Section 2.8: The Existence and Uniqueness Theorem

MATH 351

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Theorem 2.4.2 in Section 2.4

Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \tag{1}$$

- In some cases (e.g., if the DE is linear) the existence of a sol of the IVP can be established directly by actually solving the problem and exhibiting a formula for the solution.
- In general, this approach is not feasible.

For the general case, we adopt an indirect approach:

- The approach demonstrates the existence of a sol of IVP,
- The approach usually does not provide a practical means of finding it,
- In this method, we will construct a sequence of functions that converges to a limit function satisfying the IVP, although the members of the sequence individually do not.

The Initial Value Problem with I.C. $y(0) = 0$

It is sufficient to consider the problem in which the initial point (t_0, y_0) is the origin; that is, we consider the problem

$$y' = f(t, y), \quad y(0) = 0 \tag{2}$$

This is because we can always make a preliminary change of variables, corresponding to a translation of the coordinate axes, that will take the given point (t_0, y_0) into the origin.

Theorem 2.8.1

If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(0) = 0 \tag{3}$$

Sketch of the Proof

Step 1: To transform the initial vale problem (3) into a more convenient form (an **integral equation**):

Note that

- This integral equation is not a formula for the sol of the IVP.
- It provides another relation satisfied by any solution of IVP.
- The IVP and the integral equation are equivalent in the sense that any solution of one is also a solution of the other.
- It is more convenient to show that there is a unique solution of the integral equation in a certain interval $|t| \leq h$. (The same conclusion will then hold also for the IVP.)

Step 2:

To establish Theorem 2.8.1, we must answer four principal questions:

1. Do all members of the sequence $\{\phi_n\}$ exist, or may the process break down at some stage?
2. Dose the sequence converge?
3. What are the properties of the limit function? In particular, does it satisfy the integral equation and hence the initial value problem (3)?
4. Is this the only solution, or may there be others?

To answer these questions first in one example:

Example 1: Solve the IVP

$$y' = 2t(1+y), \quad y(0) = 0$$

by the method of successive approximations.

Solution:

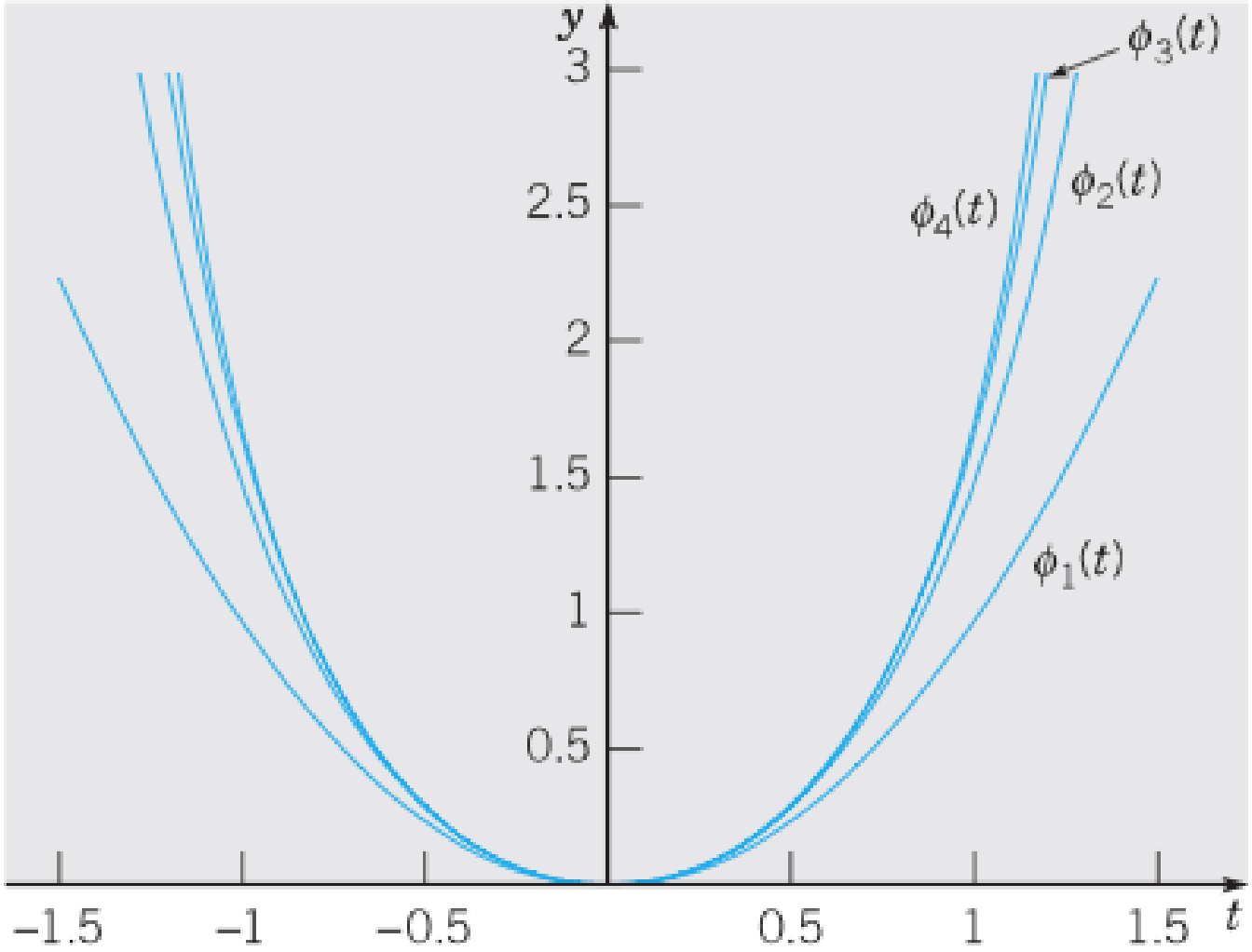

FIGURE 2.8.1 Plots of the first four Picard iterates $\phi_1(t), \ldots, \phi_4(t)$ for Example 1: $dy/dt = 2t(1+y)$, $y(0) = 0$.

The convergence of the sequence:

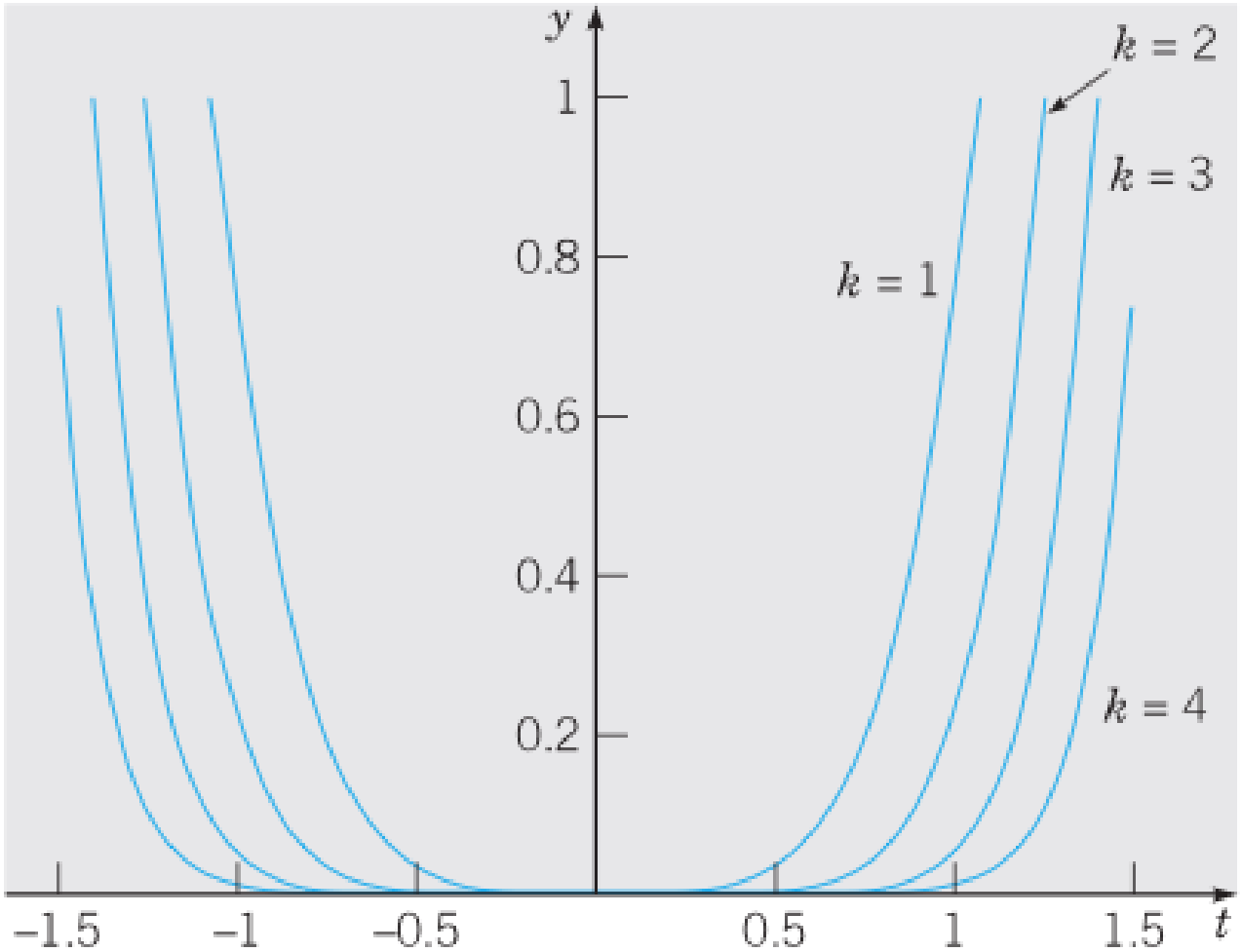

FIGURE 2.8.2 Plots of $\phi(t) - \phi_k(t)$ for Example 1 for $k = 1, \ldots, 4$.

Remark: This figure clearly shows that the gradually increasing interval over which successive iterates provide a good approximation to the solution of the IVP.

The question of uniqueness

Let us suppose that the IVP has two **different** solutions ϕ and ψ. Then both should satisfy the integral equation, that is,

$$\phi = \int_0^t 2s\phi(s)ds$$

and

$$\psi = \int_0^t 2s\psi(s)ds$$

Returning to the general problem of solving the integral equation,

1. Do all members of the sequence $\{\phi_n\}$ exist?

Note that:

- f and $\partial f/\partial y$ are continuous only in the rectangle $R : |t| \leq a, |y| \leq b$ (see the figure next page.)
- the members of the sequence cannot as a rule be explicitly determined.
- at some stage, say, for $n = k$, the graph of $y = \phi_k(t)$ may contain points that lie outside the rectangle R. (in other words, the computation of $\phi_{k+1}(t)$ might be impossible.)

Now to avoid this danger, it may be necessary to **restrict** t to a smaller interval than $|t| \leq a$.

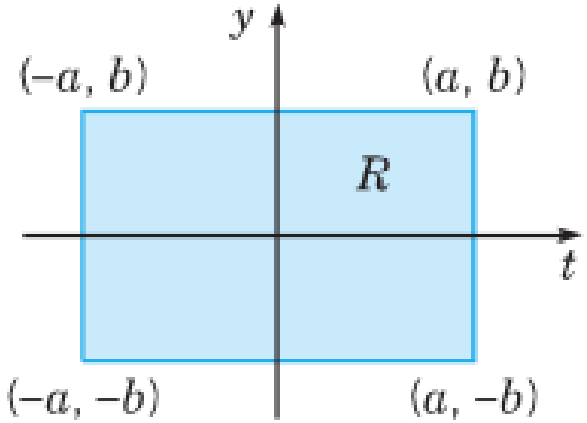

FIGURE 2.8.3 Region of definition for Theorem 2.8.1.

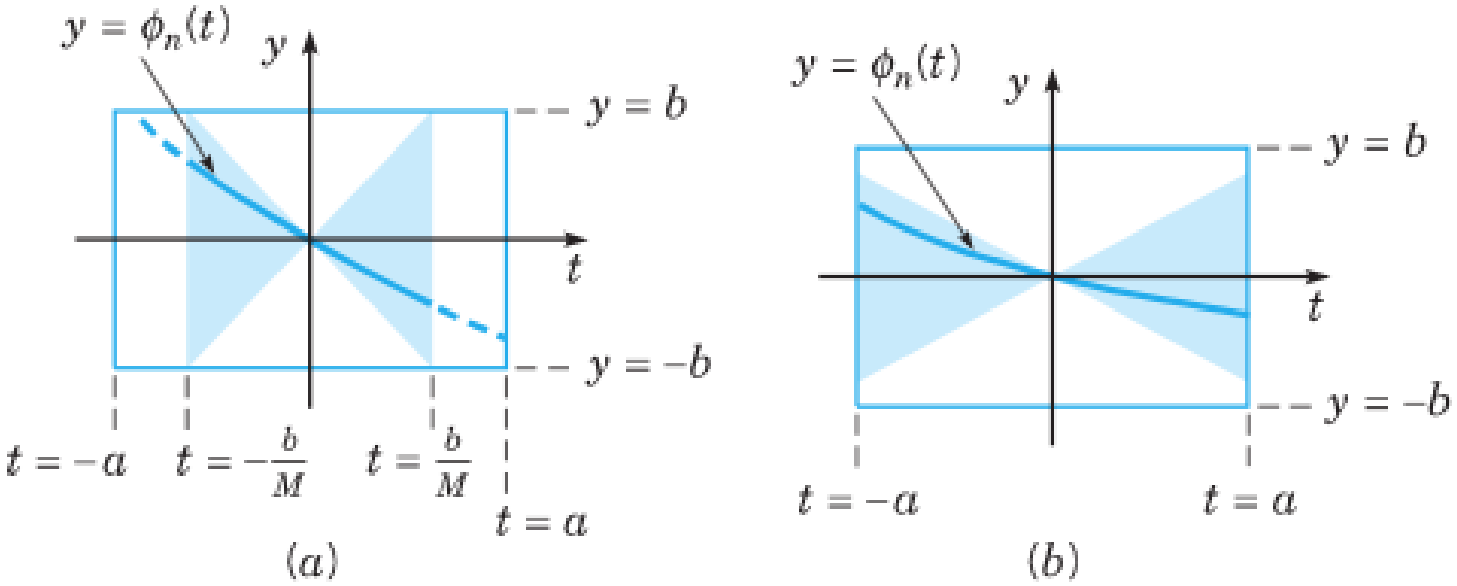

FIGURE 2.8.4 Regions in which successive iterates lie. (*a*) $b/M < a$; (*b*) $b/M > a$.

2 Does the sequence $\{\phi_n(t)\}$ converge?

Remark:

- The convergence of the sequence $\{\phi_n(t)\}$ is established by showing that the series in the above equation converges.
- To do this, it is necessary to estimate the magnitude $|\phi_{k+1}(t) - \phi_k(t)|$ of the general term.
- The argument by which this is done indicated in Problems 15 through 18 in the problems of Section 2.8.

In Problems 15 through 18, steps to prove that the sequence $\{\phi_n(t)\}$ converges.

15 If $\partial f/\partial y$ is continuous in the rectangle D, show that there is a positive constant K such that

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|,$$

where (t, y_1) and (t, y_2) are any two points in D having the same t coordinate. This inequality is known as a **Lipschitz condition**.

Proof:

16 If $\phi_{n-1}(t)$ and $\phi_n(t)$ are members of the sequence $\{\phi_n(t)\}$, use the result of Problem 15 to show that

$$f[t, \phi_n(t)] - f[t, \phi_{n-1}(t)] \leq K|\phi_n(t) - \phi_{n-1}(t)|.$$

Proof:

17 (a) Show that if $|t| \leq h$, then

$$|\phi_1(t)| \leq M|t|,$$

where M is chosen so that $|f(t, y)| \leq M$ for (t, y) in D.

Proof:

(b) Use the results of Problem 16 and part (a) of Problem 17 to show that

$$|\phi_2(t) - \phi_1(t)| \leq \frac{MK|t|^2}{2}.$$

Proof:

(c) Show, by mathematical induction, that

$$|\phi_n(t) - \phi_{n-1}(t)| \leq \frac{MK^{n-1}|t|^n}{n!} \leq \frac{MK^{n-1}h^n}{n!}.$$

Proof:

18 Note that

$$\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)].$$

(a) Show that

$$|\phi_n(t)| \leq \phi_1(t)| + |\phi_2(t) - \phi_1(t)| + \cdots + |\phi_n(t) - \phi_{n-1}(t)|.$$

Proof:

(b) Use the results of Problem 17 to show that

$$|\phi_n(t)| \leq \frac{M}{K}\left[(Kh) + \frac{(Kh)^2}{2!} + \cdots + \frac{(Kh)^n}{n!}\right]$$

.

Proof:

(c) show that the sum in part (b) converges as $n \to \infty$ and, hence, the sum in part (a) also converges as $n \to \infty$. Conclude therefore that the sequence $\{\phi_n(t)\}$ converges since it is the sequence of partial sums of a convergent infinite series.

Proof:

The second question is answered.

Now we assume that the sequence converges, we denote the limit function by ϕ, so that

$$\phi(t) = \lim_{n\to\infty} \phi_n(t).$$

3 What are the properties of the limit function ϕ?

- In the first place, we would like to know that ϕ is continuous.

- ϕ satisfies the integral equation. (Thus ϕ is also a sol of the IVP (3).)

4 Are there other solutions of the integral equation besides $y = \phi(t)$?

Proof: To show the uniqueness of the solution $y = \phi(t)$, we can proceed much as in the example.

First, assume the existence of another solution $y = \psi(t)$. It is then possible to show (see Problem 19) that the difference $\phi(t) - \psi(t)$ satisfies the inequality

$$|\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)| ds$$

for $0 \leq t \leq h$ and a suitable positive number A.

Similar to what we did in the example, we can then conclude that there is no solution of the IVP other than the one generated by the method of successive approximations.

19 In the problem we deal with the question of uniqueness of the solution of the integral equation

$$\phi(t) = \int_0^t f[s, \phi(s)]ds.$$

(a) Suppose that ϕ and ψ are two solutions of the above integral equation. Show that, for $t \geq 0$,

$$\phi(t) - \psi(t) = \int_0^t \{f[s, \phi(s)] - f[s, \psi(s)]\}\, ds.$$

(b) Show that

$$|\phi(t) - \psi(t)| \leq \int_0^t |f[s, \phi(s)] - f[s, \psi(s)]|ds.$$

(c) Use the result of Problem 15 to show that

$$|\phi(t) - \psi(t)| \leq K \int_0^t |\phi(s) - \psi(s)| ds$$

where K is an upper bound for $|\partial f / \partial y|$ in D. This is the same as the one in the example, and the rest of the proof may be constructed as the example as well.

Proof: