Wind Probability Distributions

Larry Caretto
Mechanical Engineering 483

Alternative Energy Engineering II

February 24, 2010

Outline

- Nature of probability distribution functions
- Probability distribution functions for wind velocity
  – Weibull distribution
  – Rayleigh distribution
- Calculations of average power in the wind

2

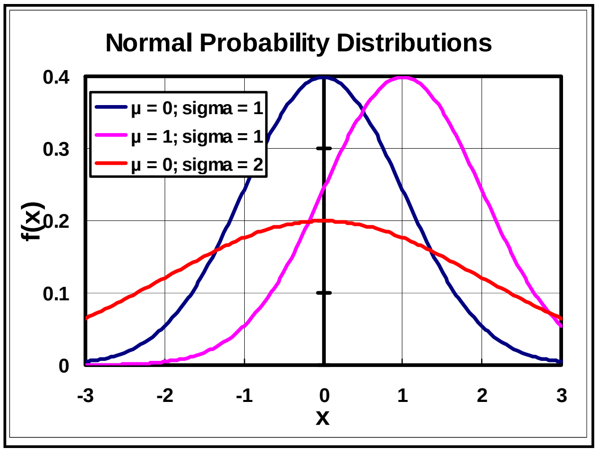
Probability Distributions

- Applied to variation of wind over time
- Best known example of probability distribution is the normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty \le x < \infty$$

- This is a two-parameter distribution
  – Mean = μ
  – Variance = σ^2

3



Probability Distributions II

- What does a probability distribution function (pdf) represent?
  – Probability that the random variable x (wind speed for our interest here) lies in a certain range $a \le x \le b$ is integral of pdf

$$P(a \le x \le b) = \int_a^b f(x)dx$$

- Note that integral over all x must equal 1 (P = 1 is certainty)

5

Distribution Dimensions

- f(x)dx represents a probability which is a dimensionless quantity
- The random variable x will usually have some dimension
- For f(x)dx to be dimensionless, f(x) must have the same dimensions as 1/x
- Can you see this for normal distribution?

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

6

Cumulative Distribution

- Define F(b) = P(x ≤ b)
- Use P(a ≤ x ≤ b) from a previous slide

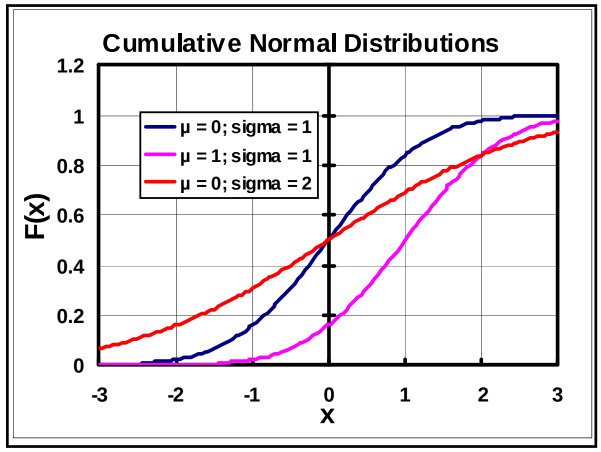
$$P(a \le x \le b) = \int_a^b p(x)dx \quad \Rightarrow \quad F(b) = P(x \le b) = \int_{-\infty}^b p(x)dx$$

- With this definition we can write

$$P(a \le x \le b) = \int_a^b p(x)dx = \int_{-\infty}^b p(x)dx - \int_{-\infty}^a p(x)dx = F(b) - F(a)$$

- Use equations or tables for F(b) to find P

California State University Northridge 7



Excel NORMDIST Function

- Normal distribution computed in Excel by the function NORMDIST(x, μ, σ, q)
  - x is the value for which the distribution value f(x) or F(x) is required
  - μ and σ are the distribution parameters
  - q is true to get F(x), false to get f(x)
    - q may be omitted when finding F(x)
- What is the probability that a variable x from a normal distribution with μ = 3 and σ = 2 lies between 1 and 2?

California State University Northridge 9

Excel NORMDIST Function II

- The answer to the question on the last slide is found by getting F(2) – F(1)
- We need to evaluate the NDIST(x, μ, σ, q) function for x = 1 and x = 2
  - Given μ = 3 and σ = 2
  - Value of NDIST(1, 3, 2, TRUE) = 0.158655
  - Value of NDIST(2, 3, 2, TRUE) = 0.308538
  - Probability x lies between 1 and 2 is 0.308538 – 0.158655 = 0.149882

California State University Northridge 10

Mean and Variance

- For any pdf we define the mean, μ, and the variance σ^2 as follows

$$\mu = \int_{-\infty}^{\infty} x f(x)dx \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

- This general expression uses the limits of –∞ and ∞ for random variable, x
  - Other upper and lower limits are substituted for specific distributions
    - For wind speed pdfs lower limit is zero

California State University Northridge 11

More on Variance

- Computational formula for variance

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} \left(x^2 - 2x\mu + \mu^2\right) f(x)dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - 2\mu \int_{-\infty}^{\infty} x f(x)dx + \mu^2 \int_{-\infty}^{\infty} f(x)dx$$

Definition of mean

Integral over all x is 1

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - 2\mu\mu + \mu^2 = \boxed{\int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2}$$

California State University Northridge 12

Average Value of g(x)

- Define $\overline{g}$ as the mean value of some function, g(x), of the random variable x

$$\overline{g} = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- For wind applications we are interested in the mean power which is the mean V^3
  - We will also be interested in how much power is within a certain range of wind velocities

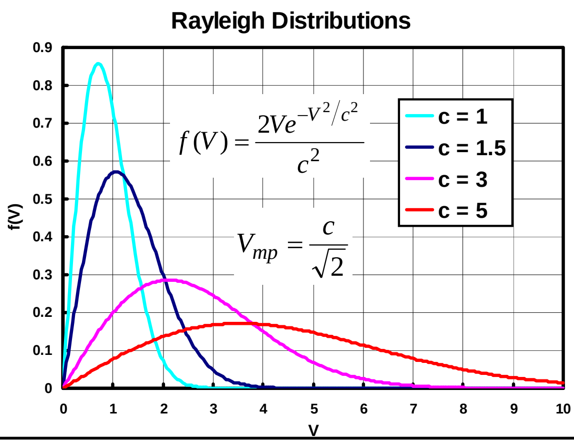
California State University Northridge 13

Rayleigh Distribution

- Probability distribution of wind over time
- A one-parameter distribution using scale parameter, c
- Mean = $c\pi^{1/2}/2$
- Variance = $c^2(2 - \pi/4)$
- Most probable V (pdf maximum): $V_{mp} = \frac{c}{\sqrt{2}}$

$$f(V) = \frac{2Ve^{-V^2/c^2}}{c^2} \qquad 0 \le V < \infty$$

California State University Northridge 14



Rayleigh Distribution Forms

- At least three variations are used

$$V_{mp} = \beta = \frac{c}{\sqrt{2}} \qquad f(V) = \frac{Ve^{-V^2/2\beta^2}}{\beta^2} \qquad 0 \le V < \infty$$

$$2\beta^2 = c^2 \qquad f(V) = \frac{2Ve^{-V^2/c^2}}{c^2} \qquad 0 \le V < \infty$$

$$\overline{V} = \beta\sqrt{\frac{\pi}{2}} = \frac{c}{2}\sqrt{\pi} \qquad f(V) = \frac{\pi Ve^{-\pi V^2/4\overline{V}^2}}{2\overline{V}} \qquad 0 \le V < \infty$$

California State University Northridge 16

Cumulative Distribution

- Find $F(V_0) = P(V \le V_0) = 1 - e^{-V_0^2/c^2}$
  - Variable transformation: $y = V^2/c^2$ so that $V = cy^{1/2}$ and $dV = (1/2)cy^{-1/2}dy$
    - $y = 0$ when $V = 0$; $y = V'_0 = V_0^2/c^2$ when $V = V_0$

$$F(V_0) = \int_0^{V_0} f(V)dV = \frac{2}{c^2}\int_0^{V_0} Ve^{-V^2/c^2}dV = \frac{2}{c^2}\int_0^{V_0'} cy^{1/2}e^{-y}\left[\frac{1}{2}cy^{-1/2}dy\right]$$

$$= \frac{c \cdot c}{c^2}\int_0^{V_0'} e^{-y}dy = -e^{-y}\Big|_0^{V_0'} = -\left(e^{-V_0'} - e^{-0}\right) = \boxed{1 - e^{-V_0^2/c^2}}$$

- As $V_0 \to \infty$, $F(V_0) \to 1$ as required

California State University Northridge 17

Typical Problem

- Given a Rayleigh wind distribution with c = 5 m/s find the fraction of time the wind velocity is between V_1 = 3 m/s and V_2 = 7 m/s

$$P(V_1 \le V \le V_2) = F(V_2) - F(V_1) = \left[1 - e^{-V_2^2/c^2}\right] - \left[1 - e^{-V_1^2/c^2}\right]$$

$$P(V_1 \le V \le V_2) = e^{-V_1^2/c^2} - e^{-V_2^2/c^2} = e^{-(3m/s)^2/(5m/s)^2} - e^{-(8m/s)^2/(5m/s)^2} = 0.6977 - 0.5273 = 0.1704$$

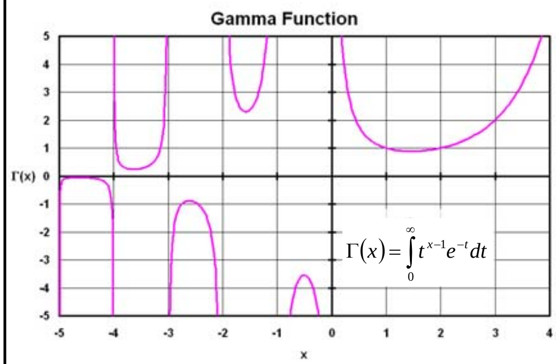
California State University Northridge 18

Gamma Function

- Used in probability integrals
- Defined as integral $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$
- For integers, $\Gamma(n) = (n-1)!$
  - $\Gamma(1) = 1$, $\Gamma(2) = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 6$
- For any argument, $\Gamma(x+1) = x\Gamma(x)$
- $\Gamma(1/2) = (\pi)^{1/2}$ **What is $\Gamma(5/2)$?**
  - $\Gamma(3/2) = (1/2)\Gamma(1/2) = (\pi)^{1/2}/2$
- Tables/software for non-integer values

$\Gamma(5/2) = (3/2)\,\Gamma(3/2) = 3(\pi)^{1/2}/4$

19



20

Mean Velocity

- Use same variable transformation
  - Define $y = V^2/2\beta^2$ so that $V = \beta(2y)^{1/2}$ and $dV = \beta(2y)^{-1/2}dy$
    - $y = 0$ when $V = 0$ and $y = \infty$ when $V = \infty$

$$\mu = \int_0^\infty V f(V) dV = \frac{1}{\sigma^2}\int_0^\infty V^2 e^{-V^2/2\beta^2} dV = \frac{1}{\beta^2}\int_0^\infty \beta^2 (2y) e^{-y}\left[\beta(2y)^{-1/2} dy\right]$$

$$= \frac{\beta^2 \cdot \beta}{\beta^2}\int_0^\infty (2y)^{1/2} e^{-y} dy = \frac{\beta}{\sqrt{2}}\int_0^\infty y^{1/2-1} e^{-y} dy = \frac{\beta}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right) = \beta\sqrt{\frac{\pi}{2}}$$

$$\boxed{\mu = c\frac{\sqrt{\pi}}{2}}$$

21

Range Near Mean

- Given a Rayleigh wind distribution with a certain value of c, find the fraction of time the wind velocity is between one half and twice the mean velocity
- Mean velocity = $c\pi^{1/2}/2$ $\quad \bar{V}^2/c^2 = \pi/4$

$$P(V_1 \le V \le V_2) = \left(1 - e^{-V_2^2/c^2}\right) - \left(1 - e^{-V_1^2/c^2}\right) = e^{-V_1^2/c^2} - e^{-V_2^2/c^2}$$

$$P(\bar{V}/2 \le V \le 2\bar{V}) = e^{-(\bar{V}/2)^2/c^2} - e^{-(2\bar{V})^2/c^2} = e^{-\pi/16} - e^{-\pi}$$

$$= .8217 - 0.0432 = 0.7785$$

22

Variance

- Use equations for σ^2 and y variable

$$\sigma^2 = \int_0^\infty V^2 p(V) dV - \mu^2 \qquad \int_0^\infty V^2 p(V) dV = \frac{1}{\beta^2}\int_0^\infty V^3 e^{-V^2/2\beta^2} dV$$

$$\frac{1}{\beta^2}\int_0^\infty V^3 e^{-V^2/2\beta^2} dV = \frac{1}{\beta^2}\int_0^\infty \beta^3 (2y)^{3/2} e^{-y}\left[\beta(2y)^{-1/2} dy\right]$$

$$= \frac{\beta^3 \cdot \beta}{\beta^2}\int_0^\infty 2y e^{-y} dy = 2\beta^2 \int_0^\infty y^{2-1} e^{-y} dy = 2\beta^2 \Gamma(2) = 2\beta^2$$

$$\sigma^2 = \int_0^\infty V^2 p(V) dV - \mu^2 = 2\beta^2 - \left[\beta\sqrt{\frac{\pi}{2}}\right]^2 = \beta^2\left(2 - \frac{\pi}{2}\right)$$

$$\boxed{\sigma^2 = c^2\left(1 - \frac{\pi}{4}\right)}$$

23

Calculating c from Wind Data

- For normal distribution the arithmetic mean, $\bar{x}$, is an estimate of the true mean, μ
  - Maximum Likelihood Estimator (MLE)

$$\bar{x} = \frac{1}{N}\sum_{i=1}^N x_i \qquad s^2 = \frac{1}{N-1}\sum_{i=1}^N (x_i - \bar{x})^2 \; MLE \; for \; \sigma^2$$

- For the Rayleigh distribution the MLE for the parameter c is given by

$$\hat{c} = \sqrt{\frac{1}{N}\sum_{i=1}^N V_i^2}$$

24

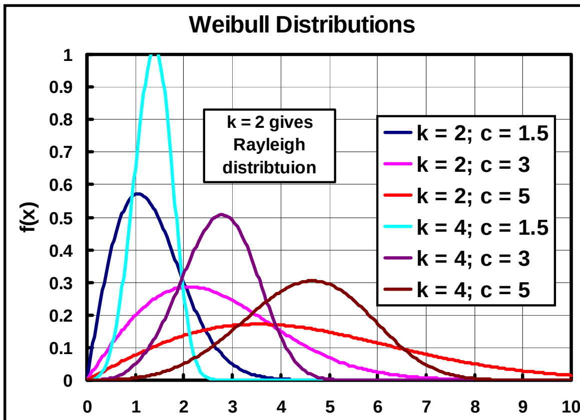
Weibull Distribution

- A two-parameter distribution with shape parameter, k, and scale parameter, c
- Rayleigh distribution is Weibull distribution with k = 2
- Mean = $c\Gamma(1 + k^{-1})$
- Variance = $c^2[\Gamma(1 + 2k^{-1}) - \Gamma^2(1 + k^{-1})]$

Γ is the gamma function

$$f(V) = \frac{k}{c}\left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} \qquad 0 \le V < \infty$$

California State University Northridge 25



k = shape parameter c = scale parameter

Weibull Parameters from Data

- First solve equation for k estimator by trial and error (use k = 2 for initial guess)
- Then solve equation for c estimator

$$\hat{k} = \frac{1}{\dfrac{\sum_{i=1}^{N} V_i^{\hat{k}} \ln(V_i)}{\sum_{i=1}^{N} V_i^{\hat{k}}} - \dfrac{1}{N}\sum_{i=1}^{N}\ln(V_i)} \qquad \hat{c} = \sqrt[\hat{k}]{\frac{1}{N}\sum_{i=1}^{N} V_i^{\hat{k}}}$$

- Function wblfit in MATLAB finds k and c

California State University Northridge 27

Wind Power

- Instantaneous wind power: $P_0 = \rho V^3 A/2$
- Mean wind power: $\overline{P_0} = \frac{\rho A \overline{V^3}}{2} = \frac{\rho A}{2}\int_0^\infty V^3 f(V)\,dV$
- Using Weibull distribution for f(V)
  - $y = (V/c)^k$; $V = cy^{1/k}$; $dV = (c/k)y^{(1-k)/k}$

$$\overline{V^3} = \int_0^\infty V^3 \frac{k}{c}\left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV = k\int_0^\infty V^2 \left(\frac{V}{c}\right)^{k} e^{-(V/c)^k} dV$$

$$= k\int_0^\infty \left(cy^{\frac{1}{k}}\right)^2 ye^{-y}\frac{c}{k}y^{\frac{1-k}{k}}dy = c^3\int_0^\infty y^{\frac{3}{k}}e^{-y}dy = c^3\Gamma\left(\frac{3}{k}+1\right)$$

California State University Northridge 28

Wind Power II

- Average wind power
  - Weibull $\overline{P_0} = \frac{\rho A \overline{V^3}}{2} = \frac{\rho A c^3}{2}\Gamma\left(\frac{3}{k}+1\right)$
  - Rayleigh $\overline{P_0} = \frac{\rho A \overline{V^3}}{2} = \frac{\rho A c^3}{2}\Gamma\left(\frac{3}{2}+1\right) = \rho A c^3 \frac{3\sqrt{\pi}}{8}$
  - Distribution equation (details next slide)

$$\overline{P} \text{ between } V_1 \text{ and } V_2 = \frac{\rho A c^3}{2}\int_{(V_1/c)^k}^{(V_2/c)^k} y^{\frac{3}{k}} e^{-y} dy$$

California State University Northridge 29

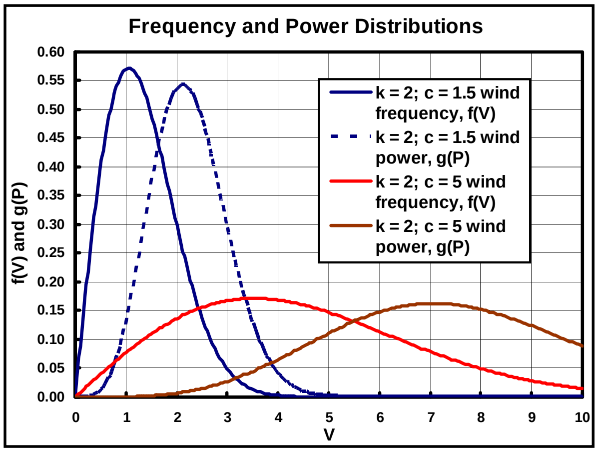
Wind Power Distribution

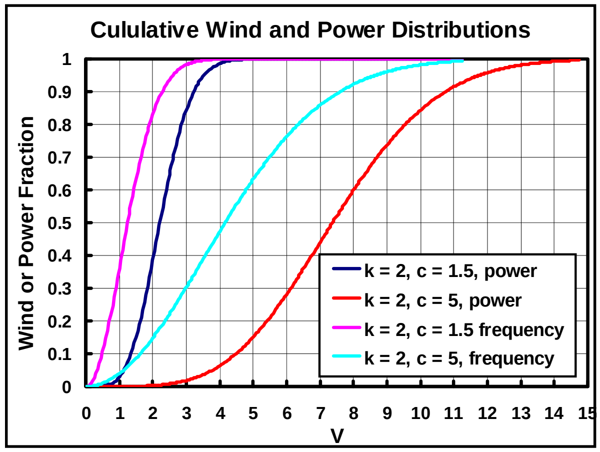
- Wind power in a differential range, dV about a velocity V is $(\rho V^3 A/2)f(V)dV$
- Average wind power in a velocity range between V_1 and V_2 is found as follows
  - $y = (V/c)^k$; $V/c = y^{1/k}$; $d(V/c) = (1/k)y^{(1-k)/k}dy$

$$\frac{\rho A}{2}\int_{V=V_1}^{V=V_2} V^3 \frac{k}{c}\left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV = \frac{\rho A}{2}k\int_{V=V_1}^{V=V_2} V^2\left(\frac{V}{c}\right)^{k} e^{-(V/c)^k} dV = \frac{\rho A}{2}k\int_{V=V_1}^{V=V_2}\left[c^2\left(\frac{V}{c}\right)^2 \bullet\right.$$

$$\left.\left(\frac{V}{c}\right)^k e^{-(V/c)^k} c\,d\left(\frac{V}{c}\right)\right]dV = \frac{\rho A c^3}{2}k\int_{y=(V_1/c)^k}^{y=(V_2/c)^k} y^{\frac{2}{k}} ye^{-y}\frac{1}{k}y^{\frac{1}{k}-1}dy = \frac{\rho A c^3}{2}\int_{y=(V_1/c)^k}^{y=(V_2/c)^k} y^{\frac{3}{k}}e^{-y}dy$$

California State University Northridge 30





Difference Between $\bar{V}^3$ and $\overline{V^3}$

- Difference in order of operations
  - Take average then cube it – $\bar{V}^3$
  - Cube each data point then average – $\overline{V^3}$
  - Example: $V_1 = 1$, $V_2 = 2$, and $V_3 = 3$
    - Average V = (1 + 2 + 3) / 3 = 2; $\bar{V}^3 = 8$
    - Average V^3 = (1 + 8 + 27)/3 = 12 = $\overline{V^3}$
- For Rayleigh distribution

$$\overline{V^3} = c^3\frac{3\sqrt{\pi}}{4} \qquad \bar{V}^3 = \mu^3 = \left(\frac{c\sqrt{\pi}}{2}\right)^3 \Rightarrow \overline{V^3} = \frac{6}{\pi}\bar{V}^3$$

California State University Northridge

33

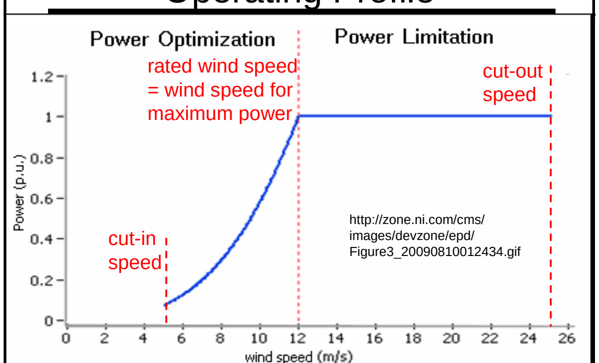
Wind Turbine Operation

- No operation until wind velocity reaches a minimum called the cut-in velocity
- Then operate at full turbine output power until turbine output is greater than generator can accept
- Limit turbine output power to full generator power at high wind speeds
- No operation above maximum velocity called cut-out velocity

California State University Northridge

34

Operating Profile



Average Operating Power

- Generator accepts all turbine power between $V_{cut\text{-}in}$ and maximum power (P_{max}) velocity, $V_{Pmax} = [2P_{max}/(c_p\rho A)]^{1/3}$
  - Power coefficient c_p = generator power divided by wind power
- Between V_{Pmax} and $V_{cut\text{-}out}$ operate at maximum power

$$\bar{P}_{operation} = \int_{V_{cut-in}}^{V_{P\max}} \frac{c_p \rho A V^3}{2} f(V)dV + \int_{V_{P\max}}^{V_{cut-out}} P_{\max} f(V)dV$$

California State University Northridge

36

Average Operating Power II

- Modify integrals from last slide
  - Previous result for power between V_1 and V_2

$$\int_{V_{cut-in}}^{V_{P\max}} \frac{c_p\rho A V^3}{2} f(V)dV = \frac{c_p \rho A c^3}{2}\int_{y=(V_{cut-in}/c)^k}^{y=(V_{P\max}/c)^k} y^{\frac{3}{k}} e^{-y} dy$$

  - Cumulative distribution function constant P_{max}

$$\int_{V_{P\max}}^{V_{cut-out}} P_{\max} f(V)dV = P_{\max}\left[\left(1-e^{-(V_{cut-out}/c)^k}\right)-\left(1-e^{-(V_{P\max}/c)^k}\right)\right]$$

$$\overline{P}_{operation} = \frac{c_p\rho A c^3}{2}\int_{(V_{cut-in}/c)^k}^{(V_{P\max}/c)^k} y^{\frac{3}{k}} e^{-y}dy + P_{\max}\left(e^{-(V_{P\max}/c)^k} - e^{-(V_{cut-out}/c)^k}\right)$$

California State University Northridge

37

Excel Wind Power Tool

- Excel workbook on course web site
- Has various VBA functions for Weibull distributions
- Can compute average annual power for inputs of k, c, c_p, r, and A
  - Can enter area or diameter
  - Two sets of units possible

$$\overline{P}_{operation} = \frac{c_p\rho A c^3}{2}\int_{(V_{cut-in}/c)^k}^{(V_{P\max}/c)^k} y^{\frac{3}{k}} e^{-y}dy + P_{\max}\left(e^{-(V_{P\max}/c)^k} - e^{-(V_{cut-out}/c)^k}\right)$$

California State University Northridge

38