MATH 581 **Numerical Methods for Linear Systems** **Fall 2011**

Review Sheet for Midterm Exam

Problem 1. Determine whether the following statements are True or False. Justify your answer.

(a) For $A \in \mathbb{R}^{m\times n}$, $\text{range}(A) = \text{range}(A^\top A)$

(b) If the vectors a_1, a_2 and b are orthogonal, then the projection of b onto the plane containing a_1 and a_2 equals $\|b\|_2$

(c) $(AB)^+ = B^+A^+$

(d) The system $Ax = b$ with

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -3 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

has unique *least squares* solution.

(e) If $q_1, q_2, \dots q_k$ are orthogonal vectors in $\mathbb{R}^n$, and $v \in \mathbb{R}^n$ then

$$r = v - \sum_{i=1}^{k} (v^\top q_i)\, q_i,$$

is orthogonal to q_i, $i = 1, 2, \dots, k$.

(f) If $AA^\top = A^\top A$, then $\|Ax\|_2 = \|A^\top x\|_2$

(g) Given a vector $\mathbf{u} \in \mathbb{R}^n$, the trace of the matrix $P = \frac{\mathbf{u}\mathbf{u}^\top}{\mathbf{u}^\top\mathbf{u}}$ always equals 1

Problem 2. Suppose the singular value decomposition of $A \in \mathbb{R}^{m\times n}$, $m \geq n$, is $A = U\Sigma V^\top$, and denote by A^+ the pseudoinverse of A

(a) Show that the SVD of A^+ is $A^+ = V\Sigma^+U^\top$

(b) Describe Σ^+ (*i.e.*, size and entries $\sigma^+_{i,j}$)

(c) What is the pseudoinverse of A^+?

(d) Show that AA^+ is a projector. What subspace does AA^+ projects onto?

Problem 3. Find the projection matrix P onto the space spanned by $\mathbf{u} = (1, 0, 1)^\top$ and $\mathbf{v} = (1, 1, -1)^\top$

Problem 4. What multiple of $a = (1, 1, 1)$ is closest to the point $b = (2, 4, 4)$?

Problem 5. In n dimensions, what angle does the vector $(1, 1, 1, \dots, 1)^\top$ make with the coordinate axis? What is the projection matrix P onto that vector?

Problem 6. Suppose P is the projection matrix onto the subspace S and Q the projection matrix onto the orthogonal complement $S^\perp$

(a) What is $P+Q$?

(b) What is PQ?

(c) Show that $P-Q$ is its own inverse

Problem 7. Find the matrix that projects every point in the plane onto the line $x+3y=0$.

Problem 8. If the vectors a_1, a_2 and b are orthogonal, what are $A^\top A$ and $A^\top b$?

Problem 9. If V is the subspace spanned by $(1,1,0,1)^\top$ and $(0,0,1,0)^\top$, find

(a) a basis for the orthogonal complement $V^\perp$

(b) the projection matrix P onto V

(c) the vector in V closest to the vector $b=(0,1,0,-1)^\top$ in $V^\perp$

Problem 10. Let $a_1=(\frac{2}{3},\frac{2}{3},-\frac{1}{3})^\top$, $a_2=(-\frac{1}{3},\frac{2}{3},\frac{2}{3})^\top$, $a_3=(\frac{2}{3},-\frac{1}{3},\frac{2}{3})^\top$, and $b=(0,3,0)^\top$

(a) Project b onto a_1 and a_2 and find its projection onto the plane containing a_1 and a_2.

(b) Find the projection of b onto a_3, then add up the three one-dimensional projections. Why is $P=a_1a_1^\top+a_2a_2^\top+a_3a_3^\top=I$?

Problem 11. Let

$$A=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

(a) Compute $\|A^{-1}\|_2=\frac{1}{\lambda_1}$, $\|A\|_2=\lambda_2$, $\kappa(A)=\frac{\lambda_2}{\lambda_1}$

(b) Find a right side b and a perturbation δb so that the error, $\frac{\|\delta x\|}{\|x\|}=\kappa\frac{\|\delta b\|}{\|b\|}$, is largest

Problem 12. Give an asymptotic estimate of the number of *flops* ($+$, $-$, $\times$, and $\div$) in algorithms 10.1, 10.2, and 10.3 in the textbook.