that $\mathbf{AX} = \mathbf{X\Lambda}$. To do this, we examine the matrix product **XΛ** below, where we regard the subscript that identifies the particular eigenvector as a column index.

---

[7] A unit vector, **u**, is one for which the two norm, $||\mathbf{u}||_2 = \sqrt{\sum u_i^2} = 1$. If we have a vector, **v**, that is not a unit vector we can convert it into a unit vector by dividing each component by $||\mathbf{v}||_2$. This would give the components of the new unit vector by the following equation: $u_i = v_i \Big/ \sqrt{\sum v_k^2}$

$$\mathbf{X\Lambda} = \begin{bmatrix} x_{(1)1} & x_{(2)1} & x_{(3)1} & \cdots & \cdots & x_{(n)1} \\ x_{(1)2} & x_{(2)2} & x_{(3)2} & \cdots & \cdots & x_{(n)2} \\ x_{(1)3} & x_{(2)3} & x_{(3)3} & \cdots & \cdots & x_{(n)3} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ x_{(1)n} & x_{(2)n} & x_{(3)n} & \cdots & \cdots & x_{(n)n} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix} \quad [72]$$

Carrying out the indicated matrix multiplication gives the following result.

$$\mathbf{X\Lambda} = \begin{bmatrix} \lambda_1 x_{(1)1} & \lambda_2 x_{(2)1} & \lambda_3 x_{(3)1} & \cdots & \cdots & \lambda_n x_{(n)1} \\ \lambda_1 x_{(1)2} & \lambda_2 x_{(2)2} & \lambda_3 x_{(3)2} & \cdots & \cdots & \lambda_n x_{(n)2} \\ \lambda_1 x_{(1)3} & \lambda_2 x_{(2)3} & \lambda_3 x_{(3)3} & \cdots & \cdots & \lambda_n x_{(n)3} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \lambda_1 x_{(1)n} & \lambda_2 x_{(2)n} & \lambda_3 x_{(3)n} & \cdots & \cdots & \lambda_n x_{(n)n} \end{bmatrix} \quad [73]$$

$$= \begin{bmatrix} \lambda_1 \mathbf{x}_{(1)} & \lambda_1 \mathbf{x}_{(2)} & \lambda_1 \mathbf{x}_{(3)} & \cdots & \cdots & \lambda_1 \mathbf{x}_{(n)} \end{bmatrix} = \mathbf{AX}$$

Equation [73] gives the desired result that **XΛ** = **AX**; we can premultiply this equation by $\mathbf{X}^{-1}$ to obtain the following result.

$$\mathbf{\Lambda} = \mathbf{X}^{-1}\mathbf{AX} \quad [74]$$

That is, we can use the matrix created by using the eigenvectors as matrix columns to produce a diagonal matrix from the square matrix, **A**. The nonzero components of the diagonal matrix are the eigenvalues of the **A** matrix. This type of transformation is important in many advanced applications of matrix theory. An example of this, the use of matrix eigenvalues in the solution of a system of ordinary differential equations, is shown in the next section.

In the example started on page 21, we found the eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 1$, and the eigenvectors $\mathbf{x}_{(1)} = \begin{bmatrix} 5\alpha \\ \alpha \end{bmatrix}$ and $\mathbf{x}_{(2)} = \begin{bmatrix} \beta \\ 0 \end{bmatrix}$ for the matrix, $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$. Using these eigenvectors, we find $\mathbf{X} = \begin{bmatrix} 5\alpha & \beta \\ \alpha & 0 \end{bmatrix}$. We use equation [56] to get $\mathbf{X}^{-1} = \dfrac{1}{(5\alpha)(0)-(\alpha)(\beta)} \begin{bmatrix} 0 & -\beta \\ -\alpha & 5\alpha \end{bmatrix}$

Substituting $\mathbf{X}^{-1}$, **A**, and **X** into equation [74] gives the following result.

$$\mathbf{X}^{-1}\mathbf{AX} = -\frac{1}{\alpha\beta} \begin{bmatrix} 0 & -\beta \\ -\alpha & 5\alpha \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5\alpha & \beta \\ \alpha & 0 \end{bmatrix} =$$

$$= -\frac{1}{\alpha\beta}\begin{bmatrix} 0 & -\beta \\ -\alpha & 5\alpha \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} (1)(5\alpha)+(5)(\alpha) & (1)(\beta)+(5)(0) \\ (0)(5\alpha)+(2)(\alpha) & (0)(\beta)+(2)(0) \end{bmatrix} \qquad [75]$$

$$= -\frac{1}{\alpha\beta}\begin{bmatrix} 0 & -\beta \\ -\alpha & 5\alpha \end{bmatrix}\begin{bmatrix} 10\alpha & \beta \\ 2\alpha & 0 \end{bmatrix} = -\frac{1}{\alpha\beta}\begin{bmatrix} -2\alpha\beta & 0 \\ 0 & -\alpha\beta \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

This $\mathbf{X}^{-1}\mathbf{AX}$ produce produces the expected result: a diagonal matrix with the eigenvalues of **A** on the diagonal. We also see that the result does not depend on the arbitrary multiplicative constant used in each eigenvector.

Similarity transformations

An important concept in the computation of matrix eigenvalues is that of a similarity transformation. For a square matrix, **A**, the transformation, $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$, where **P** is ***any invertable*** square matrix, produces a new matrix, **B** that has the same eigenvalues as **A**. We can prove this by starting with the eigenvalue equation for **B**, $\mathbf{Bx} = \lambda\mathbf{x}$, and substituting the transformation $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$.

$$\mathbf{Bx} = \mathbf{P}^{-1}\mathbf{APx} = \lambda\mathbf{x} \qquad [76]$$

If we premultiply each side of the last equation by **P**, we can manipulate the result as follows.

$$\mathbf{PP}^{-1}\mathbf{APx} = \mathbf{IAPx} = \mathbf{APx} = \mathbf{P}\lambda\mathbf{x} \qquad [77]$$

Since λ is a scalar, we can write $\mathbf{P}\lambda\mathbf{x} = \lambda\mathbf{Px}$, so that the last equation in [77] becomes an eigenvalue equation for **A**.

$$\mathbf{A}(\mathbf{Px}) = \lambda(\mathbf{Px}) \qquad [78]$$

Equation [78] tells us that we can multiply the vector, **Px**, by the matrix, **A**, and obtain the same vector, multiplied by λ. This is an eigenvalue/eigenvector equation where the eigenvalue for the **A** matrix is the same as the eigenvalue for the **B** matrix. The eigenvectors of the **A** and **B** matrices, related by a similarity transformation, $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$, satisfy the following relationships: $\mathbf{x_A} = \mathbf{Px_B}$ or, equivalently, $\mathbf{x_B} = \mathbf{B} = \mathbf{P}^{-1}\mathbf{x_A}$.

Application of Matrix Eigenvalues and Eigenvectors to a System of Differential Equations

This transform may be used in the solution of simultaneous differential equations. A general system of linear, first-order differential equations for variables, $y_i(t)$, can be written as follows:

$$\frac{dy_i}{dt} + \sum_{j=1}^{N} a_{ij} y_j = r_i(t) \qquad i = 1,\ldots N \qquad [79]$$

In this equation, the values of a_{ij} are constant. If we define **y** and **r** as column matrices, we can rewrite this system of differential equations as a matrix equation:*

* If **y** is a matrix, with components y_{ij}, then the derivative $d\mathbf{y}/dx$ is a matrix whose components are dy_{ij}/dx.

$$\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = \mathbf{r} \qquad [80]$$

We can use the matrix, **X**, whose columns are eigenvectors of the **A** matrix to define a new set of dependent variables, $s_i(t)$ whose components define the column matrix, **s**, by the following equation.

$$\mathbf{y} = \mathbf{X}\mathbf{s} \qquad or \qquad \mathbf{s} = \mathbf{X}^{-1}\mathbf{y} \qquad [81]$$

With this definition, the original matrix differential equation [80] can be written as a differential equation in **s**.

$$\frac{d\mathbf{X}\mathbf{s}}{dt} + \mathbf{A}\mathbf{X}\mathbf{s} = \mathbf{r} \qquad \Rightarrow \qquad \mathbf{X}\frac{d\mathbf{s}}{dt} + \mathbf{A}\mathbf{X}\mathbf{s} = \mathbf{r} \qquad [82]$$

The second equation is possible because we have assumed that all the values of a_{ij} are constant; *i.e.*, they do not depend on time. This means that the eigenvectors in the **X** matrix will be constants as well. If we premultiply the second equation in [82] by $\mathbf{X}^{-1}$, we obtain the following result, using equation [74] to replace $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ by the eigenvalue matrix, **Λ**.

$$\mathbf{X}^{-1}\mathbf{X}\frac{d\mathbf{s}}{dt} + \mathbf{X}^{-1}\mathbf{A}\mathbf{X}\mathbf{s} = \mathbf{X}^{-1}\mathbf{r} \qquad \Rightarrow \qquad \mathbf{I}\frac{d\mathbf{s}}{dt} + \mathbf{\Lambda}\mathbf{s} = \mathbf{X}^{-1}\mathbf{r} \qquad [83]$$

If we define a new right-hand side column matrix, $\mathbf{p}(t) = \mathbf{X}^{-1}\mathbf{r}(t)$ our system of differential equations can be written as follows.

$$\frac{d\mathbf{s}}{dt} + \mathbf{\Lambda}\mathbf{s} = \mathbf{p}(t) \qquad [84]$$

Since **Λ** is a diagonal matrix, the differential equation for each component, $s_i(t)$ depends only on s_i and p_i. It does not depend on other components of the **s** array. Thus, we have a set of independent scalar differential equation to solve.

$$\frac{ds_i}{dt} + \lambda_i s_i = p_i(t) \qquad [85]$$

This differential equation has the form of the general first-order differential equation,

$$\frac{dx}{dt} + f_1(t)x = f_2(t) \qquad [86]$$

The solution to this general equation is shown below.

$$x = e^{-\int f_1 dt}\left[\int e^{\int f_1 dt} f_2 dt + C\right] \qquad [87]$$

The constant, C, is found from the initial condition on the dependent variable. Equation [85] can be placed in the general form of equation [86] if we define $f_1 = \lambda_i$ and $f_2 = p_i(t)$. Since λ_i is not a function of time, we can write the solution to equation [85], replacing x by s_i as the dependent variable, as follows.

$$s_i = e^{-\lambda_i t}\left[\int e^{\lambda_i t} p_i dt + C_i\right] = C_i e^{-\lambda_i t} + e^{-\lambda_i t}\int e^{\lambda_i t} p_i dt = C_i e^{-\lambda_i t} + q_i(t) \qquad [88]$$

Here we have defined $q_i(t)$, which depends on the problem-specific value for $\mathbf{p} = \mathbf{X}^{-1}\mathbf{r}$.

$$q_i(t) = e^{-\lambda_i t}\int e^{\lambda_i t} p_i dt \qquad [89]$$

If p_i is a constant, we have the following result for q_i.

$$\left[q_i(t)\right]_{p_i \text{ constant}} = e^{-\lambda_i t}\int e^{\lambda_i t} p_i dt = e^{-\lambda_i t} p_i \int e^{\lambda_i t} p_i dt = e^{-\lambda_i t} p_i \frac{e^{\lambda_i t}}{\lambda_i} = \frac{p_i}{\lambda_i} \qquad [90]$$

In this case, the solution for y_i becomes

$$y_i = C_i e^{-\lambda_i t} + \frac{p_i}{\lambda_i} \qquad [91]$$

If we know the initial values of y_i at t = 0, denoted as $y_i(0)$, we could find the corresponding initial values of s_i from the definition of **s** in equation [81]. If we denote the initial values of s_i as $s_i(0)$, we would find these from the initial y values as follows.

$$\mathbf{s}(0) = \mathbf{X}^{-1}\mathbf{y}(0) \qquad \mathbf{y}(0) = \mathbf{X}\mathbf{s}(0) \qquad [92]$$

With this definition of **s**(0), with components, $s_i(0)$, we can solve for the constant, C_i, in equation [88] as follows.

$$s_i(0) = C_i e^{-\lambda_i(0)} + q_i(0) \qquad \Rightarrow \qquad C_i = s_i(0) - q_i(0) \qquad [93]$$

With this value for C_i, the solution to our differential equation becomes:

$$s_i = \left[s_i(0) - q_i(0)\right]e^{-\lambda_i t} + q_i(t) \qquad [94]$$

For the special case where all the p_i are constant we have $q_i = p_i/\lambda_I$, and since q_i is not a function of time in this case, $q_i(0) = p_i/\lambda_I$.

$$s_i = \left[s_i(0) - \frac{p_i}{\lambda_i}\right]e^{-\lambda_i t} + \frac{p_i}{\lambda_i} \qquad [95]$$

We can rewrite equation [94] as a matrix equation if we define the matrix **E** as the following diagonal matrix.

$$\mathbf{E} = [\delta_{ij} e^{-\lambda_i t}] = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{-\lambda_3 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-\lambda_N t} \end{bmatrix} \qquad [96]$$

In order to rewrite equation [94] (or equation [95]) as a matrix equation, we have to rearrange the order of the exponential terms to give the correct results from matrix multiplication.

$$s_i = \left[s_i(0) - q_i(0)\right]e^{-\lambda_i t} + q_i(t) = e^{-\lambda_i t}\left[s_i(0) - q_i(0)\right] + q_i(t) \qquad [97]$$

$$\mathbf{s}(t) = \mathbf{E}\left[\mathbf{s}(0) + \mathbf{q}(0)\right] = \mathbf{q}(t) \qquad [98]$$

If we substitute the definitions of **s** from equation [81] and the definition of **s**(0) from equation [92] we obtain a matrix equation for the original **y** variable.

$$\mathbf{s}(t) = \mathbf{X}^{-1}\mathbf{y} = \mathbf{E}\left[\mathbf{s}(0) + \mathbf{q}(0)\right] + \mathbf{q}(t) = \mathbf{E}\left[\mathbf{X}^{-1}\mathbf{y}(0) + \mathbf{q}(0)\right] + \mathbf{q}(t) \qquad [99]$$

Multiplying this equation by **X** gives the solution for **y**(t) as follows.

$$\mathbf{y}(t) = \mathbf{X}\mathbf{E}\left[\mathbf{X}^{-1}\mathbf{y}(0) + \mathbf{q}(0)\right] + \mathbf{q}(t) \qquad [100]$$

Equation [95] gives the solution for constant p_i. This equation contains the term p_i/λ_i which can be written as the product of two matrices as shown below. This equation uses the result that the inverse of a diagonal matrix $[a_i\delta_{ij}]$ is the diagonal matrix $[\delta_{km}/a_k]$.[8]

$$\begin{bmatrix} p_1/\lambda_1 \\ p_2/\lambda_2 \\ p_3/\lambda_3 \\ \vdots \\ p_N/\lambda_N \end{bmatrix} = \begin{bmatrix} 1/\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & 1/\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/\lambda_N \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_N \end{bmatrix} = \Lambda^{-1}\mathbf{p} \qquad [101]$$

Using this result and the steps that led to equation [99], we can rewrite equation [95] as shown below.

$$\mathbf{s} = \mathbf{E}\mathbf{s}(0) - \mathbf{E}\Lambda^{-1}\mathbf{p} + \Lambda^{-1}\mathbf{p} = \mathbf{E}\mathbf{s}(0) + \left(\mathbf{I} - \mathbf{E}\right)\Lambda^{-1}\mathbf{p} \qquad [102]$$

If we substitute the definitions of **s** from equation [81] and the definition of **s**(0) from equation [92] we obtain a matrix equation for the original **y** variable.

$$\mathbf{y} = \mathbf{X}^{-1}\mathbf{E}\mathbf{X}\mathbf{y}(0) + \mathbf{X}^{-1}\left(\mathbf{I} - \mathbf{E}\right)\Lambda^{-1}\mathbf{p} \qquad [103]$$

In the simplest case where all the right-hand-side terms $r_i(t)$ in the original equation [79] are zero, we will have **p** = 0 and the solution becomes.

$$\mathbf{y} = \mathbf{X}^{-1}\mathbf{E}\mathbf{X}\mathbf{y}(0) \qquad [104]$$

This equation tells us that the behavior of the solution depends on exponential terms whose time coefficients are the eigenvalues of the matrix from the original set of differential equations.

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[8] The usual formula for the product of two matrices, $c_{im} = \Sigma_k a_{ik}b_{km}$ gives the following result for the product of the original matrix and the proposed inverse: $c_{im} = \Sigma_k a_i\delta_{ik}\delta_{km}/a_m = \delta_{im}$. Here we use the fact that the product $\delta_{ik}\delta_{km}$ is zero unless i = k and m = k. The result $c_{im} = d_{im}$ is the required unit matrix for the product of a matrix and its inverse.

Special matrices and quadratic forms

A set of column matrices, $\mathbf{e}^{(i)}$, is said to be **orthogonal** if scalar product of a given matrix with any other matrix, except itself, is zero. We can write the definition of an orthogonal set of matrices as follows:

$$\textit{The set } \mathbf{e}^{(i)} \textit{ is orthogonal if } \left(\mathbf{e}^{(i)}\right)^T \mathbf{e}^{(j)} = d_i \delta_{ij} \qquad [105]$$

If all the values of d_i are one, the set of matrices is **orthonormal**. Any set of orthogonal matrices may be converted to an orthonormal set by dividing each matrix, $\mathbf{e}^{(i)}$ by $\sqrt{d_i}$.

An **orthogonal matrix** is one for which the inverse of the matrix equals its transpose. That is, the matrix, **A**, is orthogonal if $\mathbf{A}^{-1} = \mathbf{A}^T$. One consequence of this definition is that the rows (and columns) of an orthogonal matrix form an orthogonal set of row (or column) matrices.

Matrix elements may be complex numbers as well as real numbers. We have already defined the transpose, $\mathbf{B} = \mathbf{A}^T$, of a matrix, **A**, as one for which $b_{ij} = a_{ji}$. For complex matrices (*i.e.*, matrices with complex components, we define the **adjoint matrix**, $\mathbf{A}^\dagger$, as the transpose of its complex conjugate.

$$\mathbf{B} = \mathbf{A}^\dagger = \left(\mathbf{A}^\mathbf{T}\right)^* = \left(\mathbf{A}^*\right)^T \qquad \Rightarrow \qquad b_{ij} = a^*_{ji} \qquad [106]$$

We have used the notation that a* is the complex conjugate of a; some texts use the notation $\bar{a}$ to denote the complex conjugate[9] of a. The matrix, **A***, is obtained from the matrix **A**, by replacing each component, a_{ij}, by its transpose complex conjugate, a_{ij}*. Matrix notation varies among sources; some authors use the notation **A*** or $\mathbf{A}^H$ for an adjoint matrix. These authors then use the notation, $\bar{\mathbf{A}}$, for the complex conjugate of matrix **A**.

A unitary matrix is a generalization of the orthogonal matrix for complex-valued matrices. A unitary matrix, **U**, is one for which the adjoint, $\mathbf{U}^\dagger$, equals the inverse, $\mathbf{U}^{-1}$.

A **self-adjoint matrix** is one for which $\mathbf{A} = \mathbf{A}^\dagger$. Such a matrix is also called a **Hermitian matrix**. A real symmetric matrix is a self-adjoint or Hermitian matrix.

A **normal matrix** is defined as one for which the product of a matrix with its adjoint does not depend on the order of the multiplication. This means that $\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$, if **A** is a normal matrix. Both Hermitian matrices and real symmetric matrices are normal. **The important feature of normal matrices is that their eigenvectors form a complete orthogonal set.** This means that the **X** matrix, described before equation [72] and defined implicitly in that equation, will have an inverse. In addition, it is simple to determine the inverse of the eigenvector matrix, because it must be an orthogonal matrix. From the definition of orthogonal matrices this means that $\mathbf{X}^{-1} = \mathbf{X}^T$. Many engineering applications yield symmetric matrices, which are guaranteed to provide an orthogonal eigenvector matrix.

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[9] If z is a complex number whose real part is x and whose imaginary part is y, we write $z = x + iy$ where $i = \sqrt{-1}$. We can also write $z = re^{i\theta}$, where $r^2 = x^2 + y^2$, and $\theta = \tan^{-1}(y/x)$. The complex conjugate, $\bar{z} = z^* = x - iy = re^{-i\theta}$ The product of a complex number with its complex conjugate equals the magnitude of the complex number: $|z|^2 = z\bar{z} = z^*z = r^2 = x^2 + y^2$.

To illustrate this important result for normal matrices, consider the following problem that determines the eigenvectors used in the transformation $\mathbf{X}^{-1}\mathbf{AX}$ for the following real symmetric matrix: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 1 \end{bmatrix}$.

We first obtain the eigenvalues by solving the equation that $Det(\mathbf{A} - \mathbf{I}\lambda) = 0$. This equation gives

$$Det(\mathbf{A} - \mathbf{I}\lambda) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 0-\lambda & 3 \\ 1 & 3 & 1-\lambda \end{vmatrix} = \\ (1-\lambda)(-\lambda)(1-\lambda) + (2)(3)(1) + (1)(2)(3) \\ -(1)(-\lambda)(1) - (2)(2)(1-\lambda) - (1-\lambda)(3)(3) = 0 \qquad [107]$$

Doing the indicated algebra gives the following cubic equation for λ.

$$(1-\lambda)^2(-\lambda) + 6 + 6 + \lambda - 4(1-\lambda) - 9(1-\lambda) \\ = (1-\lambda)^2(-\lambda) + \lambda - 13(1-\lambda) + 12 = 0 \qquad [108]$$

The goal seek tool of Excel was used to find the eigenvalues of this equation. The three eigenvalues are λ_1 = 4.71319670582861, λ_2 = -2.78926346205648, and λ_3 = 0.0760667562278632. You can verify that these values are the solutions to the eigenvalue equation.

The eigenvectors are found by solving the matrix equation $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = \mathbf{0}$. For the first eigenvalue, we have to solve the following matrix equation.

$$\begin{bmatrix} -3.7131967 & 2 & 1 \\ 2 & -4.7131967 & 3 \\ 1 & 3 & -3.7131967 \end{bmatrix} \begin{bmatrix} x_{(1)1} \\ x_{(1)2} \\ x_{(1)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [109]$$

Applying Gauss elimination to this system of equations gives the following result.

$$\begin{bmatrix} -3.7131967 & 2 & 1 \\ 0 & -3.635958 & 3.538619 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{(1)1} \\ x_{(1)2} \\ x_{(1)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [110]$$

As we expected, our eigenvector equation has infinite solutions. If we pick a value of one for $x_{(1)3}$, we obtain $x_{(1)2}$ = [0 – 3,538619(1)]/(-3.635958) = 0.973229 and $x_{(1)1}$ = [0 – (1)(1) – (2)(0.973229)] /(-3.7131967) = 0.79350979. The norm of this eigenvector is the square root of the following sum: $(1)^2 + (0.973229)^2 + (0.79350979)^2 = 2.5768232$. Thus the norm of this first eigenvector, $||\mathbf{x}_{(1)}||_2 = (2.5768232)^{1/2} = 1.605252$. If we divide each component by this norm, we will obtain the following normalized eigenvector $\mathbf{x}_{(1)}$ = [0.494321 0.606278 0.622955]^T. You should be able to verify that the two norm of this eigenvector is one and that it satisfies the equation that $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = \mathbf{0}$. We can repeat this process for the other two eigenvectors and then we can obtain the **X** matrix in which each column is one of the eigenvectors. The inverse of this matrix is also shown below. This was found using the MINVERSE function of Excel.

$$\mathbf{X} = \begin{bmatrix} 0.494321 & 0.270183 & -0.826225 \\ 0.606278 & -0.788297 & 0.104950 \\ 0.622955 & 0.552801 & 0.553478 \end{bmatrix}$$
$$\mathbf{X}^{-1} = \begin{bmatrix} 0.494321 & 0.606278 & 0.622955 \\ 0.270183 & -0.788297 & 0.552801 \\ -0.826225 & 0.104950 & 0.553478 \end{bmatrix} \qquad [111]$$

We see that the inverse matrix in this case is the same as the transpose of the **X** matrix. This confirms that the set of eigenvectors is orthogonal. (You should also be able to show that any pair of eigenvectors has dot product (inner product) that is zero.) This example illustrates the fact that a symmetric matrix produces an orthogonal set of eigenvectors. This greatly simplifies the $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ transformation process since $\mathbf{X}^{-1} = \mathbf{X}^{T}$ for an orthogonal matrix.

A positive definite matrix is a Hermitian matrix that satisfies the following relationship for any column matrix (column vector) that is not **0**. (You should keep in mind that a Hermitian matrix whose components are real is simply a symmetric matrix.)

$$Positive\ Definite\ \mathbf{A}: \qquad \mathbf{x}^{\dagger}\mathbf{A}\mathbf{x} > 0 \qquad [112]$$

If all the components of **x** are real, $\mathbf{x}^{\dagger}$ is simply $\mathbf{x}^{T}$. A positive semi-definite matrix is one for which the greater-than sign is replaced by a greater than or equal to sign.

$$Positive\ Semidefinite\ A: \qquad \mathbf{x}^{\dagger}A\mathbf{x} \geq 0 \qquad [113]$$

The product $\mathbf{x}^{\dagger}\mathbf{A}\mathbf{x}$ is illustrated below.

$$\mathbf{x}^{\dagger}\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1^* & x_2^* & x_3^* & \cdots & \cdots & x_n^* \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \qquad [114]$$

Taking the **Ax** product gives the following intermediate result.

$$\mathbf{x}^\dagger \mathbf{A}\mathbf{x} = \begin{bmatrix} x_1^* & x_2^* & x_3^* & \cdots & \cdots & x_n^* \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots + a_{2n}x_n \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots\cdots + a_{3n}x_n \\ \vdots \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots\cdots + a_{nn}x_n \end{bmatrix} \quad [115]$$

The final product of a row matrix times a column matrix yields a (1 by 1) matrix, which is essentially a scalar. This scalar is given by the following matrix multiplication.

$$\mathbf{x}^\dagger \mathbf{A}\mathbf{x} = \begin{array}{c} x_1^*\left(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n\right) + \\ x_2^*\left(a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots + a_{2n}x_n\right) + \\ x_3^*\left(a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots\cdots + a_{3n}x_n\right) + \\ \cdots \\ \cdots \\ x_n^*\left(a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots\cdots + a_{nn}x_n\right) \end{array} \quad [116]$$

We can express the result of this matrix multiplication by a single scalar, using summation notation.

$$\mathbf{x}^\dagger \mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i^* \sum_{j=1}^{n} a_{ij}x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i^* x_j = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i=1}^{n}\sum_{j=1,\, j\neq i}^{n} a_{ij}x_i^* x_j \quad [117]$$

We see that the right hand side is a sum of pure quadratic terms, $x_i^* x_i$ and mixed quadratic terms, $x_i^* x_j$. Each pure quadric term, x_i^2, occurs only once and it is multiplied a_{ii}. As usual, the product of a number, x_i with its complex conjugate x_i^* gives a real number that is the magnitude of the original complex number. Also, for a Hermitian matrix, where $a_{ij}^* = a_{ji}$, the elements on the principal diagonal, a_{ii}, must be real numbers for the general definition of a Hermitian matrix to apply.

A given mixed quadratic term occurs twice: in the form $x_i^* a_{ij} x_j$, and once in the form $x_j^* a_{ji} x_i$. The addition of these two terms can be simplified by the use of the basic relationship for a Hermitian matrix that $a_{ij}^* = a_{ji}$. As shown below, these two terms lead to the sum of two times the real value of the each individual term.[10]

$$a_{ij}x_i^* x_j + a_{ji}x_j^* x_i = a_{ij}x_i^* x_j + a_{ij}^* x_j^* x_i = a_{ij}x_i^* x_j + \left(a_{ij}x_i^* x_j\right)^* = 2\,\mathrm{Re}(a_{ij}x_i^* x_j) \quad [118]$$

Using this relationship we can reduce the number of terms in the final summation of equation [117] by summing only over values of j that are greater than i.

[10] This is based on the result that the sum of a complex number, z = x + iy, and its complex conjugate, z* = x – iy, equals 2x, two times the real component. Similarly, z – z* = 2iy, 2i times the imaginary component.

$$\mathbf{x}^\dagger \mathbf{A}\mathbf{x} = \sum_{i=1}^{n} a_{ii} x_i^2 + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathrm{Re}(a_{ij} x_i^* x_j) \qquad [119]$$

If the matrices A and **x** are real we can write equation [119] as follows.

$$\mathbf{x}^{\mathrm{T}} \mathbf{A}\mathbf{x} = \sum_{i=1}^{n} a_{ii} x_i^2 + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_{ij} x_i x_j \qquad [120]$$

If **A** is positive definite, we are assured that any general quadratic form like the ones in equations [119] or [120] will be positive. (If **A** is positive semi-definite we know that the sums will be greater than or equal to zero.) We can show that a matrix is positive definite or semi-definite by finding all its eigenvalues. A Hermitian matrix is positive definite if all its eigenvalues are positive; it is positive semi-definite if all its eigenvalues are zero or positive.

Vector spaces, norms, and inner products

Several concepts that we introduced with vectors can be generalized to other areas of engineering analysis. This general picture is called an abstract vector space. The name "abstract" is used because the things that we represent in such spaces may not be the same as the traditional vectors we are used to in mechanics. (However, traditional vectors are one item that is represented in a vector space.)

We recognize that vectors are usually represented by two or three components in a two- or three-dimensional space. We define a dot product for vectors and we define the magnitude of the vector as the sum of the squares of its components. We know that we cannot represent a three-dimensional vector in a two-dimensional space, and we know that we can represent a vector in more than one way by using a different coordinate system. When we represent a vector, we like to use an orthogonal coordinate system as the basic way to represent the vector. In this case, we know that the dot products of unit vectors in different coordinate directions are zero. All the ideas in this paragraph should be familiar to you for vectors representing force, velocity, acceleration and the like that you have encountered in your engineering and physics courses. We want to generalize these ideas to systems that can have any number of components, not just two or three. In addition, we want to consider functions as well as numbers as the components.

We also want to consider the possibility that the elements that we represent may have complex values. That is a quantity, z, may be represented as the sum of a real part, x, and an imaginary part, y; we write this as z = x + iy, where i = $\sqrt{-1}$. We define the complement of this complex number (with the notation z* or $\bar{z}$) as x – iy. That is, we change the sign of the imaginary part and use an asterisk or a line over the complex variable to denote the complex conjugate. A system of definitions that is developed for complex variables can be readily applied to real variables by setting the complex part equal to zero. Furthermore, the complex conjugate of a real number is just that number. If we set y = 0 in the definition, $z^* = \bar{z} = x - iy$, we get $z^*|_{y=0} = x$. A vector that consists of complex components, **z**, will have a complex conjugate, **z***, whose components are the complex conjugates of the components of the original vector, **z**.

The basic definition of a (linear) vector space simply states that vectors must satisfy certain simple properties. These are listed below.

1. If **x** and **y** are vectors in the space then **x** + **y** is also a vector in the space.[11]

[11] Sometimes the symbol ⊕ is used to emphasize that the "addition" operation for vectors may be different from the usual operation we expect of adding each component the vector, **a**, to the corresponding component of the vector **b** to get the components of the vector sum **a** + **b**. We will not be considering such

2. The addition operation in the previous statement is commutative and associative. That is, **x** + **y** = **y** + **x** and **x** + **y** + **z** = (**x** + **y**) + **z** = **x** + (**y** + **z**).

3. The space contains a null element, **0**, such that **x** + **0** = **0** + **x** = **x**.

4. For each vector, **x**, in the space there is another vector, **-x**, such that **x** + (**-x**) = **0**.

5. Vectors can be multiplied by scalars. If **x** and **y** are vectors in the space and α and β are scalars, all the following relationships hold:

   i) $\alpha\mathbf{x}$, $\beta\mathbf{x}$, $\alpha\mathbf{y}$, and $\beta\mathbf{y}$ are all vectors in the space

   ii) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

   iii) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

   iv) $\alpha\beta\mathbf{x} = (\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$

   v) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

   vi) $1\mathbf{x} = \mathbf{x}$

6. The norm of a vector, **x**, expressed as ||**x**||, is a measure of the size of the vector. This is a generalization of the usual definition of the length of a vector, |**x**| = $\sqrt{x_1^2 + x_2^2 + x_3^2}$. There are many possible norms. Any definition of a norm must satisfy the following relationships:

   i) $||\alpha\mathbf{x}|| = |\alpha|\ ||\mathbf{x}||$ $\qquad |\alpha| = \sqrt{\alpha\bar{\alpha}}$ for complex α

   ii) $||\mathbf{x}|| > 0$ if $\mathbf{x} \neq \mathbf{0}$

   iii) $||\mathbf{x}|| = 0$ if $\mathbf{x} = \mathbf{0}$

   iii) $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$

7. A common definition of the norm has the form $\|\mathbf{x}\| = \left[\sum |x_i|^q\right]^{1/q}$. This is called the "q norm" and is usually written as $||\mathbf{x}||_q$. In this notation, the usual definition of a vector length is the "two norm", $||\mathbf{x}||_2$. Other common norms are the one norm, which is simply the sum of absolute values and the "infinity norm" which is the element which has the maximum absolute value.

8. The vector dot product is generalized for abstract vector spaces using the notation (**x**,**y**). (Sometimes the notation <**x**,**y**> is used.) These inner products satisfy the following relationships.

   i) $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})^*$

   ii) $(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$

   iii) $(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$

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abstract operations here. In a similar sense the symbol $\otimes$ is used to represent a general operation of multiplication which may be different from normal multiplication of two numbers.

iv) $(\mathbf{x}, \mathbf{x}) > 0$ unless $\mathbf{x} = 0$

v) $(\mathbf{x}, \beta\mathbf{y}) = (\beta\mathbf{y}, \mathbf{x})^* = [\beta(\mathbf{y}, \mathbf{x})]^* = \beta^*[(\mathbf{y}, \mathbf{x})]^* = \beta^*(\mathbf{y}, \mathbf{x})^* = \beta^*(\mathbf{x}, \mathbf{y})$

9. We can form linear combinations of any number, k, vectors in the space. The linear combination is defined in terms of a set of k scalars, α_1, α_2, …, α_k, such that our linear combination is given by the equation $\sum_{i=1}^{k} \alpha_i \mathbf{x}_{(i)}$

10. A set of vectors is said to be ***linearly dependent*** if $\sum_{i=1}^{k} \alpha_i \mathbf{x}_{(i)} = \mathbf{0}$, where at least one of the a_i is not equal to zero.

11. A ***linearly independent*** set of vectors is one that is not linearly dependent.

12. The vector space is said to be ***n-dimensional*** if a set of n linearly independent vectors exists in the space, but no set of n+1 linearly independent vectors exists in that space.

13. Any vector in an n-dimensional space can be represented by a linearly independent combination of n vectors. Such a set of vectors is called a **basis set** and is said to **span the space**.

14. Two vectors whose inner product equals zero are said to be ***orthogonal***. That is, **x** and **y** are orthogonal if $(\mathbf{x}, \mathbf{y}) = 0$.

15. A set of n vectors, $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, … , $\mathbf{e}_{(n)}$, are said to be orthogonal if the inner product of any unlike pair of vectors vanishes. That is if $(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = 0$ for any i and j such that $i \neq j$, the set of vectors is orthogonal.

16. A set of n vectors, $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, … , $\mathbf{e}_{(n)}$, are said to be ***orthonormal*** if the inner product of any unlike pair of vectors vanishes and the inner product of like vectors equals one. For an orthonormal set, then, $(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = \delta_{ij}$.

17. We can convert any orthogonal set of vectors into an orthonormal set by dividing each component of the orthogonal vector with index i, by the inner product for that vector with itself, $(\mathbf{e}_{(i)}, \mathbf{e}_{(i)})$.

The statements above summarize several definitions about abstract vector spaces, norms and inner products. A major idea that is used throughout engineering analysis courses is the idea that we can represent a "vector" in terms of a basis set, which is a linearly independent set of vectors that span the space. This means that any "vector" in the space can be expressed as a linear combination of the basis set.

We know that this is true for our conventional vectors where we use the basis set **i**, **j**, and **k**. The possible dot products of this basis set can be seen to form an orthonormal set. As noted earlier, these possible dot products are $\mathbf{i}\bullet\mathbf{i} = \mathbf{j}\bullet\mathbf{j} = \mathbf{k}\bullet\mathbf{k} = 1$ and $\mathbf{i}\bullet\mathbf{j} = \mathbf{i}\bullet\mathbf{k} = \mathbf{j}\bullet\mathbf{i} = \mathbf{j}\bullet\mathbf{k} = \mathbf{k}\bullet\mathbf{i} = \mathbf{k}\bullet\mathbf{j} = 0$. In a more general notation we can represent this basis set as $\mathbf{e}_{(1)} = \mathbf{i}$, $\mathbf{e}_{(2)}$, $= \mathbf{j}$, and $\mathbf{e}_{(3)} = \mathbf{k}$. This basis set is orthonormal since $(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = \delta_{ij}$.

We will extend the notation of inner products and orthogonality to functions. For functions, the inner product is defined in terms of an integral. If we have a set of functions, $f_i(x)$ defined on an interval $a \le x \le b$, we define the inner product for these functions as follows.

$$\left(f_i, f_j\right) = \int_a^b f_i^*(x)\, f_j(x)\rho(x)dx \qquad [121]$$

In this definition, the function $\rho(x)$ is called the weighting function. In many cases, $\rho(x)$ =1 and is not considered in the definition of the inner product of functions. The functions $f_i(x)$ are orthogonal if the following relationship holds.

$$\left(f_i, f_j\right) = \int_a^b f_i^*(x)\, f_j(x)\rho(x)dx = a_i\delta_{ij} \qquad [122]$$

We can define a set of ***orthonormal*** functions by the following equation.

$$\left(f_i, f_j\right) = \int_a^b f_i^*(x)\, f_j(x)\rho(x)dx = \delta_{ij} \qquad [123]$$

Any set of orthogonal functions $g_i(x)$ can be converted to a set of orthonormal functions $f_i(x)$ by dividing by the square root of the inner product, (g_i,g_i).

$$f_i(x) = \frac{g_i(x)}{\sqrt{(g_i, g_i)}} = \frac{g_i(x)}{\sqrt{\int_a^b g_i^*(x) g_j(x)\rho(x)dx}} \qquad [124]$$

We can show that the functions $f_i(x)$ form an orthonormal set by constructing the inner product (f_i, f_i).

$$(f_i, f_i) = \left(\frac{g_i}{\sqrt{(g_i, g_i)}}, \frac{g_i}{\sqrt{(g_i, g_i)}}\right) = \frac{(g_i, g_i)}{\left(\sqrt{(g_i, g_i)}\right)^2} = 1 \qquad [125]$$

Note that the inner product (g_i,g_i) is a constant so that we can bring it outside the inner product calculation for (f_i,f_i). We see that the inner product of f_i with itself is equal to one. The inner product (f_i, f_j) is proportional to (g_i, g_j) which is zero. Thus the functions f_i form an orthonormal set.