MATH 255 **Applied Honors Calculus III** **Winter 2005**

Final Exam Review Sheet – Part I

1. True or False. Determine whether the following statements are true or false and provide an explanation for your answer (5 points each).

(a)

$$\int_0^1 \int_{-1}^1 e^{x^2+y^2} \sin x \, dx \, dy = 0$$

(b)

$$\int_0^9 \int_0^1 (x^2 + \sqrt{y}) \, \sin(x^2 y^2) \, dx \, dy \le 36$$

(c) The function $f(x,y,z) = e^x \sin y + z$ satisfies Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

(d) If D is the disk given by $x^2 + y^2 \le 4$, then

$$\iint_D \sqrt{4 - x^2 - y^2} \, dx \, dy = \frac{16\pi}{3}.$$

(e) If C is a smooth simple closed curve in the xy-plane, then the area of the domain inside C is given by

$$\oint_C x \, dy$$

where the integration is counterclockwise.

(f) The function

$$f(x,y) = \begin{cases} \frac{x^2+\cos^2 y}{2(x^2+y^2)} & \text{if} \quad (x,y) \ne (0,0) \\ \frac{1}{2} & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

(g) The parametric equations

$$r(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle, \qquad 0 \le \phi \le \frac{\pi}{4}, \qquad 0 \le \theta \le 2\pi$$

describe the portion of the unit sphere centered at the origin inside the cylinder $x^2 + y^2 = 1$.

(h)

$$\int_{-2}^2 \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} dy \, dx = 2\pi$$

2. (25 points) Consider the two dimensional potential fluid flow with velocity field,

$$\mathbf{v}(x,y) = \langle v_1(x,y), v_2(x,y)\rangle = \nabla\Phi(x,y)$$

where $\Phi(x,y)$ is a smooth scalar function.

(a) Show that $\nabla \times \mathbf{v} = 0$ for any choice of the function $\Phi(x,y)$, and explain why this implies that the circulation, $\oint_C \mathbf{v}\cdot d\mathbf{r}$, is equal to zero for *any* simple closed curve C.

(b) A fluid is called *incompressible* if $\nabla\cdot\mathbf{v} = 0$. Show that in that case the velocity potential satisfies laplace's equation,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0.$$

(c) Compute the velocity field of the potential flow with $\Phi(x,y) = x^2 - y^2$, and show that the fluid is incompressible.

(d) Consider the motion of a fluid particle in the velocity field from part (c). Let $\mathbf{r}(t)$ be the position of the particle at time t, and $\mathbf{r}(0) = \langle 2, 3\rangle$. Write the system of differential equations describing the position of the particle. Show that the fluid particle is moving on the hyperbola $xy = 6$.

3. (15 points) Compute the flux, $\iint_{\partial E}\mathbf{F}\cdot d\mathbf{S}$, of the vector field

$$\mathbf{F}(x,y,z) = \frac{x^3}{3}\mathbf{i} + \frac{y^3}{3}\mathbf{j} + 4\mathbf{k}$$

through the boundary ∂E of the domain E bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

4. (15 points) Let S be a simple closed surface with outward unit normal $\mathbf{n}$, and let $\mathbf{r} = \langle x, y, z\rangle$ be a parametric representation of S.

(a) Show that the volume of the solid contained in the surface S is given by the integral

$$V = \frac{1}{3}\iint_S \mathbf{r}\cdot\mathbf{n}\, dS.$$

(b) Calculate the volume of a cone of height H and base area A by putting the vertex of the cone at the origin and using the formula found in part (a).

5. (20 points) Let the vector field $\mathbf{F}$ in $\mathbb{R}^3$ be defined by

$$\mathbf{F}(x,y,z) = \langle z^2, 2yz, 2xz + y^2\rangle.$$

(a) Show that the vector field $\mathbf{F}$ is conservative, and find a scalar function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

(b) If $\mathbf{F}$, as given above, is the force acting on a particle, find the work of this force in moving the particle from the initial ($t = 0$) to the final ($t = 1$) points of the curve

$$\mathbf{r}(t) = \left\langle \sqrt{t}, 3^t - 1, 3\sin\frac{2t}{\pi}\right\rangle, \quad t\in[0,1]$$

6. (20 points) One way to approximate a function $f(x)$ consist on replacing the function by its tangent line at some points of interest. this type of approximation, however, works well only near that point, and can be very inaccurate over an entire *interval*.

One way to approximate a function $f(x)$ by a linear function $g(x) = ax + b$, over the interval $[x_0, x_1]$ is to choose values of the constants a and b that minimize the value of the "error" given by the integral

$$\int_{x_0}^{x_1} \big[f(x) - (ax + b)\big]^2 \, dx.$$

Determine the constants a and b that minimize the error for the function $f(x) = x^2$ over the interval $[0, 1]$.

Hint: You may find the following formula useful: $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma$.