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Territorial developments based on graffiti: A statistical mechanics approach

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ABSTRACT

We study the well-known sociological phenomenon of gang aggregation and territory formation through an interacting agent system defined on a lattice. We introduce a two-gang Hamiltonian model where agents have red or blue affiliation but are otherwise indistinguishable. In this model, all interactions are indirect and occur only via graffiti markings, on-site as well as on nearest neighbor locations. We also allow for gang proliferation and graffiti suppression. Within the context of this model, we show that gang clustering and territory formation may arise under specific parameter choices and that a phase transition may occur between well-mixed, possibly dilute configurations and well separated, clustered ones. Using methods from statistical mechanics, we study the phase transition between these two qualitatively different scenarios. In the mean-fields rendition of this model, we identify parameter regimes where the transition is first or second order. In all cases, we have found that the transitions are a consequence solely of the gang to graffiti couplings, implying that direct gang to gang interactions are not strictly necessary for gang territory formation; in particular, graffiti may be the sole driving force behind gang clustering. We further discuss possible sociological—as well as ecological—ramifications of our results.

1. Introduction

Lattice models have been extensively used in the physical sciences over the past decades to describe a wide variety of condensed matter equilibrium and non equilibrium phenomena, see e.g., the reviews in Ref. [1–3]. Magnetization was the original application, but the list has grown to include structural transitions in DNA [4–6], polymer coiling [7,8], cellular automata [9,10], and gene regulation [11–13] to name a few. The resulting models are certainly simplified, but what they lack in detail is compensated by their amenability to analytical and computational treatment—and, occasionally, to exact solution. Moreover, at least for the behavior in the vicinity of a continuous transition, the simplifications inherent in these approximate models may be presumed to be inconsequential. In short, lattice models have proved extremely useful in the context of the physical, biological and even chemical sciences. In more recent years, lattice models have also been applied to study social phenomena [14–16], such as racial segregation [17,18], voter preferences [19–21], opinion formation in financial markets [22–24], and language changes in society [25–27], offering insight into socioeconomic dynamics and equilibria. In this paper we consider the problem of gang aggregation via graffiti in what is—to the best of our knowledge—the first application of lattice model results to the emergence of gang territoriality.

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Scratching words or painting images on visible surfaces is certainly not a new phenomenon. Wall scribblings have survived from ancient times and have been used to reconstruct historical events and to understand societal attitudes and values. Today, graffiti (from the Italian *graffiare*, to scratch) is a pervasive characteristic of all metropolitan areas [28]. Several types of graffiti exist. Some are political in nature, expressing activist views against the current establishment; others are expressive or offensive manifestations on love, sex or race. At times, the graffiti is a mark of one's passage through a certain area, with prestige being attributed to the most prolific or creative tagger or to one who is able to reach inaccessible locations. The mark can be anything from a simple signature to a more elaborate decorative aerosol painting [29,30]. All of these types of graffiti are usually scattered around the urban landscape and do not appear to follow any predetermined spatio-temporal pattern of evolution. They affect the quality of life simply as random defacement of property, although sometimes they are considered art [31].

On the other hand, *gang* graffiti represents a much more serious threat to the public, since it is usually a sign of the presence of criminal gangs engaged in illegal or underground activities such as drug trafficking or extortion [32,33]. Street gangs are extremely territorial, and aim to preserve economic interests and spheres of influence within the neighborhoods they control. A gang's "turf" is usually marked in a characteristic style, recognizable to members and antagonists [34,35] with incursions by enemies often resulting in violent acts. The established boundaries between different gang factions are sometimes respected peacefully, but more often become contested locations where it is not uncommon for murders and assaults to occur [36]. It is here, on the boundaries between gang turfs, that the most intense graffiti activity is usually concentrated.

Several criminological and geographical studies have been presented connecting gang graffiti and territoriality in American cities [31,30,35]. In particular, it is now considered well-established that the spatial extent of a gang's area of influence is strongly correlated to the spatial extent of that particular gang's graffiti style or language. Furthermore, it is known that the incidence of gang graffiti may change in time, reflecting specific occurrences or neighborhood changes. For example, rival gangs may alternate between periods of truce and hostility, the latter being triggered by arrests or shootings. Similarly, boundaries may shift locations when the racial or socio-economic makeup of a neighborhood changes, creating new tensions, or when gang members migrate to new communities [28]. In all these cases, periods of more intense gang hostility are usually accompanied by intense graffiti marking and erasing by rival factions in contested or newly settled boundary zones [35].

The purpose of this paper is to present a mathematical model that includes relevant sociological and geographical information relating gang graffiti to gang activity. In particular, we study the segregation of individuals into well defined gang clusters as driven by gang graffiti, and the creation of boundaries between rival gangs. We use a spin system akin to a 2D lattice Ising model to formulate our problem through the language of statistical mechanics. In this context, the site variables s_i have two constituents which represent 'gang' and 'graffiti' types, respectively, and *phase separation* is assumed to be the proxy for gang clustering. For the purpose of simplicity, we consider only two gangs, hereafter referred to as the red and blue gang, whose members we refer to as *agents*. Lattice sites may be occupied by agents of either color or be void. Since gang members are assumed to tag their territory with graffiti of their same color, we also assign a graffiti index to each site representing the preponderance of red or blue markings.

In particular, agents are attracted to sites with graffiti of their same color, and avoid locations marked by their opponents. We deliberately avoid including direct interactions between gang members, so that "ferromagnetic" type gang–gang attractions exist only insofar as they are mediated by the graffiti. On one hand this is mathematically interesting: in the broader context of physical systems, interactions are often mediated but rarely are indirect interactions the subject of mathematical analysis. On the other hand, by excluding direct gang interactions, we can specifically focus on the role of graffiti in gang dynamics and segregation. Furthermore, as will be later discussed, under certain conditions, gang–gang couplings may be unimportant, and one of the primary conclusions of this work is that they appear to be unnecessary to account for the observed phenomena of gang segregation. In any case, we informally state without proof that all the results of this work also hold if explicit agent–agent interactions are included.

We thus write $s_i = (\eta_i, g_i)$, representing the agents and graffiti configuration at site i, respectively. The former component η_i is discrete allowing, for simplicity, at most one agent on each site. The latter g_i is continuous and, in principle, unbounded. We let $\mathbf{s}$ denote a spin configuration on the entire lattice, and in Section 2, propose a Hamiltonian, $\mathscr{H}(\mathbf{s})$, to embody all relevant sociological information. Once $\mathscr{H}(\mathbf{s})$ has been determined, the probability for the occurrence of a spin configuration $\mathbf{s}$ on a finite connected lattice $\Lambda \subset \mathbb{Z}^2$ is determined by the corresponding Gibbs distribution $\mathbb{F}(\mathbf{s})$. Note that due to the choices made on the range of the η_i, g_i values, $\mathbb{F}(\mathbf{s})$ is discrete in the η variables and continuous in the g ones. It is given by

$$\mathbb{F}(\mathbf{s}) = \frac{1}{\mathcal{Z}} \exp(-\mathscr{H}(\mathbf{s})),$$

where $\mathcal{Z}$ is the partition function for the finite lattice Λ formally provided by the expression

$$\mathcal{Z} = \sum_{\mathbf{s}\in\mathbb{S}} \exp(-\mathscr{H}(\mathbf{s})).$$

Here, $\mathbb{S}$ denotes the set of all possible configurations on Λ and the summation symbol is understood to be a summation over the discrete components and an integration over the continuous ones. As usual, we begin with a finite lattice and its associated boundary conditions, and obtain infinite volume results by taking the appropriate limits. Using techniques from

statistical mechanics, we prove that our system undergoes a phase transition as the coupling parameters are varied. In the unconstrained ensemble, certain parameter choices lead to predominance of either the red or blue gang, indicating that for configurations where the red to blue gang ratio is fixed at unity, a phase separation will occur. Conversely, in other regions of parameter space, there is no dominance of either gang type, indicating that the two are well-mixed and/or dilute. In this work we will investigate under which conditions to expect phase separation or gang dilution.

Our paper is organized as follows: in Section 2, we give details of the model and in Section 3, we prove that a phase transition exists as a function of the relevant parameters. Since information on the location of *all* transition points is, by necessity, incomplete we consider an approximation in the form of a simplified mean field version of our Hamiltonian and derive the corresponding mean field equations in Section 4. Here, we show that the mean field Hamiltonian also exhibits a phase transition and we further prove that the latter is continuous in one specified region of parameter space and first order in another. Finally, in Section 5 we end with a discussion of potential sociological and ecological implications of our results.

2. The Hamiltonian

Let us define a spin system on a finite lattice $\Lambda \subset \mathbb{Z}^2$. Here, the spin at each site $i \in \Lambda$ is denoted by $s_i = (\eta_i, g_i)$ and, we reiterate, η_i denotes the *agent spin* and g_i represents the *graffiti field*. We allow the agent spin to be in the set $\{0, \pm 1\}$; $\eta_i = -1$ if the agent at site i belongs to the blue gang, $\eta_i = 0$ if there is no agent, and $\eta_i = +1$ if the agent is a red gang member. The graffiti field is in the set of real numbers: $g_i > 0$ indicates an excess of red graffiti, $g_i < 0$ an excess of blue graffiti, and, in either case, $|g_i|$ indicates the magnitude of the excess. We now introduce the formal Hamiltonian $\mathscr{H}(\mathbf{s})$

$$-\mathscr{H}(\mathbf{s}) = J \sum_{\langle i,j \rangle} \eta_i g_j + K \sum_i \eta_i g_i + \alpha \sum_i \eta_i^2 - \lambda \sum_i g_i^2, \tag{1}$$

where $\mathbf{s}$ is a given configuration on the full Λ lattice, i and j index its sites and $\sum_{\langle i,j \rangle}$ is the sum taken over every bond between nearest neighbor sites belonging to Λ. We discuss the role of spins on the lattice boundary Λ^c in Proposition 3.4 and the following sections. The expression in (1) will be referred to as the GI–Hamiltonian (graffiti interaction Hamiltonian) and its corresponding partition function will be denoted by an unadorned $\mathcal{Z}$. Note that since η_i is either 0 or ± 1, $\eta_i^2 = |\eta_i|$; however, we choose to display the above form to leave open the possibility of $\eta_i \in \mathbb{Z}$. As discussed earlier, there are no *explicit* agent–agent interactions in this model; indeed, the structure of the Hamiltonian assumes that gang members interact with each other only via the graffiti tagging. As a result, occupation at site i by a gang member is "energetically" favored only if nearest-neighbor and on-site graffiti are predominantly of its same color. The two coupling constants, J for nearest-neighbor interactions and K for on-site occupation, reflect this trend. The $\alpha \eta_i^2$ term represents the proclivity of a given site to be occupied by agents regardless of color, implying that gang members carry a strong tendency to occupy unclaimed turf if $\alpha \gg 1$, while $\alpha \ll -1$ represents a natural paucity of gangs altogether. Finally, we assume graffiti imbalance of either color to be energetically unfavorable via the $-\lambda g_i^2$ term. This can be interpreted as natural decay of graffiti due to the elements, or to police or community intervention. For purposes of stability, λ must be positive. Although the interactions J, K are tacitly assumed to be positive, generalizations to negative values may be possible, and a corresponding analysis may be undertaken given the proper sociological interpretations.

3. Phase transition in the GI-system

3.1. Low temperature phase

The basic strategy we follow to demonstrate an ordered, "low temperature" phase is a *contour* argument, here illustrated: Suppose that η_i, the agent spin at site i, differs from the agent spin η_j at a different site j. The two agent spins can differ either by color, representing two different gang affiliations, or by occupation, where one site is occupied and the other is void. At the scale of nearest neighbors, each edge in the lattice can be defined as either a *coherent* bond, where the adjoining lattice sites are occupied and their agent spins are identical, or as an *incoherent* bond if this condition does not hold. Thus, explicitly, $(\eta_i, \eta_j) = (1, 1)$ or $(-1, -1)$ are coherent, and all the other types are not.

Let us now consider any path on the lattice that joins sites i and j. Since i and j have agent spins which are not identical, it must be the case that on any path between i and j, there is an incoherent bond. Furthermore, these incoherent bonds must form a closed contour on the dual lattice that separates i from j. In the following subsections, we derive a bound on the probability of any such incoherent bonds and their aggregation into contours. When these probabilities are small enough—which happens in certain regions of the parameter space—we can establish a low temperature phase. For example, the presence of a red agent at the origin will imply that, with significant probability, the majority of the other sites will also be occupied by red agents, showing the existence of a red phase. Similarly, a blue phase can be shown to exist.

To achieve all of these ends, we will employ the methods of *reflection positivity* described in Ref. [37,38] which contain a detailed account of useful techniques along with relevant classic references. In this paper, we will be working on the $L \times L$ *diagonal* 2D torus—the SST—which we denote by $\mathbb{T}_L$. We will often refer to the Gibbsian probability measure on $\mathbb{T}_L$ associated with the Hamiltonian in Eq. (1) which we denote by $\mathbb{P}_L(\cdot)$.

3.1.1. Reflection positivity

By means of the reflection positivity of the Gibbs distribution we can easily bound the expectation of an observable which depends only on the spin at any two neighboring lattice points. This result will be used to build the contour argument that will lead us to prove the existence of a low temperature phase. We thus briefly introduce the concept of reflection positivity, referring the interested reader to Ref. [37] for a more detailed discussion of these topics.

Consider a plane of reflection p which intersects the torus in a path running through next nearest (diagonal) pairs of sites. Let ϑ_p be the reflection operator through p. On the SST, this plane p divides the lattice into two halves, identified as $\mathbb{T}_L^+$ and $\mathbb{T}_L^-$, such that $\mathbb{T}_L^+ \cap \mathbb{T}_L^- = p$. Let $\mathscr{U}_p^+$ denote the set of functions which depend only on the spin variables in $\mathbb{T}_L^+$ and similarly for $\mathscr{U}_p^-$. The *reflection map*, ϑ_p, which, in a natural fashion identifies sites in $\mathbb{T}_L^+$ with those in $\mathbb{T}_L^-$ via a reflection through p, can also be used to define maps between $\mathscr{U}_p^+$ and $\mathscr{U}_p^-$: Specifically, if $f \in \mathscr{U}_p^+$, we define $\vartheta_p f \in \mathscr{U}_p^-$ to be the function f evaluated on the configuration reflected from $\mathbb{T}_L^-$.

A measure μ is *reflection positive* with respect to ϑ_p if for every $f, g \in \mathscr{U}_p^+$, or $\mathscr{U}_p^-$, the following two properties hold

1. $\mathbb{E}_\mu(f \vartheta_p f) \geq 0$,
2. $\mathbb{E}_\mu(f \vartheta_p g) = \mathbb{E}_\mu(g \vartheta_p f)$.

It is known (e.g., see Ref. [37]) that $\mathbb{P}_L$ is reflection positive with respect to ϑ_p for every p of the above described type. We next use reflection positivity to find an upper bound on the expectation of observables defined on bonds. In doing so, we use the following lemmas:

Lemma 3.1. *Let $\langle i, j\rangle$ denote a bond of $\mathbb{T}_L$ and let α_i and γ_j denote site events at the respective endpoints of the bond. Let $\mathcal{Z}_{\mathbb{T}_L}^{(\alpha,\gamma)}$ denote the partition function (on $\mathbb{T}_L$) which has been constrained so that at each site with the parity of i, the translation of the event α_i occurs and similarly for γ. Then, for $L = 2^k$ for some integer k,*

$$\mathbb{P}_L(\alpha_i \cap \gamma_j) \leq \left[\frac{\mathcal{Z}_{\mathbb{T}_L}^{(\alpha,\gamma)}}{\mathcal{Z}_{\mathbb{T}_L}}\right]^{\frac{1}{2V}},$$

where $V = L^2$ is the volume of the torus.

Proof. The result from this lemma dates back to the original papers on the subject. In particular, the use of bond events on the SST was highlighted in Ref. [38]. A modern and complete derivation is contained in Ref. [37], Section 5.3. □

For a slightly more general scenario, let us consider the bond $\langle i, j\rangle$ and various events $\alpha_i^1, \gamma_j^1, \ldots, \alpha_i^n, \gamma_j^n$ and let us denote by $b_1 = \alpha_i^1 \cap \gamma_j^1 \ldots b_n = \alpha_i^n \cap \gamma_j^n$ the corresponding bond events as described. Letting $b = \cup_{j=1}^n b_j$ we find

$$\mathbb{P}_L(b) \leq \sum_{j=1}^n \left[\frac{\mathcal{Z}_{\mathbb{T}_L}^{(\alpha_j,\gamma_j)}}{\mathcal{Z}_{\mathbb{T}_L}}\right]^{\frac{1}{2V}} := \sum_{j=1}^n \left[\frac{\mathcal{Z}_{\mathbb{T}_L}^{(b_j)}}{\mathcal{Z}_{\mathbb{T}_L}}\right]^{\frac{1}{2V}}.$$

Finally, we have

Lemma 3.2. *Let $r_1, \ldots, r_m$ denote translations of the bond $\langle i, j\rangle$ and b_{r_j} the translation of the bond event(s) b described above. Then*

$$\mathbb{P}_L(\cap_{j=1}^m b_{r_j}) \leq \left[\sum_{j=1}^n \left[\frac{\mathcal{Z}_{\mathbb{T}_L}^{(b_j)}}{\mathcal{Z}_{\mathbb{T}_L}}\right]^{\frac{1}{2V}}\right]^m.$$

Proof. Again, we refer the reader to Ref. [37], Section 5.3. □

3.1.2. A bound on the incoherent bond probabilities

In order to prove a phase transition by a contour argument, we must place an upper bound on the probability for the occurrence of any type of incoherent bond where agent spins of neighboring sites are different. There are four types of incoherent bonds, namely $(\eta_i, \eta_j) = (-1, 1), (-1, 0)$, and $(1, 0)$, and $(0, 0)$, regardless of order. Let us introduce the following notation: consider undirected bonds between two particular neighboring lattice sites, $\langle i, j\rangle$ and let $(\cdot, \cdot)$ denote the event of any of the nine coherent or incoherent bonds so that

$$(\cdot, \cdot) \in \{(+, +), (-, -), (+, -), (-, +), (+, 0), (0, +), (-, 0), (0, -), (0, 0)\}.$$

Similarly, let $Z_{\mathbb{T}_L}^{(\cdot,\cdot)}$ denote the partition function restricted to configurations where all agent spins are frozen in accord with the above described (chessboard) pattern and the rest of the statistical mechanics is provided by the graffiti field against this background [37,39]. The following is readily obtained:

Proposition 3.3. *The above described (agent-constrained) partition functions are given by*

$$\mathcal{Z}_{\mathbb{T}_L}^{(0,0)} = \left[\frac{\sqrt{\pi}}{\sqrt{\lambda}}\right]^V,$$
$$\mathcal{Z}_{\mathbb{T}_L}^{(-,-)} = \mathcal{Z}_{\mathbb{T}_L}^{(+,+)} = \left[\frac{e^{\alpha}\sqrt{\pi}}{\sqrt{\lambda}} e^{\frac{1}{4\lambda}[4J+K]^2}\right]^V,$$
$$\mathcal{Z}_{\mathbb{T}_L}^{(+,-)} = \mathcal{Z}_{\mathbb{T}_L}^{(-,+)} = \left[\frac{e^{\alpha}\sqrt{\pi}}{\sqrt{\lambda}} e^{\frac{1}{4\lambda}[-4J+K]^2}\right]^V,$$
$$\mathcal{Z}_{\mathbb{T}_L}^{(0,+)} = \cdots = \mathcal{Z}_{\mathbb{T}_L}^{(-,0)} = \left[\frac{e^{\frac{1}{2}\alpha}\sqrt{\pi}}{\sqrt{\lambda}} e^{\frac{1}{8\lambda}K^2}\right]^V.$$

Proof. Since the agent variables are frozen, the g_i Gaussian variables are independent and the above amount to straightforward Gaussian integrations. □

Using Lemma 3.1 and the fact that the full partition function satisfies $\mathcal{Z}_{\mathbb{T}_L} \geq \mathcal{Z}_{\mathbb{T}_L}^{(+,+)}$, we can write

$$\begin{aligned}
&\mathbb{P}_L(0,0) \leq \mathrm{e}^{-\frac{1}{2}\alpha}\mathrm{e}^{-\frac{1}{8\lambda}[4J+K]^2},\\
&\mathbb{P}_L(+,-) = \mathbb{P}_L(-,+) \leq \mathrm{e}^{-\frac{2JK}{\lambda}},\\
&\mathbb{P}_L(+,0) = \cdots = \mathbb{P}_L(0,-) \leq \mathrm{e}^{-\frac{1}{4}\alpha}\mathrm{e}^{-\frac{2J^2+JK+\frac{1}{16}K^2}{\lambda}}.
\end{aligned} \tag{2}$$

We denote by $\varepsilon = \varepsilon(J, K, \lambda, \alpha)$ the sum of the estimates for the probabilities provided by the right hand sides of the preceding display. For fixed α and $K > 0$, note that as $J\lambda^{-1/2} \to \infty$ (or, better yet, $J\lambda^{-1/2}$ and $K\lambda^{-1/2}$ both tending to infinity) the quantity ε tends to zero. This implies the suppression of all incoherent bonds so that the lattice must be almost fully tiled with coherent ones. In particular, the lattice is nearly filled with agents, which, at least locally, are mostly of the same type. As will be demonstrated below, this implies the existence of distinctive red and blue phases, i.e., in the language of statistical mechanics, of a "low temperature" regime. We formalize this result in the next subsection.

3.1.3. The contour argument

We have now established all the tools we need to complete the contour argument. Accordingly, we now show that two well-separated lattice sites must, with probability tending to one, have identical agent spins in the limit $\varepsilon \ll 1$. This in turn will imply the existence of a low temperature phase.

Theorem 3.1. *Consider the GI-system on $\mathbb{Z}^2$ and let $\varepsilon(J, K, \lambda, \alpha)$ denote the quantity described in the last paragraph of the previous subsection. Then, if the parameters are such that ε is sufficiently small, there are at least two distinct limiting Gibbs states characterized, respectively, by the abundance of red agents and the abundance of blue agents. Moreover, this property holds in any limiting shift invariant Gibbs state.*

Proof. Let us start on $\mathbb{T}_L$ with $L = 2^k$. For $i, j \in \mathbb{T}_L$ where i and j are well separated, let us consider the event $\upsilon_B := \{\eta_i \neq \eta_j\} \cup \{\eta_i = 0\}$. We will show, under the stated conditions, that uniformly in L this probability vanishes as $\varepsilon \to 0$. As discussed previously, in order for this event to occur, the sites i and j must be separated by a closed contour consisting of bonds dual to incoherent bonds. For $\ell = 4, 6, \ldots$ let $\mathfrak{N}_\ell = \mathfrak{N}_\ell(i - j, L)$ denote the *number* of such contours of length ℓ on $\mathbb{T}_L$. Then we claim that uniformly in L and $i - j$,

$$\mathfrak{N}_\ell \leq 2\ell^2\lambda_2^\ell$$

where λ_2 (with $\lambda_2 \approx 2.638\ldots < 3$) is the connectivity constant for $\mathbb{Z}^2$ [40]. A word of explanation may be in order. The λ_2^ℓ generously accounts for walks of length ℓ in the vicinity of site i and the factor of two for walks in the vicinity of site j. Finally, the factor of ℓ^2 accounts for the origin of the walk. Note this is an over-counting, e.g., contours which wind the torus but do not necessarily "enclose" i or j are counted twice. Using Lemma 3.2 we may now write

$$\mathbb{P}_L(\upsilon_B) \leq \sum_\ell \mathfrak{N}_\ell \varepsilon^\ell \leq 2 \sum_{\ell:\mathfrak{N}_\ell \neq 0} \ell^2[\lambda_2\varepsilon]^\ell.$$

The above obviously tends to zero as $\varepsilon \to 0$ demonstrating that in finite volume, the lattice is either populated with mostly red agents *or* mostly blue agents depending—with high probability—on what is seen at the origin. The implication of this result is that, for ε sufficiently small, there are at least two infinite volume Gibbs states—which can be realized as the limits of the appropriately conditioned $\mathbb{T}_L$'s. These states have one of the two mutually exclusive characteristics: a preponderance of red agents or a preponderance of blue agents. The fact that the above must also hold in any shift-invariant Gibbs state is the subject of Theorem 2.5 and its Corollary in Ref. [41] with a slight extension provided by Corollary 5.8 in Ref. [42]. □

3.2. High temperature phase

As is sometimes (e.g., historically) the case in statistical mechanics, it can be an intricate job to establish a high temperature phase—a region of parameters where the limiting Gibbs measure is unique and correlations decay rapidly. Typically, one calls upon the Dobrushin uniqueness criterion [43]. However for us, this route is interdicted by the unbounded nature of the g_i graffiti field. The strategy here will be percolation based: First we establish the so-called FKG property for all the associated Gibbs measures. Then what follows will be a relatively standard argument through which we show that the necessary and sufficient condition for uniqueness is that the average of η_i—akin to a magnetization—vanishes in the state designed to optimize this quantity. Then, finally, we will develop a random cluster-type expansion demonstrating that under the expected high-temperature conditions for the couplings, e.g., $\lambda \gg 1$, the stated condition on this magnetization is satisfied. In addition, high-temperature behavior should also be achieved under the condition that agents are sparse. This requires an alternative percolation criterion used in conjunction with the above mentioned expansion. In both scenarios, the rapid decay of correlations arises as an automatic byproduct.

3.2.1. FKG properties

In this paragraph we will demonstrate that the *FKG Lattice Condition* (see e.g., Ref. [44] Page 78) is satisfied by any finite volume Gibbs measure associated with the GI–Hamiltonian. Let us start by noting that we can define a natural partial ordering on the pair of states s_i and s_i' via the notation

$$s_i \succeq s_i', \quad \text{if } \eta_i \geq \eta_i' \quad \text{and} \quad g_i \geq g_i'.$$

Further we introduce the notation $\mathbf{s} \succeq \mathbf{s}'$ to signify that the above holds for all the s_i, s_i' at each $i \in \Lambda$. For individual spins s_i and s_i', we also denote $s_i \vee s_i' := (\max\{\eta_i, \eta_i'\}, \max\{g_i, g_i'\})$ and similarly for the "minimum" $s_i \wedge s_i'$. Finally, for spin configurations $\mathbf{s}$ and $\mathbf{s}'$, the configurations $\mathbf{s} \vee \mathbf{s}'$ and $\mathbf{s} \wedge \mathbf{s}'$ are defined as the sitewise maximum and minimum, respectively. The FKG lattice condition—conveniently stated for finite volume measures—is that for all $\mathbf{s}, \mathbf{s}'$, the following inequality holds:

$$\mathbb{F}_\Lambda(\mathbf{s} \vee \mathbf{s}')\mathbb{F}_\Lambda(\mathbf{s} \wedge \mathbf{s}') \geq \mathbb{F}_\Lambda(\mathbf{s})\mathbb{F}_\Lambda(\mathbf{s}'). \tag{3}$$

The well known consequence of the above is that any pair of random variables that are both increasing with respect to the partial order described above are positively correlated.

Proposition 3.4. *The finite volume Gibbs measures associated with the GI–Hamiltonian satisfy the FKG lattice condition.*

Proof. We consider an arbitrary graph and, as will be made evident, the proof automatically accounts for any fixed boundary conditions. Now, as is well known, it is sufficient to establish that the lattice condition Eq. (3) holds when differences between configurations are exhibited only on a pair of spin-variables. The fixed boundary spins thus may be regarded as part of the background which is common to all four possible agent-graffiti spin configurations in question. Let us thus assume that the differences between two configurations occur at sites a and b in the graph where certain specified variables have been "raised" above a base configuration level $\mathbf{s}$. We denote the single raise configurations by $\mathbf{s}_a$ and $\mathbf{s}_b$ and the double raise by $\mathbf{s}_{ab}$. Thus, it is sufficient to show $\mathbb{F}(\mathbf{s}_{ab})\mathbb{F}(\mathbf{s}) \geq \mathbb{F}(\mathbf{s}_a)\mathbb{F}(\mathbf{s}_b)$. All told, there are three possibilities to consider: graffiti–graffiti, gang–graffiti and gang–gang raises on the a and b sites. For the mixed gang–graffiti case we must also consider the $a = b$ possibility where the gang and graffiti spins have been "raised" at the same site. We need not consider the normalization constant in any of these cases, since it appears in identical roles on both sides of the purported inequality; consideration of the Boltzmann factors is sufficient. Let us introduce, in the setting of our general graph, the interaction

$$-\mathscr{H}(\mathbf{s}) = \sum_{\langle i,j \rangle} J_{i,j}\eta_i g_j - \sum_i [\alpha_i \eta_i^2 + \lambda_i g_i^2],$$

where the first sum now extends over all edges considered to be part of the graph and our only stipulation is that $J_{i,j} > 0$. Also, we may formally include $i = j$ in this sum. Let us denote the "raised" graffiti variables via the positive increments δg_a and δg_b so that, in the graffiti–graffiti case, at sites a and b $g_a \to g_a + \delta g_a$ and $g_b \to g_b + \delta g_b$. It is straightforward to see that $\mathscr{H}(\mathbf{s}_{ab}) + \mathscr{H}(\mathbf{s}) = \mathscr{H}(\mathbf{s}_a) + \mathscr{H}(\mathbf{s}_b)$ and the desired inequality holds as an identity. Similarly for the gang–gang case. We can now consider the mixed case where, without loss of generality, $g_a \to g_a + \delta g_a$ and $\eta_b \to \eta_b + \delta\eta_b$, and for us, $\delta\eta_b \equiv 1$. Here

$$-(\mathscr{H}(\mathbf{s}_a) - \mathscr{H}(\mathbf{s})) = \sum_{i \neq b} J_{i,a}\eta_i \delta g_a + J_{a,b}\eta_b \delta g_a - \lambda \delta {g_a}^2,$$

while

$$-(\mathscr{H}(\mathbf{s}_b) - \mathscr{H}(\mathbf{s})) = \sum_{j \neq b} J_{b,j} g_j \delta\eta_b + J_{a,b} g_a \delta\eta_b + \alpha \delta {\eta_b}^2.$$

However

$$\begin{aligned} -(\mathscr{H}(\mathbf{s}_{ab}) - \mathscr{H}(\mathbf{s})) &= \left[\sum_{i \neq b} J_{i,a}\eta_i \delta g_a + J_{a,b}\eta_b \delta g_a - \lambda \delta {g_a}^2\right] + \left[\sum_{j \neq b} J_{b,j} g_j \delta\eta_b + J_{a,b} g_a \delta\eta_b + \alpha {\eta_b}^2\right] + J_{a,b}\delta g_a \delta\eta_b \\ &\geq 2\mathscr{H}(\mathbf{s}) - \mathscr{H}(\mathbf{s}_a) - \mathscr{H}(\mathbf{s}_b). \end{aligned}$$

Combining the above results we find that indeed

$$\mathscr{H}(\mathbf{s}_{ab}) + \mathscr{H}(\mathbf{s}) \leq \mathscr{H}(\mathbf{s}_a) + \mathscr{H}(\mathbf{s}_b).$$

The same inequality can be easily shown in the mixed gang–graffiti case for $a = b$, by assuming $g_a \to g_a + \delta g_a$ and $\eta_a \to \eta_a + \delta\eta_a$ and by following the same steps as above. This completes the proof. □

As an immediate consequence, we can identify boundary conditions on Λ which most favor the dominance of the red gang. Indeed, it is now seen—as was anyway clear heuristically—that we must make the boundary spins "as red as possible" in order for a predominance of Λ sites to be occupied by red agents. This amounts, somewhat informally, to setting $g_i \equiv +\infty$ and $\eta_i \equiv 1$ (which is anyway automatic if $K \neq 0$) all along the boundary. This "specification" which seems a bit arduous to work with is not nearly as drastic as it sounds. Let us start with some notation: For Λ a finite subset of $\mathbb{Z}^2$, let us define $\partial\Lambda$ as those sites in Λ^c with a neighbor in Λ and $\mathrm{D}\Lambda$ as those sites in Λ with a neighbor in Λ^c. Clearly the only immediate consequence of the "drastic" boundary condition is to force $\eta_i \equiv 1$ for $i \in \mathrm{D}\Lambda$ and to bias, by at most $(3J + K)g_i$ the *a priori* Gaussian distribution of the g's. We shall do that—and a bit more—on $\partial\Lambda$ arguing that this, at most, is the result of the "drastic" boundary condition on $\partial(\Lambda \cup \partial\Lambda)$. Precisely, we define the *red* boundary condition on Λ as $\eta_i \equiv 1$ and g_i independently distributed as normal random variables with variance $1/2\lambda$ and mean $(4J + K)/2\lambda$ for each $i \in \partial\Lambda$. By the established monotonicity properties these are exactly the boundary conditions imposed on the slightly larger lattice that will optimize the average of η_i and g_i for any $i \in \Lambda$.

3.2.2. A uniqueness criterion

It is not hard to show, by monotonicity, that a limiting *red measure* exists along any thermodynamic sequence of volumes and that the limit is independent of the sequence and therefore translation invariant. We shall denote this measure by $\mu_{\mathrm{R}}(\cdot)$ and by $\mathbb{E}_{\mathrm{R}}(\cdot)$ the corresponding expectations. Similarly for the blue measure we introduce $\mu_{\mathrm{B}}(\cdot)$ and $\mathbb{E}_{\mathrm{B}}(\cdot)$. We can thus state

Proposition 3.5. *The necessary and sufficient condition for uniqueness among the limiting Gibbs states for the GI-system is that* $\mathbb{E}_R(\eta_0) = 0$*, where* η_0 *is the spin at the lattice origin.*

Proof. For two measures μ_1 and μ_2 e.g., on $\{-1, 0, 1\}^{\mathbb{Z}^2}$, we use the notation $\mu_1 \geq \mu_2$ to indicate that for any random variable X which is increasing in all coordinates, the expected values $\mathbb{E}_1(X)$, calculated via the μ_1 measure are always greater than the those obtained via μ_2:

$$\mathbb{E}_1(X) \geq \mathbb{E}_2(X).$$

This is known as *stochastic dominance*. Consider $\mu_{\mathrm{R}}(\cdot)$ which, by slight abuse of notation, we temporarily take to be the restriction of μ_R to agent events. Suppose that $\mathbb{E}_{\mathrm{R}}(\eta_0) = 0$. Then, by translation invariance, we have $\mathbb{E}_{\mathrm{R}}(\eta_i) = 0$ for all i. Similar considerations apply to the corresponding $\mu_{\mathrm{B}}(\cdot)$. It is immediately clear—by symmetry or stochastic dominance—that $\mathbb{P}_{\mathrm{R}}(\eta_0 = 0) = \mathbb{P}_{\mathrm{B}}(\eta_0 = 0)$ and thus the single site distributions are identical. By the corollary to the Strassen theorem [45,46] since $\mu_{\mathrm{R}} \geq \mu_{\mathrm{B}}$ *and* these measures have identical single site distributions they must be identical probability measures. Similar considerations apply to the full measures since the distribution of the g_i is determined by their conditional distributions given the local configuration of the η_i's. Uniqueness is established since, if $\mu_{\odot}(\cdot)$ denotes any other infinite volume measure associated to the GI–Hamiltonian, we have $\mu_{\mathrm{R}} \geq \mu_{\odot} \geq \mu_{\mathrm{B}}$ which implies equality in light of $\mu_{\mathrm{R}} = \mu_{\mathrm{B}}$. □

3.2.3. Proof of a high-temperature phase

We shall develop a graphical representation for the GI-system akin to the FK representation for the Potts model [47] that, for all intents and purposes, is the same as the one used in Ref. [48], where only the case of bounded fields is explicitly analyzed. Let us then consider the GI–Hamiltonian in finite volume with all notation pertaining to boundary conditions temporarily suppressed. For fixed $\mathbf{s}$, we may decompose the graffiti fields and agents according to affiliation:

$$\begin{aligned} g_i &= q_i\vartheta_i; \qquad \vartheta_i = \pm 1, \qquad q_i = |g_i|, \\ \eta_i &= r_i\sigma_i; \qquad \sigma_i = \pm 1, \qquad r_i = |\eta_i|, \end{aligned}$$

where the σ's and ϑ's have the definitive character of *Ising* variables. We can now write

$$\mathrm{e}^{J_{i,j}g_i\eta_j} = \mathrm{e}^{-J_{i,j}q_ir_j}(R_{i,j}\delta_{\vartheta_i,\sigma_j} + 1),$$

where $R_{i,j} = R(J_{i,j}, q_i, r_j) := \mathrm{e}^{2J_{i,j}q_ir_j} - 1$. In our case, we have $J_{i,j} = J$ if i and j are neighboring pairs and $J_{i,i} = K$; which we will not yet distinguish notationally and consider a general $J_{i,j}$ label. Thus

$$\mathrm{e}^{-\mathscr{H}(\mathbf{s})} = \prod_{(i,j)} \mathrm{e}^{-J_{i,j}q_ir_j}(R_{i,j}\delta_{\vartheta_i,\sigma_j} + 1).$$

Opening the product, we select one term for each "edge": If the $R_{i,j}$ term is selected, we declare the edge to be *occupied*, otherwise it is *vacant*. It is noted here that the edges should be interpreted as *directed*: all edges appear twice and we must

regard $\langle i, j\rangle$ as distinctive from $\langle j, i\rangle$; moreover, for $K \neq 0$, the above is understood to include $i = j$. The configurations of occupied edges will, generically, be denoted by ω. Summing over the Ising variables, we acquire the weights

$$W(\omega) = 2^{C(\omega)} \sum_{\mathbf{q},\mathbf{r}} \prod_{(i,j)\in\omega} R_{i,j}(J_{i,j}q_i, r_j),$$

where, as before the summation notation also indicates integration over the continuous variables. In the above, $C(\omega)$ denotes the number of connected components of ω; here connectivity deduced according to the *directed* nature of the edges or via a double covering of the lattice. Normalizing these weights by the partition function we obtain a probability measure on the bond configurations ω. As will be made explicit below, this probability measure on bond configurations is well defined in finite volume. Let us denote the probability measure on the bond configurations ω by $\mathbb{P}^{\odot}_{\Lambda}(\cdot)$, where the $\odot$ now denotes boundary conditions accounted for in a routine fashion. Then, for each ω consisting of appropriate edges, $\mathbb{P}^{\odot}_{\Lambda}(\omega) \in (0, 1)$. We shall not discuss the problem of infinite volume limits which would take us too far astray but be content with statements that are uniform in volume. With regards to the latter, and of crucial importance for our purposes is the connection back to the spin-measure inherent in this representation. For the Potts models, this was first elucidated in Ref. [49] with the complete picture emerging in Ref. [50]. In particular, for any site, the contribution to the magnetization vanishes if the site belongs to a cluster that is isolated from the boundary. The principal objective for this representation is the following claim:

Proposition 3.6. *Let $\Lambda \subset \mathbb{Z}^2$ be a finite connected set and consider the above described representation in Λ with boundary condition $\odot$ on $\partial\Lambda$. Let $\langle a, b\rangle$ denote an edge with $a \neq b$ and both a and b not belonging to $\partial\Lambda$. Let $\mathbf{e}_{ab}$ denote the event that this edge is occupied and let ω denote a configuration on the compliment of $\langle a, b\rangle$. Then, for fixed α and K, there is an $\varepsilon(J, \lambda)$ with $\varepsilon \to 0$ as $J^2/\lambda \to 0$ such that uniformly in Λ, ω and $\odot$—as well as K and α,*

$$\mathbb{P}^{\odot}_{\Lambda}(\mathbf{e}_{ab}) < \frac{\varepsilon}{1+\varepsilon}.$$

Proof. Let $W^{\odot}_{\Lambda}(\cdot)$ denote the configurational weights with associated boundary conditions as described above. Then it is seen that

$$\frac{\mathbb{P}^{\odot}_{\Lambda}(\omega \vee \mathbf{e}_{ab})}{1 - \mathbb{P}^{\odot}_{\Lambda}(\omega \vee \mathbf{e}_{ab})} = \frac{W^{\odot}_{\Lambda}(\omega \vee \mathbf{e}_{ab})}{W^{\odot}_{\Lambda}(\omega)}.$$

Our goal is to estimate the right hand side of the above which thereby generates the quantity ε featured in the statement of this proposition. Noting the positivity and product structure of the numerator and denominator, we may regard the object on the right as the expectation with respect to a weighted measure of the quantity R_{ab} and we shall denote this by $\mathbb{E}_{\omega}(R_{a,b})$. The latter will be estimated via conditional expectation: Let $Q_{\hat{q}_a}$ denote a specification of the q-fields and agent occupation variables except for q_a and let

$$\varepsilon := \sup_{\omega, Q_{\hat{q}_a}} \mathbb{E}_{\omega}(R_{ab} \mid Q_{\hat{q}_a}).$$

Obviously, $\varepsilon \geq \sup_{\omega} \mathbb{E}_{\omega}(R_{ab})$. As for the complimentary fields, there is not a great deal of dependence: In particular, all that is needed is that $r_c = 1$ for all c such that $\langle a, c\rangle \in \omega$. Concerning the optimizing ω, non-local considerations dictate simply, that ω be such that $\langle a, b\rangle$ does not *reduce* the number of components. Locally, as can be explicitly checked, or derived from monotonicity principals, the optimal scenario is when all bonds emanating from a are present in the configuration. Thus we have

$$\varepsilon = \frac{\int \mathrm{e}^{-(4J+K)q} R^4(J) R(K) \mathrm{e}^{-\lambda q^2} dq}{\int \mathrm{e}^{-(3J+K)q} R^3(J) R(K) \mathrm{e}^{-\lambda q^2} dq} = 2\frac{\int \mathrm{e}^{-\lambda q^2} \sinh^4(Jq) \sinh(Kq) dq}{\int \mathrm{e}^{-\lambda q^2} \sinh^3(Jq) \sinh(Kq) dq},$$

where in the first line $R(J) := R(J, q, 1)$. We claim that the final ratio is bounded by $J/\lambda^{1/2}$ multiplied by a constant that may be proportional to the ratio $K/\lambda^{1/2}$. Indeed let us substitute $\omega = \lambda^{1/2} q$ and $\kappa := K/\lambda^{1/2}$. The above quantity can thus be rewritten as

$$\varepsilon = 2\frac{\int \mathrm{e}^{-\omega^2} \sinh \kappa\omega (\sinh \omega J/\lambda)^4 d\omega}{\int \mathrm{e}^{-\omega^2} \sinh \kappa\omega (\sinh \omega J/\lambda)^3 d\omega}.$$

Our claim is obvious if $\kappa \to 0$ but we may wish to consider cases where κ stays bounded away from zero. In general, the integrands are not dominated by large ω and we may expand the factors $\sinh \omega J/\lambda$ with the result

$$\varepsilon \to \frac{2J}{\lambda^{1/2}} \frac{\int \mathrm{e}^{-\omega^2} \sinh \kappa\omega \cdot \omega^4 d\omega}{\int \mathrm{e}^{-\omega^2} \sinh \kappa\omega \cdot \omega^3 d\omega}.$$

We finally claim that the right side is bounded by a linear function of κ:

$$\frac{1}{1+\kappa}\frac{\int e^{-\omega^2}\sinh\kappa\omega\cdot\omega^4 d\omega}{\int e^{-\omega^2}\sinh\kappa\omega\cdot\omega^3 d\omega} < B,$$

for some $B < \infty$. This is indeed true as $\kappa \to 0$. We only need to show that the inequality holds in the case $\kappa \to \infty$. But here the factor $e^{-\omega^2}\sinh\kappa\omega$ is, essentially, a Gaussian in the variable $\omega - \kappa$ and the desired result follows. □

Theorem 3.2. *Consider the GI-system and let ε denote the quantity described in Proposition 3.6. Then for $\varepsilon < \varepsilon_0$, given by*

$$\varepsilon_0 + \frac{1}{2}\varepsilon_0^2 = \frac{1}{2}$$

there is a unique limiting Gibbs state featuring rapid decay of correlations.

Proof. Using the result of Proposition 3.6, we shall compare the described graphical representation with independent bond percolation on $\mathbb{Z}^2$. We start with a well known—and readily derivable—result: Let $Y_1, \ldots, Y_N$ denote an array of Bernoulli random variables with collective behavior described by the measure $\mu_{\mathbf{Y}}$ and let $\mathbf{Y}_{\hat{Y}_j}$ denote a configuration on the compliment of Y_j. Let us now introduce

$$p_j = \max_{\mathbf{Y}_{\hat{Y}_j}} \mathbb{P}_{\mathbf{Y}}(Y_j = 1 \mid \mathbf{Y}_{\hat{Y}_j})$$

to denote the maximal conditional probability of observing $\{Y_j = 1\}$. Finally, let $X_1, \ldots, X_N$ denote a collection of independent Bernoulli random variables with parameters $p_1, \ldots, p_N$. Then, denoting the independent measure by $\mu_{\mathbf{X}}$, we have

$$\mu_{\mathbf{X}} \geq \mu_{\mathbf{Y}}.$$

Thus we may bound the probabilities of increasing events in the graphical representation by the corresponding probabilities from independent percolation on $\mathbb{Z}^2$ with bond occupation probabilities determined by the ε from Proposition 3.6. However, we must note that the relevant percolation problem has multiple types of edges. The ε_0 in the statement of this proposition bounds the probability of the event $\mathbf{e}_{ab} \cup \mathbf{e}_{ba}$ by $\frac{1}{2}$. If the clusters of the featured representation fail to percolate, then, as $\Lambda \nearrow \mathbb{Z}^2$, the origin is disconnected from the boundary with a probability tending to one. As discussed just prior to Proposition 3.6, this implies $\mathbb{E}_{\mathrm{R}}(\eta_0) = 0$ and by Proposition 3.5, uniqueness is established. Under the condition $\varepsilon < \varepsilon_0$, exponential decay of correlations can also be established. We will be content with the decay of the two point function. The problem of general correlations under these conditions has been treated elsewhere [51,52]. In particular, for $i, j \in \mathbb{Z}^2$, $\mathbb{E}(\eta_i\eta_j)$ in the unique infinite volume measure is bounded, in finite volume approximations by the probability that i and j reside in the same cluster. For $\varepsilon < \varepsilon_0$, this decays exponentially in $|i - j|$ uniformly in Λ for $|\Lambda|$ sufficiently large. □

We now turn our attention to an alternative criterion for high temperature behavior which may also be of relevance in a sociological context: Sparsity of agents. Mathematically, this pertains to the situation where α is large and negative ($-\alpha \gg 1$) which *a priori* suppresses the fraction of agent occupied sites. Our arguments will initially be based on more primitive notions of percolation and, following the methods of Ref. [53] (see also [54,55]) could, perhaps, be completed along these lines. However, it turns out to be far simpler to appeal to the graphical representation just employed for the final stage of the argument. We start with the relevant notion of percolation and connection. In the context of site percolation on $\mathbb{Z}^2$, we may define various notions of connectivity [56]. Here we define $\diamond$-connectivity to indicate connection between sites that are no more than two lattice sites away. This is not to be confused with $*$-connectivity which does not consider a pair of sites to be connected if they are separated by two units in the vertical or horizontal direction. We denote by $p_c^{\diamond}$ the threshold for $\diamond$-percolation on $\mathbb{Z}^2$. Standard arguments dating to the beginning of the subject show that $p_c^{\diamond} \in (0, 1)$; in particular, $p_c^{\diamond}$ is less than the threshold for ordinary, or even $*$-connected, percolation and mean-field type bounds readily demonstrate that $p_c^{\diamond} > \frac{1}{12}$.

The next proposition concerns the relative abundance of, e.g., red sites under the condition $-\alpha \gg 1$ with the other parameters fixed.

Proposition 3.7. *Consider the GI-system with parameters λ, K and J fixed. Then there is a $\delta_\alpha = \delta_\alpha(J, K, \lambda)$ with $\delta_\alpha \to 0$ as $\alpha \to -\infty$ such that uniformly in volume and boundary conditions, for any site i that is away from the boundary*

$$\mathbb{P}_{\Lambda}^{\odot}(\eta_i = 1) < \delta_\alpha.$$

Proof. Here we employ the preliminary (red $\succeq$ blue) FKG properties that were established earlier, in 3.2.1. We start with a $\gamma > 0$ (and somewhat "large") and, for $j \in \Lambda$ not too near the boundary, we consider $\mathbb{P}_{\Lambda}^{\odot}(g_j > \gamma)$. By the FKG property, this

probability is less than the corresponding conditional one given that $\eta_j = 1$ and that $\eta_k = 1$ for all k that are neighbors of j. This conditional probability is given by a definitive expression:

$$\mathbb{P}^{\odot}_{\Lambda}(g_j > \gamma) \leq \frac{\int_{g>\gamma} \mathrm{e}^{+(4J+K)g}\mathrm{e}^{-\lambda g^2} dg}{\int_{g} \mathrm{e}^{+(4J+K)g}\mathrm{e}^{-\lambda g^2} dg} := \delta_{\gamma}.$$

The above can be expressed directly via the error function but in any case, as is not hard to show,

$$\delta_{\gamma} \leq \frac{1}{2}\mathrm{e}^{-\lambda[\gamma - \frac{4J+K}{2\lambda}]^2},$$

as long as $\gamma \geq (4J + K)/2\lambda$, which also quantifies how large γ must be. Provided i is a few spaces away from the boundary, we note that $1 - 5\delta_{\gamma}$ is a valid estimate of the probability that both g_i and the g-values at the neighbors of i do not exceed γ. Let us denote this (good) non-high field event by G_i. Then we may write

$$\begin{aligned}\mathbb{P}^{\odot}_{\Lambda}(\eta_i = 1) &= \mathbb{P}^{\odot}_{\Lambda}(G_i)\mathbb{P}^{\odot}_{\Lambda}(\eta_i = 1 \mid G_i) + \mathbb{P}^{\odot}_{\Lambda}(G_i^c)\mathbb{P}^{\odot}_{\Lambda}(\eta_i = 1 \mid G_i^c) \\ &\leq 5\delta_{\gamma} + \frac{\mathrm{e}^{(4J+K)\gamma}\mathrm{e}^{\alpha}}{1 + \mathrm{e}^{(4J+K)\gamma}\mathrm{e}^{\alpha} + \mathrm{e}^{-(4J+K)\gamma}\mathrm{e}^{\alpha}} := \delta_{\alpha}, \end{aligned} \tag{4}$$

where, in various stages we have employed worst case scenarios. Clearly, for fixed (J, K, λ) we may choose γ large so that δ_{γ} is small, and α negative and large in magnitude, so that δ_{α} is small. □

Theorem 3.3. *Consider the GI-System and suppose that $-\alpha$ is large enough so that $\delta_{\alpha} < p_c^{\diamond}$ as described just prior to the statement of Proposition 3.7. Then there is a unique limiting Gibbs state featuring rapid decay of correlations.*

Proof. By the dominance principle stated at the beginning of the proof of Theorem 3.2 if $\delta_{\alpha} < p_c^{\diamond}$, the red agents fail to exhibit $\diamond$-percolation regardless of boundary conditions. Now consider, in the context of the bond–representation, the event that the origin is connected to $\partial\Lambda$ in the red boundary conditions, which represents the sole non-vanishing contribution to $\mathbb{E}^{R}_{\Lambda}(\eta_0 = +1)$. The bonds of any path connecting the origin to $\partial\Lambda$ within this cluster may be envisioned as alternating connections between agents and fields; the connection to the red boundary ensures that both types of entities take on the red color. In particular, all the agents in the cluster are red so that these agents must (at least) form a $\diamond$-connected cluster. Hence, in finite volume, we may bound

$$\mathbb{E}^{\mathrm{R}}_{\Lambda}(\eta_0 = +1) \leq \mathbb{P}^{\mathrm{R}}_{\Lambda}(0 \underset{\diamond,\mathrm{R}}{\rightsquigarrow} \partial\Lambda)$$

where $\{0 \underset{\diamond,\mathrm{R}}{\rightsquigarrow} \partial\Lambda\}$ is the event of a red $\diamond$-connection between the origin and the boundary.

When the red agent occupation probabilities are dominated by independent sites with parameter $\delta_{\alpha} < p_c^{\diamond}$, such probabilities decay exponentially. Evidently, in the limiting state, the "magnetization" vanishes which by Proposition 3.5 implies a unique state. Similarly, exponential decay of correlations is implied by exponential decay of $\diamond$-connectivities. □

4. The mean field rendition

In the previous section, we showed that a phase transition between well-mixed and clustering configurations exists for the general Hamiltonian in Eq. (1). However, finding the exact or even approximate values of the J, K, α, λ parameters for which the well-mixed to clustering transition occurs is in general a difficult task. Moreover, the nature of the transition is not elucidated by the techniques of the preceding section. On the basis of informal simulations described in the Appendix and certain other considerations it appears that the transition may be discontinuous or second order depending on where the phase boundary is crossed. This cannot be proved in the context of the present model. We thus introduce a *mean-field Hamiltonian*, where instead of nearest-neighbor interactions we consider an all-to-all (interaction) coupling that is rescaled by the number of sites. Models of this sort are often referred to as *complete-graph* systems. The mean-field Hamiltonian allows us to define, in the thermodynamic limit, a simple mean field free energy per particle. This free energy can be subjected to exact mathematical analysis which provides a quantification of the phase transition. In particular, we have found that the phase boundary between the diffuse states and the gang-symmetry broken phase can indeed be of either type.

Let us thus consider a lattice of N sites—where the detailed geometry is no longer of relevance. At each site i, there is the same $s_i = (\eta_i, g_i)$ featured in the previous section. However now, the Hamiltonian reads

$$-\mathscr{H}^{\mathrm{MF}}_{N}(\mathbf{s}) = \frac{1}{N}\sum_{i,j} J\eta_i g_j + \sum_{i}(\alpha\eta_i^2 - \lambda g_i^2). \tag{5}$$

It is observed that the couplings J and K need no longer be distinguished. Indeed, for large N, the $g_i\eta_i$ interaction, and any other *particular* interaction is not of pertinence. We now introduce the relevant collective quantities, n G, and b, obtained

via η_i and g_i, which will allow for a more convenient analysis. In particular, if N^+ and N^- designate the number of red and blue lattice agents, respectively, we define

$$b := \frac{N^+ + N^-}{N} \quad \text{and} \quad n := \frac{N^+ - N^-}{N},$$

as the fraction of the lattice covered by agents of any type and the excess—positive or negative—of this fraction that is of the red type. Moreover, we introduce $G = \frac{1}{N}\sum_i g_i$ to be the average graffiti imbalance. In this context, n and G are akin to magnetizations in a standard one-component spin model, with n corresponding to magnetization in the agent variables and G in the graffiti field. For occasional use, we also define $n^\pm = N^\pm/N \leq 1$. We remark that in these definitions there is an implicit N dependence which is notationally suppressed.

4.1. The partition function

In the forthcoming, we will evaluate, asymptotically, the mean field partition function $\mathcal{Z}^{\mathrm{MF}}$ defined in accordance with the previous section as the partition sum $\mathcal{Z}_N^{\mathrm{MF}} = \sum_{\mathbf{s}} e^{-\mathscr{H}_N^{\mathrm{MF}}(\mathbf{s})}$. Here, for reasons which will soon become clear, we will treat the graffiti field variables slightly differently. We define

$$d\mu_{g_i} := \sqrt{\frac{\lambda}{\pi}}\mathrm{e}^{-\lambda g_i^2},$$

as the normalized Gaussian measure for the individual field variables. Letting $\mathbf{g}$ denote the array of these random variables we may write

$$\mathcal{Z}_N^{\mathrm{MF}} := \mathbb{E}_{\mathbf{g}}\left(\sum_{\boldsymbol{\eta}} \mathrm{e}^{\,J\sum\limits_{i,j} g_i\eta_j + \alpha\sum\limits_i \eta_i^2}\right),$$

where $\mathbb{E}_{\mathbf{g}}(\cdot)$ denotes expectation with respect to the free (independent) ensemble of Gaussian random variables and $\sum_{\boldsymbol{\eta}}$ denotes the rest of the partition sum i.e., over the agent configurations. It is acknowledged that this differs from the prior definitions by a multiplicative factor of $[\lambda/\pi]^{N/2}$ which, of course, is inconsequential.

It is at this point, with the current formulation, that the advantage of the all-to-all coupling is manifest: For any $\mathbf{s}$ (and any N) the quantity in the exponent depends only on n, b and G: $\mathcal{Z}_N^{\mathrm{MF}} = \mathbb{E}_{\mathbf{g}}(\sum_{\boldsymbol{\eta}} \mathrm{e}^{N(JnG+\alpha b)})$. Concerning the agent configurations, to perform the summation, we must multiply the integrand by the number of ways of arranging N^+ red sites and N^- blue sites among N possible positions. We denote this object by $W_N(b, n)$ which is given, explicitly, by the trinomial factor

$$W_N(b,n) = \binom{N}{N^+, N^-} = \binom{N}{\frac{1}{2}N(b+n),\ \frac{1}{2}N(b-n)}.$$

As for the graffiti field configurations, it is noted that since G is proportional to a sum of Gaussian random variables, it is itself a Gaussian. Indeed the mean of NG is zero and the variance is $N[2\lambda]^{-1}$. Thus the expectation over $\mathbf{g}$ can be replaced with the expectation over NG leading to

$$\mathcal{Z}_N^{\mathrm{MF}} = \sum_{n,b} \mathbb{E}_{NG}[W_N(b,n)\mathrm{e}^{N[JbG+\alpha b]}] \propto \sum_{n,b} W_N(n,b)\mathrm{e}^{N[JnG+\alpha b-\lambda G^2]}dG$$

with the constant of proportionality independent of N. Now, on the basis of the Stirling approximation,

$$W_N(b,n) \approx \left[\left(\frac{b+n}{2}\right)^{\frac{b+n}{2}}\left(\frac{b-n}{2}\right)^{\frac{b-n}{2}}(1-b)^{1-b}\right]^{-N}.$$

Thus, modulo lower order terms, we have $\mathcal{Z}_N^{\mathrm{MF}} \approx \sum_{n,b,G} \mathrm{e}^{-N\Phi(b,n,G)}$ where Φ, the free energy function, is given by

$$e^{-\Phi(b,n,G)} := e^{(JnG+\alpha b-\lambda G^2)}\left[\left(\frac{b+n}{2}\right)^{\frac{b+n}{2}}\left(\frac{b-n}{2}\right)^{\frac{b-n}{2}}(1-b)^{1-b}\right]^{-1}. \tag{6}$$

In accordance with standard asymptotic analysis

$$\lim_{N\to\infty} -\frac{1}{N}\log \mathcal{Z}_N^{\mathrm{MF}} = \min_{b,n,G} \Phi(b,n,G) := F_{\mathrm{MF}}$$

where $F_{\mathrm{MF}} = F_{\mathrm{MF}}(J,\alpha,\lambda)$ is the (actual) limiting free energy per site. While various aspects of the above scenario for all-to-all coupling models have been long known and certain cases explicitly proven [57], there is a general theorem to this effect that is sufficient for our purposes, presented in Section 5 of [58]. Thus the efforts of a mean-field analysis may be summarized as follows: we are to minimize $\Phi(b,n,G)$ and the values of b, n and G at the minima—as a function of the couplings—will determine the various *phases* of the system. Even in this simplified context, as will be seen, the phase transitions can be dramatic.

4.2. The mean-field equations

The free energy function is obviously well behaved except at the extreme values of the variables. In particular, we would like to assume that $0 < b < 1$ and $-b < n < +b$ where the strict inequalities imply that the function is smooth. Now a direct calculation of the asymptotics makes it clear that no minimum could possibly occur near the $b = 0, 1$ and $b = \pm n$ boundaries. Thus, we can confine attention to the interior of the above b and n intervals and proceed by differentiation of $\Phi(b, n, G)$ as defined in Eq. (6). Thus we arrive at the *mean-field equations*:

$$-\frac{\partial \Phi}{\partial G} = Jn - 2\lambda G = 0, \tag{7}$$

$$-\frac{\partial \Phi}{\partial b} = \alpha + \log(1-b) - \frac{1}{2}\log\left(\frac{b^2 - n^2}{4}\right) = 0, \tag{8}$$

$$-\frac{\partial \Phi}{\partial n} = JG - \frac{1}{2}\log\left(\frac{b+n}{b-n}\right) = 0. \tag{9}$$

Free energy minimization only occurs for values of (n, b, G) that satisfy the above system. However, other stationary points for Φ can—and, e.g., in the case of discontinuous transitions generically will – occur so we must proceed with some caution. It is noted that Eq. (7) allows us to eliminate G altogether. Defining $\mu = \frac{J^2}{2\lambda}$, we rewrite Eqs. (8) and (9) as

$$4e^{2\alpha} = \frac{b^2 - n^2}{(1-b)^2}, \tag{10}$$

$$\mu n = \frac{1}{2}\log\left(\frac{b+n}{b-n}\right). \tag{11}$$

The analysis of this system, along with the minimization it is supposed to imply will constitute the bulk of the remainder of this work. Foremost, it is noted that the presentation in Eqs. (10) and (11) are, for all intents and purposes, the same as would have been obtained from the mean-field version of the so-called BEG model [59]. As such, some aspects of the current problem have been treated in Ref. [60]. However, the specifics in Ref. [60] are not readily translated into that of the current work and, moreover, our conclusions are achieved by straightforward methods of analysis.

Our investigation will proceed as follows: it is evident from physical considerations, and the subject of an elementary mathematical theorem proved at the end of this subsection, that as the parameters sweep through their allowed values, a phase transition occurs from the circumstances where Φ is minimized by $n = 0$ to those where $n \neq 0$ is required. First, we will follow the consequences of the *assumption* that this happens continuously: i.e., that the minimizing n goes to zero continuously through small values. In the leading order, this provides a purported phase boundary which we denote by the LSP-curve. Considerations of higher order terms in the vicinity of the LSP-curve yield that for certain portions of the curve, the stipulation is self-consistent and for the rest, it is not. Detailed analysis will show that the former is completely consistent. In particular these calculations correspond to the true minima of the free energy function. By contrast, the latter (non-self-consistent) portion is a consequence of a discontinuous transition which has "already" occurred at prior values of the parameters. In particular, the perturbative analysis is highlighting a local extremum and not the true minimum.

We conclude this subsection with the derivation of the LSP-curve—as well as the introduction of notation that will be used throughout the reminder of the analysis. Assuming $n = 0$, Eq. (11) is trivially satisfied and Eq. (10) defines the "ambient" value of b which we denote by b_R:

$$b_R := \frac{2e^{\alpha}}{1 + 2e^{\alpha}}.$$

Note that $(b = b_R, n = G = 0)$ is *always* a solution to the mean-field system. For simplicity we consider b_R and μ as the relevant parameters for our system for the remainder of this paper. Let us now consider slight perturbations of b about b_R and of n about zero. We thus write $b = b_R(1 + \Delta)$ with $\Delta \ll b_R$ and $|n| > 0$ with $n \ll 1$ and obtain the following approximations by expanding Eq. (10) to lowest order

$$n^2 \approx 2\Delta \frac{b_R^2}{1 - b_R}, \tag{12}$$

while Eq. (11), written to a higher approximation than will be immediately necessary, gives us

$$\mu n \approx \frac{n}{b_R} - \frac{n\Delta}{b_R} + \frac{n^3}{3{b_R}^3}. \tag{13}$$

We pause to observe that Eq. (13) and, in general, Eq. (11), have the symmetry property that with all other quantities fixed, if n is a solution then so is $-n$. Thus, we might as well assume that $n \geq 0$. Indeed, we shall adhere to this convention

throughout. Assuming now that μ is variable while b_R is fixed, the $n \to 0$ limit of Eqs. (12) and (13) yields the tentative phase boundary

$$\mu_S(b_R) = \frac{1}{b_R}. \tag{14}$$

This *defines* the LSP-curve; the correspondingly tentative conclusion is that $n > 0$ and $b > b_R$ occurs for $\mu > \mu_S$ while for $\mu \leq \mu_S$, $n = b - b_R = 0$. However, the viability of these tentative conclusions depends, in a definitive fashion, on the value of b_R. In particular an analysis of the higher order terms in Eq. (13) testifies that this picture cannot possibly be correct for $b_R < \frac{1}{3}$; this is the subject of our next subsection. However, a more difficult analysis shows that this picture is indeed correct for $b_R \geq \frac{1}{3}$ which is the subject matter of the final subsection. First, we must attend to some necessary details.

4.2.1. Preliminary analysis

In this subsection we will establish some basic properties of the model such as the existence of high- and low-temperature phases along with various monotonicity properties. In particular we show that at fixed b_R, the quantity n, assumed to be non-negative, is non-decreasing with μ and strictly increasing whenever it is non-zero. For the benefit of our physics readership, such contentions might typically be *assumed* and consequently, the entire subsection could be skipped on a preliminary reading. However, it is remarked that in the normal (physics) course of events, such questions are most often settled by direct perturbative calculation. Even for continuous transitions, on some occasions, additional justification is actually required. Sometimes, as in the present work, when the transition is discontinuous, the relevant calculations simply cannot be done analytically and then, indeed, one must rely more heavily on abstract methods.

In what follows, we shall work with the free energy function given by Eq. (6) with G eliminated in favor of n according to Eq. (7) and working with the parameters μ and b_R. For simplicity, this will be denoted by $\Phi_{b_R,\mu}(b, n)$ but with subscripts omitted unless absolutely necessary. Thus $\Phi(b, n)$ is now notation for the function

$$\Phi_{b_R,\mu}(b, n) := -\frac{1}{2}\mu n^2 - \alpha(b_R)b + \left(\frac{b+n}{2}\right)\log\left(\frac{b+n}{2}\right) + \left(\frac{b-n}{2}\right)\log\left(\frac{b-n}{2}\right) + (1-b)\log(1-b). \tag{15}$$

It is clear that the minimum of $\Phi(b, n)$ corresponds to the minimum of the original three variable free energy function $\Phi(n, b, G)$. In the following, will use the notation $n(\mu)$ (with $n(\mu) \geq 0$) as though this defines an unambiguous function. Of course in the case of phase coexistence, this will not be true. In general, then, $n(\mu)$ will stand for a representative from the set of minimizers at parameter value μ and all of the results in this subsection hold. We start with some elementary properties of the phase diagram generated by the corresponding minimization problem.

Proposition 4.1. *Consider $\Phi(b, n)$ with b_R fixed and μ ranging in $[0, \infty)$. Then for all μ sufficiently large, $\Phi(b, n)$ is minimized by a non-zero n and for all μ sufficiently small, Φ is minimized by $(b_R, 0)$.*

Proof. We begin with the assertion, gleaned from Eq. (8), that along the curves $n = 0$, Φ is minimized by $b = b_R$. Thus we may pick any fixed, nontrivial n_0, with $0 < n_0 < b_R$, and it is sufficient to establish that $\Phi(n_0, b_R) < \Phi(0, b_R)$ once μ is sufficiently large. However, the desired inequality is manifest for large μ since the only μ dependence in Φ is in the term $-\frac{1}{2}\mu n_0^2$ which is, eventually, in excess of the differences between the μ independent term and $\Phi(0, b_R)$. The second statement is proved as follows: since $b = 0$—which necessarily implies $n = 0$—does not minimize the free energy function, we may use the variable $\theta := n/b$ so that Eq. (10) now reads

$$b\mu\theta = \frac{1}{2}\log\left(\frac{1+\theta}{1-\theta}\right).$$

As is well known from the analysis of mean-field Ising systems (and can be established, e.g., by further differentiation) the above equation has only the trivial solution if $b\mu \leq 1$. Since b cannot be greater than one, the second statement has been proved—in fact whenever $\mu \leq 1$. □

The above result establishes, in a limited sense, the existence of a phase transition. Here we will sharpen this result by proving that along the lines of fixed b_R, there is a single transition from $n \equiv 0$ to $n > 0$. This is an immediate corollary to the following lemma which we state separately for future purposes.

Lemma 4.2. *Let $\Phi_\mu(b, n)$ denote the free energy function with b_R fixed and μ (displayed) in $[0, \infty)$. Then the minimizing $n(\mu)$, if unique, is a non-decreasing function of μ. More generally, if at various values of μ, Φ_μ has a minimizing set of n's then, if $\mu' > \mu$, the minimum of the minimizers at μ' is greater than or equal to the maximum of the minimizers at μ. Thus, in general any possible "choice" of $n(\mu)$ is non-decreasing.*

Proof. Let $\mu, \mu' \in [0, \infty)$ with $\mu' > \mu$ and let us denote by (b', n') a minimizing pair for $\Phi_{\mu'}$ and similarly for (b, n) at μ. The key observation is the meager μ–dependence of the function Φ_μ. Indeed, $\Phi_{\mu'}(x, y) = \Phi_\mu(x, y) - \frac{1}{2}(\mu' - \mu)y^2$. We do this twice:

$$\begin{aligned}\Phi_{\mu'}(b', n') &= \Phi_\mu(b', n') - \frac{1}{2}(\mu' - \mu)[n']^2 \\ &\geq \Phi_\mu(b, n) - \frac{1}{2}(\mu' - \mu)[n']^2 \\ &= \Phi_{\mu'}(b, n) - \frac{1}{2}(\mu - \mu')n^2 - \frac{1}{2}(\mu' - \mu)[n']^2 \end{aligned} \tag{16}$$

leading to $\Phi_{\mu'}(b', n') \geq \Phi_{\mu'}(b, n) + \frac{1}{2}(\mu' - \mu)(n^2 - [n']^2)$. This necessarily implies that $[n']^2 \geq n^2$ since otherwise, the previous inequality would be strict implying that (b, n) would have been a "better minimizer" for $\Phi_{\mu'}$ than (b', n'). □

Using this result we may now show the following

Corollary 4.3. *Consider the mean-field model defined by the free energy function given in Eq. (6). Then for each fixed $b_R \in (0, 1]$, there is a transitional value of μ, denoted by $\mu_T(b_R)$, such that $n \equiv 0$ for $\mu < \mu_T$ and $n > 0$ for $\mu > \mu_T$.*

Proof. This follows immediately from Proposition 4.1 and Lemma 4.2 above. □

Also of interest is the following:

Corollary 4.4. *Consider the mean-field model defined by the free energy function given in Eq. (6). Let $n(\mu)$ denote any non-negative function corresponding to a minimizing n at parameter value μ (usually uniquely determined). Then for $\mu \geq \mu_T$, the function $n(\mu)$ is strictly increasing.*

Proof. It is seen that *if* $n(\mu_T) = 0$ then the statement of this corollary is self-evident at $\mu = \mu_T$. For the rest of this proof, we may simply assume that μ is such that $n(\mu) > 0$. Suppose then that $\mu' > \mu$ and that $n = n(\mu)$ is part of the minimizing pair $(b(\mu), n(\mu))$ at parameter value μ. Suppose further that at μ' the same n is also part of a minimizing pair. Then we claim that the $b(\mu)$ is *not* the partner at μ' since given n'—purportedly equal to n—then b' is uniquely determined by Eq. (11). Upon performing some algebraic manipulations the latter reads $b' = n'/\tanh \mu' n'$. Thus, the equality $n = n'$ would lead to

$$b' = \frac{n'}{\tanh \mu' n'} = \frac{n}{\tanh \mu' n} \neq \frac{n}{\tanh \mu n} = b$$

so that explicitly (b, n) cannot be a minimizer at parameter value μ'. Using the appropriate $b' \neq b$, we would have

$$F_{\mathrm{MF}}(\mu') = \Phi_{\mu'}(n, b') = \Phi_\mu(n, b') - \frac{1}{2}(\mu' - \mu)n^2 \geq \Phi_\mu(n, b) - \frac{1}{2}(\mu' - \mu)n^2 = \Phi_{\mu'}(n, b)$$

in contradiction with the fact that (b, n) is not minimized for the parameter value μ'. □

4.3. A discontinuous transition for $b_R < \frac{1}{3}$

The dividing point of $b_R = \frac{1}{3}$ along the LSP-curve $\mu = 1/b_R$ is apparent from the higher order terms in Eq. (13). Indeed, supposing $\mu = 1/b_R + \varepsilon$ we obtain, with the additional aid of Eq. (12),

$$\varepsilon n = \frac{n^3}{2b_R^3}\left(b_R - \frac{1}{3}\right) + \cdots. \tag{17}$$

For $b_R \geq \frac{1}{3}$, Eq. (17) is consistent (and, as it turns out correct) but in the case of $b_R < \frac{1}{3}$ this equation alone precludes the possibility of a continuous transition. Indeed since we cannot have $n^2 < 0$, the only logical consequence of Eq. (17) is $n \equiv 0$ for $\mu \gtrsim \mu_S(b_R)$, i.e., the transition occurs later. But the lower order term *insisted* that $\mu = \mu_S(b_R)$ was the only viable candidate for a continuous transition. Thus: the transition cannot be continuous and, at least for $b_R < \frac{1}{3}$, the preliminary assumption that n goes to zero continuously can no longer be sustained. In particular, for $b_R < \frac{1}{3}$, perturbative analysis will never be valid because the relevant quantities will *never* be small.

This leaves open the possibility of a transition at some $\mu_T(b_R)$ that is different than μ_S. We shall show that $\mu_T < \mu_S$ as a direct consequence of the following:

Proposition 4.5. *Consider the mean-field model defined via the free energy function given by Eq. (6). Then, if $b_R < \frac{1}{3}$ at $\mu = \mu_S(b_R)$ the quantity n is strictly positive.*

Proof. We expand the free energy function $\Phi(n, b)$—with G eliminated via Eq. (7)—about $b = b_R, n = 0$ and along the curve $\mu = \mu_S(b_R)$. The convenient variables are now chosen as $b = b_R(1 + \Delta)$ and $n = b_R m$. We first note that all odd terms in m must vanish. In addition, the term linear in Δ vanishes due to the stationarity of Φ along the curve $\mu = \mu_S(b_R)$ and, as it turns out, so does the term which is quadratic in m. This leaves us with

$$\Phi(b, n) = \Phi(b_R, 0) + \frac{1}{2} b_R \left[\frac{1}{6} m^4 + \frac{1}{1 - b_R} \Delta^2 - m^2 \Delta \right] + \cdots.$$

Examining the quadratic form in the variables m^2 and Δ, the condition for a local minimum is that

$$\frac{1}{6} \frac{1}{1 - b_R} > \frac{1}{4},$$

i.e., $b_R > \frac{1}{3}$. We return to $b_R \geq \frac{1}{3}$ in the next subsection. Of current relevance is the fact that for $b_R < \frac{1}{3}$, the curve $\mu = b_R^{-1}$ is of a saddle point nature. This implies that there *is* a direction of decrease which, as is easily seen, is optimized, in the physical direction, when $m^2 = 3\Delta$. It is concluded that under the stated conditions, we can produce a pair (b, n) with $n^2 > 0$ (and $b > b_R$) such that the free energy for the non-trivial pair is lower; we just make the corresponding objects small enough to withstand the higher order corrections. Thus the actual minimum also must occur for non-trivial values of the n observable. □

We now have

Theorem 4.1. *Consider the mean-field system defined by the free energy function as given in Eq. (6). Then for $b_R < \frac{1}{3}$, there is a discontinuous transition at some positive $\mu_T(b_R) < \mu_S(b_R)$.*

Proof. That a transition occurs at *some* $\mu_T > 0$ is the statement of Corollary 4.3. Moreover, Lemma 4.2 and the above analysis implies $\mu_T \leq \mu_S$. The discussion prior to Proposition 4.5 demonstrates that at $\mu = \mu_T$, the quantity n is already positive. It only remains to show that the inequality relating μ_S and μ_T is strict. To this end, let us reimplement the heretofore unnecessary notation for the full dependence of the free energies on parameters. We have learned that for $b_R < \frac{1}{3}$, there is an $n_\star > 0$ and a $b_\star$ (with $b_\star > b_R$) such that

$$F_{\mathrm{MF}}(b_R, \mu_S) = \Phi_{b_R, \mu_S}(b_\star, n_\star) < \Phi_{b_R, \mu_S}(b_R, 0).$$

Invoking Lemma 4.2, it is now sufficient to show that there is a $\delta\mu > 0$ such that for some nonzero $\tilde{n}$, and some $\tilde{b}$, the inequality $\Phi_{b_R, \mu_S - \delta\mu}(\tilde{b}, \tilde{n}) < \Phi_{b_R, \mu_S - \delta\mu}(b_R, 0)$ can be shown to hold. Once again, the key is the simple dependence of the free energy functions on the parameter μ. Indeed, using $n_\star$ and $b_\star$ as trials, we obtain

$$\Phi_{b_R, \mu_S - \delta\mu}(n_\star, b_\star) = F^{\mathrm{MF}}_{b_R, \mu_S} + \frac{1}{2}[\delta\mu] n_\star^2$$

while $\Phi_{b_R, \mu_S - \delta\mu}(b_R, 0) \equiv \Phi_{b_R, \mu_S}(b_R, 0) < F^{\mathrm{MF}}_{b_R, \mu_S}$. Thus, the desired inequality will indeed hold for all $\delta\mu$ sufficiently small. □

4.4. A continuous transition for $b_R \geq \frac{1}{3}$

The starting point in our analysis is to show that at the purported critical curve, the quantity n actually vanishes.

Proposition 4.6. *For $b_R > \frac{1}{3}$ and $\mu = b_R^{-1} =: \mu_S$, the unique solution to the mean-field equations is $n = 0$ with $b = b_R$. In particular, $\Phi_{b_R, \mu_S}(b_R, 0) < \Phi_{b_R, \mu_S}(b, n)$ for any $(b, n) \neq (b_R, 0)$.*

Proof. Assuming $n > 0$ the agent fraction b can be eliminated in favor of the ratio

$$\theta := \frac{n}{b}.$$

Note that while this is the same substitution as before, here it is b rather than n that is being eliminated. Notwithstanding, θ still satisfies $0 < \theta \leq 1$. In these variables, the mean-field equations, Eqs. (10) and (11) respectively become

$$n = \frac{R\theta}{R + \sqrt{1 - \theta^2}} \tag{18}$$

$$n = b_R \mathrm{Arctanh}\ \theta \tag{19}$$

where in the above, $R := b_R/(1 - b_R)$. Let us now define $\ell(\theta)$ as

$$\ell(\theta) := \frac{1}{b_R} \frac{R\theta}{R + \sqrt{1 - \theta^2}} = \frac{(1 + R)\theta}{R + \sqrt{1 - \theta^2}} = \frac{(1 + R)\theta}{R + Q}$$

where $Q = Q(\theta) := \sqrt{1-\theta^2}$. To prove the current proposition we need to show that for all $\theta > 0$,

$$\operatorname{Arctanh}\theta > \ell(\theta)$$

demonstrating that there cannot be a non-trivial solution to the mean-field equations under the conditions stated. Note that for $0 < \theta \ll 1$ the desired inequality can be explicitly demonstrated. In general, it is sufficient to show, for $0 < \theta \leq 1$, that $\ell'(\theta) < 1/(1-\theta^2)$, i.e., in the Q variable that

$$\frac{1}{Q^2} > \frac{(1+R)(R+Q+(1-Q^2)/Q)}{(R+Q)^2}.$$

Although both sides diverge as $Q \to 0$ the divergence on the left hand side is clearly stronger so we actually only need consider $Q > 0$ limiting us to $Q \in (0, 1)$. After some manipulation, the inequality we need to prove is equivalent to

$$(R+Q)^2 > (1+R)(RQ^2+Q) = (1+R)(RQ^2+Q^2) + (1+R)(Q-Q^2).$$

That is, we now wish to show

$$R(R+2Q+RQ)(1-Q) > (1+R)Q(1-Q).$$

Since $Q \neq 1$, the above is equivalent to

$$R^2 + QR + R^2Q > Q.$$

Finally, since also $Q < 1$ it is enough to show that $2R^2Q^2 + RQ \geq Q$ i.e., that $2R^2 + R \geq 1$ which occurs for $R \geq \frac{1}{2}$. This corresponds to $b_R \geq \frac{1}{3}$. □

We can finally show

Theorem 4.2. *Consider the mean-field GI-system defined by the free energy function given in Eq. (6). Then, for $b_R \geq \frac{1}{3}$, as a function of μ with b_R fixed, there is a continuous transition at $\mu = \mu_S = 1/b_R$, i.e., $n(\mu) \equiv 0$ for $\mu < \mu_S$ and $n(\mu) > 0$ for the n-component of any minimizing pair $(n(\mu), b(\mu))$ while, if $\mu \downarrow \mu_S$, it is found that $n(\mu) \downarrow 0$.*

Proof. We will marshal the facts at our disposal and then proceed in a more abstract vein than has been the case in the more recent of our arguments. In what is to follow, $n(\mu)$ and the corresponding $b(\mu)$ is, once again, notation for a minimizing pair without any claims to uniqueness. By the preceding proposition, we know that at $\mu = \mu_S$, the quantity $n(\mu)$ is unambiguous and vanishes for $\mu < \mu_S$ by Lemma 4.2. Conversely, for $\mu > \mu_S$ we may write, adhering to the notation in the proof of Theorem 4.1, our usual expression:

$$\Phi_{b_R,\mu}(b, n) = \Phi_{b_R,\mu_S}(b, n) - \frac{1}{2}(\mu - \mu_S)n^2.$$

For $n^2 \propto b - b_R \ll 1$ from Proposition 4.5, we know that the quantity $\Phi_{b_R,\mu_S}(b, n)$ agrees with $\Phi_{b_R,\mu_S}(b_R, 0)$ up to quartic order in n. Thus allowing $n^2 \ll 1$ with $n^2(\mu - \mu_s) \gg n^4, (b - b_R)^2$ we find a non-zero n corresponding to a free energy lower than that of $\Phi_{b_R,\mu_S}(b_R, 0)$. Therefore, again by Lemma 4.2, we have $n(\mu) > 0$ for all $\mu > \mu_S$. It remains to establish that $n \downarrow 0$ as $\mu \downarrow \mu_S$. Note, that along any decreasing sequence of μ's the corresponding possible n's must be monotone by Corollary 4.4—or even Lemma 4.2—and hence $n \downarrow 0$ as $\mu \downarrow \mu_S$. Now let us suppose otherwise: that for some sequence of μ's decreasing to μ_S there is an associated sequence of minimizers, $(b(\mu), n(\mu))$ that has $n(\mu) \downarrow n_\star > 0$. Let $b_\star$ denote the associated limit for the $b(\mu)$ along a further subsequence if necessary. Since

$$\Phi_{b_R,\mu}(b(\mu), n(\mu)) < \Phi_{b_R,\mu}(b_R, 0) \equiv \Phi_{b_R,\mu_S}(b_R, 0)$$

we would have, by continuity, $\Phi_{b_R,\mu_S}(b_\star, n_\star) \leq \Phi_{b_R,\mu_S}(b_R, 0)$ indicating that *at* $\mu = \mu_S$, there is a minimizer with positive magnetization in contradiction with Proposition 4.6 above. It follows that, under the stated condition $b_R \geq \frac{1}{3}$, the limit of $n(\mu)$ is zero as $\mu \to \mu_S$ while it vanishes below and is positive above. By this (and any other) criterion, the transition at μ_S is continuous. This completes the proof. □

5. Discussion

In this work, we have formulated a lattice model for gang territoriality where red and blue gang agents interact solely through graffiti markings. Using a contour argument, we showed that a phase transition occurs between a well mixed, "high-temperature" phase and an ordered, "low-temperature" one as the coupling parameter J between gang members and graffiti becomes stronger while the graffiti evaporation parameter λ decreases. In the mean field limit of all-to-all lattice site couplings, we can also identify the tricritical point in phase space that distinguishes the occurrence of a continuous phase transition from a first order one. We find this point to be located at $b_R = 1/3$ which corresponds, in terms of the original variables of the problem, to the gang proclivity term $\alpha = -2\log 2$. In particular, for $b_R \geq 1/3$ the phase transition is continuous and occurs at $\mu = 1/b_R$. Thus, in the mean-field limit, for fixed $\alpha \geq -2\log 2$ the ordered "low temperature"

phase arises for $J^2 > \lambda/(e^{-\alpha}+2)$, and the "high temperature" one is attained on the other side of this inequality. The transition between the two occurs in a continuous manner across the $J^2 = \lambda/(e^{-\alpha}+2)$ locus. In the opposite case of $b_R < 1/3$ (or $\alpha < -2\log 2$) the phase transition is discontinuous. Here, we also are able to prove that the transition between high and low temperature phases occurs not at $\mu = 1/b_R$, but rather along the $\mu = \mu_T < 1/b_R$ curve, so that the phase change occurs earlier in J and along a separatrix $J^2 = J_c^2 < \lambda/(e^{-\alpha}+2)$.

In the context of gang–graffiti interactions, we may identify the low temperature, clustered phase as pertaining to a high level of antagonism between between rival gangs, where segregation leads to conflict along boundaries. Vice versa, the high temperature, well mixed configuration can be interpreted as a peaceful state, where despite different affiliations, gang members share the same turf. Our mean field results indicate that the confrontational state is surely attained, whether in a continuous or first order manner, for $J^2 > \lambda/(e^{-\alpha}+2)$, which represents high gang–graffiti territoriality J, low external intervention in graffiti removal λ and high proclivity α for individuals to become gang members. Gang clustering can be avoided by intervening in all three directions: by externally eliminating graffiti (λ), but also, from a deeper sociological point of view, by decreasing the lure of graffiti tags or of joining gangs in the first place (J, α). The emergence of a (continuous or discontinuous) phase transition shows that it is possible to obtain segregation in a lattice model without invoking direct agent-to-agent coupling; it is certain that adding such coupling terms to the Hamiltonian would allow for even more favorable segregation conditions.

Although our work was conceived within the context of gang interactions, the proposed model Hamiltonian and the tools used are general enough that our fundamental results may be applicable to several other contexts where territoriality is played out through markings and not through direct contact between players. Many animals, among which wolves, foxes and coyotes, are known to scent-mark their territories as a way of warning intruders of their presence and to exchange internal communication [61]. At times, buffer zones can originate between distinct animal clusters where prey species, such as deer or moose, may thrive [62]. Insects, such as beetles and bees, are also known to avoid previously marked locations as a way to optimize foraging patterns. Similarly to the role of gang graffiti markings, foreign scents lead "others" to retreat from already occupied turf or visited patches. Our work also applies to these contexts. Although some stochastic treatments have been recently presented [63], classical ecological studies of territoriality are usually carried out via reaction–diffusion equations where focal points such as dens, burrows or nests are often included [64–66], leading to segregation. Within this work on the other hand—whether first order or continuous—agent clustering is a natural consequence of a probabilistic treatment without the need to include any anchoring sites. Finally, we are able to connect local microscopic parameters—J, K, λ, α—to the emergence of large scale territorial patterns, be they gang clusters or animal groupings.

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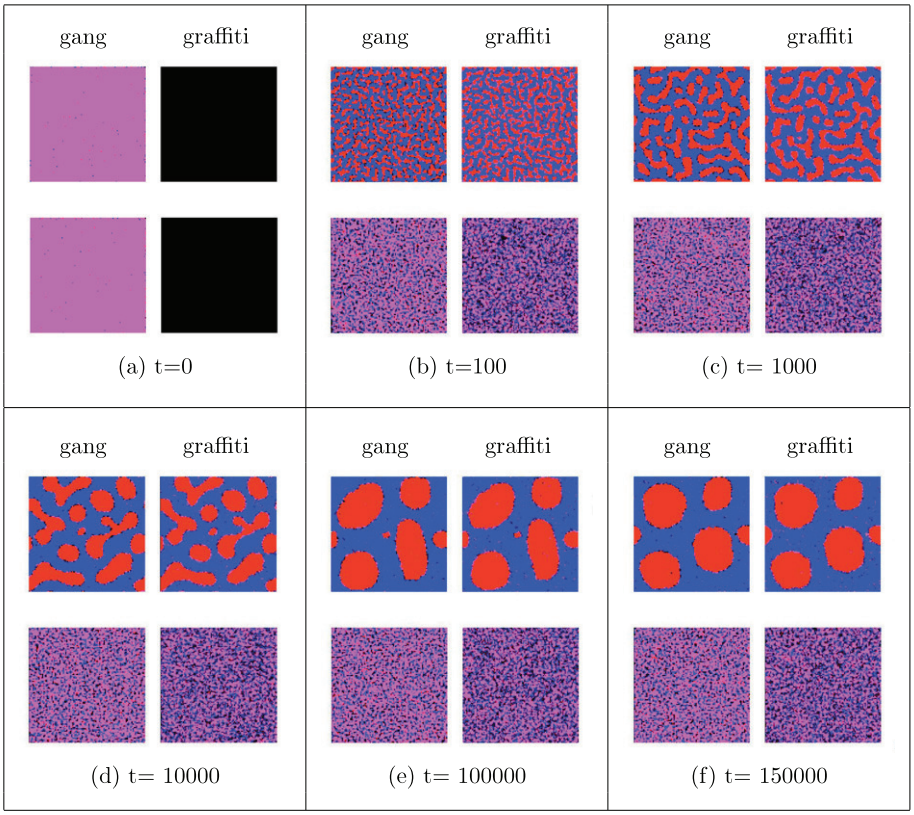
Appendix

Here we present a brief description of the informal simulations mentioned in Section 4. We consider a 100×100 square lattice with periodic boundary conditions initialized at $t = 0$ so that each site is populated with either red or blue agents, or a mixture of both. We assume a random distribution of 10^5 blue and 10^5 red agents and do not impose any restriction on the number of individuals on each site, so that multiple agents can occupy the same location at any given time. Initial conditions are completed by assuming that at $t = 0$ there is no graffiti present. At each time step of the simulation agents leave their graffiti on-site with a probability p_m that depends on the current graffiti level. In particular, $p_m = 0.1$ if the site is not marked by the opposite gang, and $p_m = 1$ otherwise. The agent then moves to any of its nearest neighbor sites j with probability $e^{-g_j}/\sum_j e^{g_j}$, where g_j is the amount of the opposite gang's graffiti at location j. Agents will thus preferentially relocate to nearest neighbor sites tagged by the least amount of the opposite gang's graffiti. Finally, at each site, graffiti is removed according to a probability p_g. Similarly to the number of agents, we impose no restriction on the amount of graffiti at each site.

While the simulation rules described here are similar in spirit to the model we analyze in this work, they do not directly lead to the Hamiltonian in Eq. (1). These informal simulations however provided us with a playing ground to investigate any phase transitions that may take place upon varying relevant parameters, such as the graffiti removal probability p_g. For example, in Table A.1, we track the dynamic progression of two sets of parameters. In the top row we set $p_g = 0.25$ so that 75% of the graffiti is retained at each iteration, while in the lower one we set $p_g = 0.75$ so that only 25% of the graffiti is kept. In Table A.1 the left and right hand side plots show agent and graffiti distributions, respectively. Red and blue pixels indicate site occupied by respective gang agents, black pixels represents no agents, and magenta shades indicate coexistence of both red and blue agents. Just as in the main body of this paper, we do not include any direct coupling between red and blue agents who interact only via the graffiti field. Similarly to what we later found in the main analysis, the degree of persistence of the graffiti field—which can be related to λ in the Hamiltonian in Eq. (1)—yields different qualitative behaviors and, if sufficiently large, may lead to aggregation patterns with distinct red and blue phases. The emergence of separate clusters from these simulations motivated the more extensive study presented in this work.

Table A.1
Snapshots of a Monte Carlo simulation of gang dynamics on a 100×100 square lattice with periodic boundary conditions. For each image in the sequence, the upper left panel represents gang agent populations while the upper right panel is the corresponding graffiti distribution. Iteration time is measured in arbitrary units. At $t = 0$, 10^5 red and 10^5 blue gang members are placed at random on the lattice with possible overlaps. Magenta indicates a mixture and black indicates a void in gang agents or graffiti. Agents tag their sites with probability $p_m = 0.1$ if the site is not marked by the opposite gang's graffiti and with probability $p_m = 1$ otherwise. In the upper panels of the table entries, $p_g = 0.25$ so that at each time step, graffiti will persist with a 75% possibility. The lower panels, where $p_g = 0.75$, mirror the upper ones but with a much lower graffiti persistence, of 25%. Note the different outcomes of the simulations at long times: when graffiti is allowed to persist longer, segregation occurs with the formation of islands of red and blue gangs. In this work, just as in our current model, there is no direct interaction between gang members, underlying the importance of the graffiti field as an indirect coupling between agents.



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