1	2	3	4	5	6

Name:________________________

Math 581, Midterm 1. November 2, 2006

1. Suppose an approximation $\hat{y}$ to $y = f(x)$ needs to be computed, where $f(x)$ is a scalar function of a scalar variable.

a) (7pt) Give the definition of the backward error for $y = f(x)$.

b) (8pt) If $f(x) = \dfrac{1}{x}$, and our calculations for $y = f(3)$ give $\hat{y} = 0.3$. What is the backward error involved in this calculation?

2. Using the backward error result for the vector inner product from Section 3.1 (formula (3.4)): for $x, y \in \mathbb{R}^n$,

$$fl(x^T y) = (x + \Delta x)^T y = x^T (y + \Delta y), \quad |\Delta x| \le \gamma_n |x|, \quad |\Delta y| \le \gamma_n |y|,$$

where $|x|$ denotes the vector with elements $|x_i|$ and inequalities between vectors hold componentwise;

a) (10pts.) prove the forward error bound (3.5):

$$|x^T y - fl(x^T y)| \le \gamma_n |x|^T |y|.$$

b) (10pts.) prove the following backward error result for matrix-vector product. Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $y = Ax$, then (see formula (3.11))

$$\hat{y} = (A + \Delta A)x, \qquad |\Delta A| \le \gamma_n |A|$$

3. (15pts.) Using the definition of γ_n from Section 3.1:

$$\gamma_n := \frac{nu}{1-nu},$$

where u is the unit roundoff, $nu \ll 1$. Prove that

$$u \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n.$$

4. (20pts.) Recall that

$$\|A\|_{\alpha,\beta} = \max_{x\neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha}.$$

Prove that for any subordinate norm $\|\cdot\|_\gamma$, the following inequality holds (see formula (6.7))

$$\|AB\|_{\alpha,\beta} \leq \|A\|_{\gamma,\beta}\|B\|_{\alpha,\gamma}.$$

Hint. One way to prove this is to use the trivial identity

$$\frac{\|ABx\|_\beta}{\|x\|_\alpha} = \frac{\|ABx\|_\beta}{\|Bx\|_\gamma}\frac{\|Bx\|_\gamma}{\|x\|_\alpha}.$$

5. (15pts.)

The quantity $y = n!$ is computed using the following algorithm:

$s = 1$
for $i = 2 : n$
$s = s * i$
end

Write this algorithm using the general framework of Section 3.8, for $n = 3$. Make sure to specify what is function $g_k(x_k)$, what are the components x_k, and what form does the matrix $\bar{I}$ takes in this particular example.

6. (15pts.) Let a real vector $x = (1/2, \sqrt{3}/2)$ be given. Find a real vector y dual to x in the $\| \cdot \|_2$-norm.

Recall that y is the dual vector to x in the 2-norm, if

$$y^T x = \|y\|_2^D \|x\|_2 = 1.$$

and the dual norm is defined by

$$\|y\|_2^D = \max_{x \neq 0} \frac{\|y^T x\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|y^T x\|_2$$