December 6 Homework Solutions

1. Use the Euler method, $\mathbf{y}_{n+1} = \mathbf{y}_n + hf(x_n, \mathbf{y}_n)$, for a system of equations to solve $y_1' = 2y_1 - 4y_2$, $y_2' = y_1 - 3y_2$, $y_1(0) = 3$, $y_2(0) = 0$. Solve for ten steps with h = 0.1 and plot the solution in the y_1-y_2 plane.

The Euler formula, in vector form, is $\mathbf{y}_{n+1} = \mathbf{y}_n + hf(x_n, \mathbf{y}_n)$. The components of the matrices in this equation are shown below, using the formula for the derivatives.

$$\mathbf{y}_{n+1} = \begin{bmatrix} y_{1,n+1} \\ y_{2,n+1} \end{bmatrix} = \mathbf{y}_n + h\mathbf{f}_n = \begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} + h\begin{bmatrix} f_{1,n} \\ f_{2,n} \end{bmatrix} = \begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} + h\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix}$$

Starting with the initial conditions at x_0 and $\mathbf{y}_0$, we get the following values for the first step.

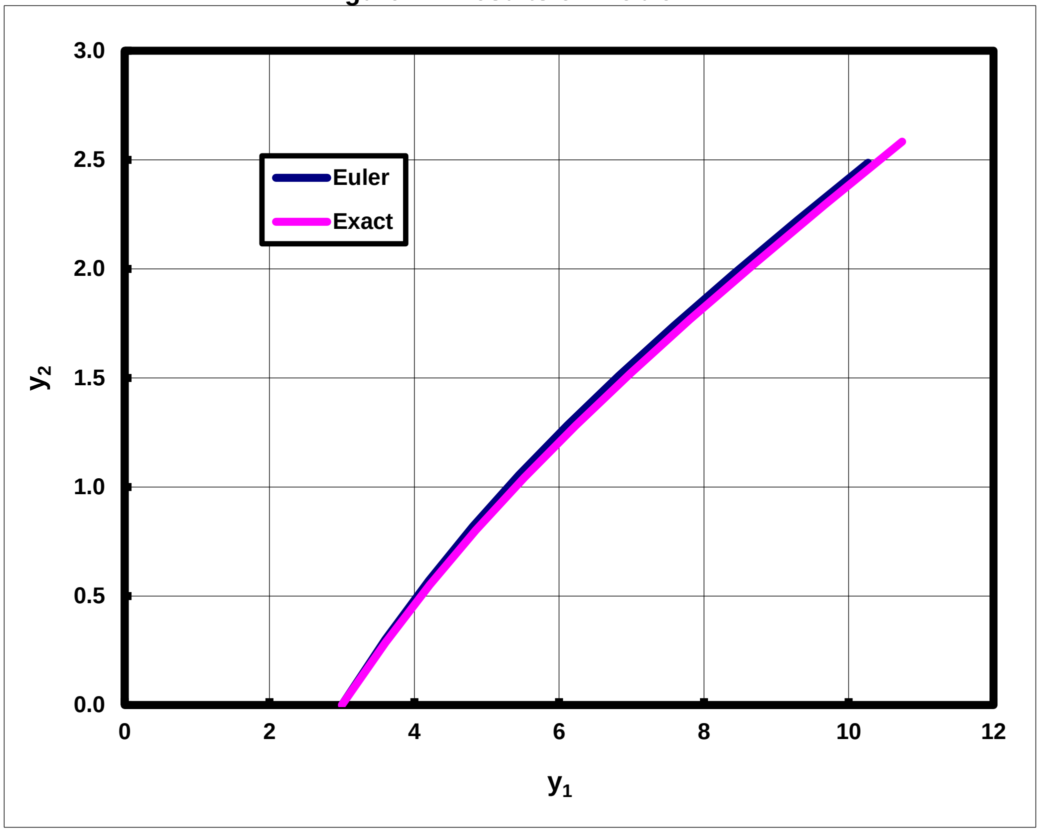
$$\mathbf{y}_1 = \begin{bmatrix} y_{1,1} \\ y_{2,1} \end{bmatrix} = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} + h\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + (0.1)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + (0.1)\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 0.3 \end{bmatrix}$$

Continuing in this fashion produces the table below. In this table, we are comparing the numerical results to the exact solution. To obtain the exact solution we can combine the two first order equations into a single equation, $y_1'' + y_1' - 2y_1 = 0$. The solution to this equation is $y_1 = C_1e^x + C_2e^{-2x}$. This gives $y_2 = (C_1/4)e^x + C_2e^{-2x}$. After finding C_1 and C_2 from the initial conditions we have the result that $y_1 = 4e^x - e^{-2x}$ and $y_2 = e^x - e^{-2x}$.

i	x_i	y_{1i}	exact y_{1i}	error y_1	y_{2i}	exact y_{2i}	error y_2	f_1	f_2
0	0.0	3	3	0	0	0	0	6	3
1	0.1	3.6	3.601953	0.001953	0.3	0.28644	-0.01356	6	2.7
2	0.2	4.2	4.215291	0.015291	0.57	0.551083	-0.01892	6.12	2.49
3	0.3	4.812	4.850624	0.038624	0.819	0.801047	-0.01795	6.348	2.355
4	0.4	5.4468	5.517970	0.071170	1.0545	1.042496	-0.01200	6.6756	2.2833
5	0.5	6.11436	6.227006	0.112646	1.28283	1.280842	-0.00199	7.0974	2.26587
6	0.6	6.824100	6.987281	0.163181	1.509417	1.520925	0.011508	7.61053	2.295849
7	0.7	7.585153	7.808414	0.223261	1.739002	1.767156	0.028154	8.21430	2.368148
8	0.8	8.406583	8.700267	0.293684	1.975817	2.023644	0.047828	8.90990	2.479133
9	0.9	9.297573	9.673114	0.375541	2.223730	2.294304	0.070574	9.70023	2.626383
10	1.0	10.26760	10.73779	0.470196	2.486368	2.582947	0.096578	10.58972	2.808491

The exact solution and the solution using Euler's method are plotted in the $y_1 - y_2$ plane on the next page. It is not easy to see on this plot, but the difference between the two lines, which is the error in the solution, is increasing as both values of y_i increase with x.

Figure 1 – Results of Problem 1



2. **Use the fourth-order Runge-Kutta method for the system of equations solved in problem 1: $y_1' = 2y_1 - 4y_2$, $y_2' = y_1 - 3y_2$, $y_1(0) = 3$, $y_2(0) = 0$. Solve for two steps with h = 0.5 and compare the results to those in problem 1.**

To apply the Runge-Kutta method to a system of equations, we have to compute the k values, for each variable, in sequence. We start by computing $\mathbf{k}_{(1)}$.for the initial step then continue with subsequent k values until we have all four that are required to compute $\mathbf{y}_{(1)}$.

$$\mathbf{k}_{(1)} = h\mathbf{f}(x_n, \mathbf{y}_n) = h\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} = (0.5)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = (0.5)\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.5 \end{bmatrix}$$

$$\mathbf{k}_{(2)} = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_{(1)}/2) = (0.5)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\left\{\begin{bmatrix} 3 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 \\ 1.5 \end{bmatrix}\right\} = (0.5)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 4.5 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.125 \end{bmatrix}$$

$$\mathbf{k}_{(3)} = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_{(2)}/2) = (0.5)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\left\{\begin{bmatrix} 3 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 \\ 1.125 \end{bmatrix}\right\} = (0.5)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 4.5 \\ 0.5625 \end{bmatrix} = \begin{bmatrix} 3.375 \\ 1.406 \end{bmatrix}$$

$$\mathbf{k}_{(4)} = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_{(3)}) = (0.5)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\left\{\begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3.375 \\ 1.406 \end{bmatrix}\right\} = (0.5)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 6.375 \\ 1.406 \end{bmatrix} = \begin{bmatrix} 3.5625 \\ 1.0781 \end{bmatrix}$$

$$\mathbf{y}_{(1)} = \mathbf{y}_{(0)} + \frac{\mathbf{k}_{(1)} + 2\mathbf{k}_{(2)} + 2\mathbf{k}_{(3)} + \mathbf{k}_{(4)}}{6} = \begin{bmatrix}3\\0\end{bmatrix} + \frac{1}{6}\left\{\begin{bmatrix}3\\1.5\end{bmatrix} + 2\begin{bmatrix}3\\1.125\end{bmatrix} + 2\begin{bmatrix}3.375\\1.406\end{bmatrix} + \begin{bmatrix}3.5625\\1.0781\end{bmatrix}\right\} = \begin{bmatrix}6.219\\1.273\end{bmatrix}$$

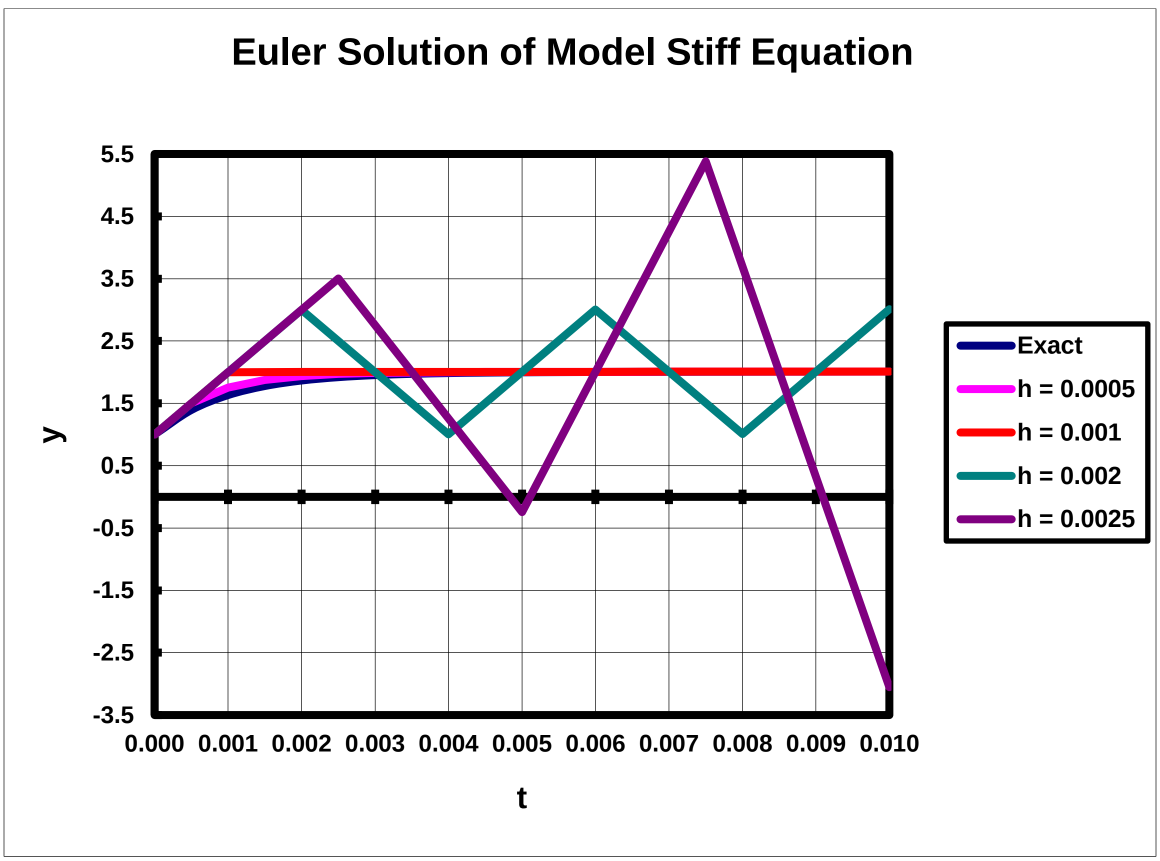
The second step in the Runge-Kutta solution is taken in the same way as the first step. The results at the end of this step, at x = 1, are compared to the results from the Euler algorithm in the table below.

Exact Solution		Runge-Kutta Solution		Euler Solution		Runge-Kutta Error		Euler Error	
y_1	y_2	y_1	y_2	y_1	y_2	y_1	y_2	y_1	y_2
10.73779	2.58295	10.72876	2.576721	10.2676	2.48637	0.00903	0.00623	0.47020	0.09658

The Runge-Kutta used eight function evaluations and the Euler used ten function evaluations. However, the Runge-Kutta produced a much smaller error.

3. Consider the model stiff ODE: y' = -1000[y –(t + 2)] + 1, y(0) = 1. Solve this ODE by the explicit Euler method from t = 0 to t = 0.01 using h = 0.0005, 0.0001, 0.002, and 0.0025. Compare the solution with the exact solution: $y = -e^{-1000t} + t + 2$

In this problem, we have dy/dt = f(t,y) = -1000[y – (t+2)] +1. Using the explicit Euler algorithm, $y_{n+1} = y_n + hf(t_n, y_n)$ gives the following result for the first step with h = 0.0005: $y_1 = y_0 + (0.0005)\{-1000[y_0 – (t_0+2)] +1\}$ = 1 + (0.0005){-1000[1 – (0+2)] +1} = 1.5005. We repeat this process until we get to t = 0.01. The entire solution is then repeated three times for the remaining step sizes. The results of these calculations are shown in the figure below and the table on the next page.



Results for Explicit Euler Calculations					
i	t_i	y_i	f_i	exact y_i	error
Results for h = 0.0005					
0	0	1	1001	1	0
1	0.0005	1.5005	501	1.393969	-0.10653
2	0.001	1.751	251	1.633121	-0.11788
3	0.0015	1.8765	126	1.77837	-0.09813
4	0.002	1.9395	63.5	1.866665	-0.07284
5	0.0025	1.97125	32.25	1.920415	-0.05083
6	0.003	1.987375	16.625	1.953213	-0.03416
7	0.0035	1.9956875	8.8125	1.973303	-0.02238
8	0.004	2.0000938	4.90625	1.985684	-0.01441
9	0.0045	2.0025469	2.953125	1.993391	-0.00916
10	0.005	2.0040234	1.976562	1.998262	-0.00576
11	0.0055	2.0050117	1.488281	2.001413	-0.0036
12	0.006	2.0057559	1.244141	2.003521	-0.00223
13	0.0065	2.0063779	1.12207	2.004997	-0.00138
14	0.007	2.006939	1.061035	2.006088	-0.00085
15	0.0075	2.0074695	1.030518	2.006947	-0.00052
16	0.008	2.0079847	1.015259	2.007665	-0.00032
17	0.0085	2.0084924	1.007629	2.008297	-0.0002
18	0.009	2.0089962	1.003815	2.008877	-0.00012
19	0.0095	2.0094981	1.001907	2.009425	-7.3E-05
20	0.01	2.009999	1.000954	2.009955	-4.4E-05
Results for h = 0.001					
0	0	1	1001	1	0
1	0.001	2.001	1	1.633121	-0.36788
2	0.002	2.002	1	1.866665	-0.13534
3	0.003	2.003	1	1.953213	-0.04979
4	0.004	2.004	1	1.985684	-0.01832
5	0.005	2.005	1	1.998262	-0.00674
6	0.006	2.006	1	2.003521	-0.00248
7	0.007	2.007	1	2.006088	-0.00091
8	0.008	2.008	1	2.007665	-0.00034
9	0.009	2.009	1	2.008877	-0.00012
10	0.01	2.01	1	2.009955	-4.5E-05
Results for h = 0.002					
0	0	1	1001	1	0
1	0.002	3.002	-999	1.866665	-1.13534
2	0.004	1.004	1001	1.985684	0.981684
3	0.006	3.006	-999	2.003521	-1.00248
4	0.008	1.008	1001	2.007665	0.999665
5	0.01	3.01	-999	2.009955	-1.00005
Results for h = 0.0025					
0	0	1	1001	1	0
1	0.0025	3.5025	-1499	1.920415	-1.58208
2	0.005	-0.245	2251	1.998262	2.243262
3	0.0075	5.3825	-3374	2.006947	-3.37555
4	0.01	-3.0525	5063.5	2.009955	5.062455

The exact solution, $y = -e^{-1000t} + t + 2$, shows a rapidly decaying exponential followed by a linear portion with a slope of 1. For the small time-scale used in this solution, the linear portion of the solution appears to be a constant at y = 2. The Euler algorithm gives a good solution in this part of the region because the solution is first order and the Euler method has a global first order error. Thus, it should be able to represent linear solutions exactly.

The exponential term has a value of -1 at t = 0, but the value soon approaches zero. At t = 0.005, the value of the exponential term is –0.007, which is a negligible part of the solution. Despite this decay of the exponential term, we still have stability problems for large h values. The have of h = 0.002 appears stable in that the solution is bounded. However, the values are obviously far from correct. The solution for h = 0.0025 is unstable.

4. Solve y' = -1000[y –(t + 2)] + 1, y(0) = 1 by the second-order Gear method from t = 0 to t = 0.1 using h = 0.01, and 0.02. The equation for the second-order Gear method is

$$y_{n+1} = \frac{2}{3}hf_{n+1} + \frac{4}{3}y_n - \frac{1}{3}y_{n-1}$$

Note that for this textbook problem you can write an equation for f_{n+1} and substitute it into the Gear algorithm to obtain an equation that is linear in y_{n+1}. You can solve that equation for y_{n+1} so that you will not have to iterate each step of the solution. Use the exact solution, $y = -e^{-1000t} + t + 2$, to get the necessary starting values.

Substituting the derivative expression for f_{n+1} into the Gear algorithm equation gives the following result.

$$y_{n+1} = \frac{2}{3}h\left\{-1000\left[y_{n+1} - \left(t_{n+1} + 2\right)\right] + 1\right\} + \frac{4}{3}y_n - \frac{1}{3}y_{n-1}$$

Although the implicit form of Gear's method usually requires a trial-and-error solution for y_{n+1}, in this textbook problem, with a linear expression for f_{n+1}, we can solve for y_{n+1}, explicitly. If we start by multiplying the entire equation by three, we get the following result.

$$3y_{n+1} = -2000hy_{n+1} + 2000h\left(t_{n+1} + 2\right) + 2h + 4y_n - y_{n-1}$$

$$y_{n+1} = \frac{2000h\left(t_{n+1} + 2\right) + 2h + 4y_n - y_{n-1}}{3 + 2000h}$$

For h = 0.01, we have the values of $y_0 = 1$ and $y_1 = y(0.01) = 2.009955$ from the exact solution. We then compute the value of y_2 from the general equation as follows.

$$y_2 = \frac{2000h\left(t_2 + 2\right) + 2h + 4y_1 - y_0}{3 + 2000h} = \frac{2000(0.01)\left(0.02 + 2\right) + 2(0.01) + 4(2.009955) - 1}{3 + 2000(0.01)}$$

$y_2 = 2.06347$

The complete results for this step size are shown in the table below.

i	t_i	y_i	f_i	exact y_i	error
0	0	1	1001	1	0
1	0.01	2.009955	1.0454	2.009955	0
2	0.02	2.063470	-42.4704	2.020000	-4.35E-02
3	0.03	2.037562	-6.56204	2.030000	-7.56E-03
4	0.04	2.039425	1.574879	2.040000	5.75E-04
5	0.05	2.049571	1.428763	2.050000	4.29E-04
6	0.06	2.059950	1.049573	2.060000	4.96E-05
7	0.07	2.070010	0.989979	2.070000	-1.00E-05
8	0.08	2.080004	0.996102	2.080000	-3.90E-06
9	0.09	2.090000	0.999758	2.090000	-2.42E-07
10	0.10	2.100000	1.000127	2.100000	1.27E-07

For h = 0.02, we obtain the following results.

i	t_i	y_i	f_i	exact y_i	error
0	0	1	1001	1	0
1	0.02	2.020000	1.00000	2.020000	0
2	0.04	2.063256	-22.25581	2.040000	-2.33E-02
3	0.06	2.062163	-1.16333	2.060000	-2.16E-03
4	0.08	2.079660	1.33959	2.080000	3.40E-04
5	0.10	2.099918	1.08190	2.100000	8.19E-05

We see that doubling the step size increases the order by a factor of 643. This is much more than a second order error.

We see that the Gear method allows much larger steps than the explicit Euler method while providing a smaller error.

5. Solve the following ODE y'' + (1+x)y' +(1+x)y = 1 with y(0) = 1 and y'(1) = 0, by the shooting method using the second-order modified Euler method, given by the equations below, with h = 0.125.

$$y_{n+1}^P = y_n + hf(x_n, y_n) \qquad y_{n+1}^C = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^P)\right]$$

To use the modified Euler method, we have to convert the second order ODE to a pair of first order ODEs. We do this by defining z = y' so that the original ODE becomes z' = y'' = 1 – (1 + x)(y' + y) = 1 – (1 + x)(z + y) = f_z. To use the shoot-and-try method, we guess z(0) = 0, for an initial guess. This gives the following calculations for the initial step.

$$y_1^P = y_0 + hf_y(x_0, y_0, z_0) = y_0 + hz_0 = 1 + (0.125)(0) = 1$$

$$z_1^P = z_0 + hf_z(x_0, y_0, z_0) = z_0 + h[1 - (1 + x_0)(y_0 + z_0)] = 0 + (0.125)[1 - (1+0)(1+0)] = 0$$

$$y_{n+1}^C = y_0 + \frac{h}{2}\left[f_y(x_0, y_0, z_0) + f_y(x_1, y_1^P, z_1^P)\right] = 1 + \frac{0.125}{2}\left[0 + z_1^P\right] = 1 + \frac{0.125}{2}[0+0] = 1$$

$$z_1^C = z_0 + \frac{h}{2}\left[f_z(x_0, y_0, z_0) + f_z(x_1, y_1^P, z_1^P)\right] = 0 + \frac{0.125}{2}[0 + 1 - (1 + 0.125)(1+0)] = -\frac{(0.125)^2}{2}$$

Continuing in this fashion until x = 1 gives a value of y'(1) = z(1) = -0.268540. Using Excel for the calculations, it is possible to use the goal seek function to determine the value of z(0) that will produce the desired value of y'(1) = 0. The results for this case are shown in the table below. In this table, the predictor (P) and corrector(C), for a given x value, are shown on separate rows.

i	P/C	x_i	y_i	z_i	dy/dx	dz/dx
0		0	1	-0.686317	-0.686317	0.686317
1	P	0.125	0.91421	-0.600527	-0.600527	0.647107
1	C	0.125	0.833783	-0.517188	-0.517188	0.643831
2	P	0.250	0.769134	-0.436709	-0.436709	0.584469
2	C	0.250	0.709515	-0.359941	-0.359941	0.563031
3	P	0.375	0.664523	-0.289562	-0.289562	0.484428
3	C	0.375	0.623929	-0.224095	-0.224095	0.450229
4	P	0.500	0.595917	-0.167817	-0.167817	0.357850
4	C	0.500	0.571423	-0.117312	-0.117312	0.318834
5	P	0.625	0.556759	-0.077458	-0.077458	0.221136
5	C	0.625	0.544585	-0.043710	-0.043710	0.186077
6	P	0.750	0.539122	-0.020450	-0.020450	0.092324
6	C	0.750	0.535112	-0.003050	-0.003050	0.068892
7	P	0.875	0.534731	0.005562	0.005562	-0.013048
7	C	0.875	0.534888	0.009052	0.009052	-0.019886
8	P	1.000	0.536019	0.006566	0.006566	-0.085170
8	C	1.000	0.536995	2.08E-16	2.08E-16	-0.073990

For h = 0.125, then, the value of the initial slope, dy/dx z is -0.686317 and the final value of y is 0.536995; repeating the calculation with h = 0.005 (details not shown) gives an initial slope, dy/dx = -0.796480 and a final y value of 0.513456.