V1

1	2	3	4	5	6	7	8	$\sum$

Last Name:________________

First Name:________________

ID:________________ Section:____

Math 581 Final. December 21, 2006

Attention! Because this is an open book test, the emphasis will be given to ability to understand and implement the concepts. You are allowed to use a calculator, including a graphing calculator, however the numbers were kept simple to allow calculations by hand. In all problems, please, show all important steps in you solution/proof.

1. a) (2pt) How many significant digits are in the number 03.5000?

The two definitions of accuracy are given:

Def. A. An approximation $\hat{x}$ to x has p correct significant digits if $\hat{x}$ and x round to the same number to p significant digits. Rounding is the act of replacing a given number by the nearest p significant digits number with some rule for breaking ties when the are two nearest.

Def. B. $\hat{x}$ agrees with x to p significant digits if $|x - \hat{x}|$ is less than half a unit in the pth significant digit of x.

b) (4pt) To how many significant digits does 3.4991 approximate 3.5000 according to the **Def. A**

c) (4pt) To how many significant digits does 3.4991 approximate 3.5000 according to the **Def. B**

2. a) (10pt) Perform Rounding Error analysis for the problem of computing determinant of a 2×2 matrix in floating point arithmetic:

$$d = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Assume that the calculations in the determinant formula $\hat{d} = fl(\det(A)) := fl(a_{11}a_{22} - a_{12}a_{21})$ are done from left to right.

b) (5pt) Based on your answer in part a), is it possible to derive a backward error result for computing the quantity $d = det(A)$ in the form

$$\hat{d} = \det(A + \Delta A)?$$

3. (15pt) Prove that the subordinate matrix norm for the vector norm $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ is given by the formula ($A \in \mathbb{R}^{m\times n}$)

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

4. (15pt) (Refers to page 123) Use Theorem 7.4 to prove formula (7.11).

Hint. Recall Theorem 7.4:

Theorem 7.4. Let $Ax = b$ and $(A + \Delta A)y = (b + \Delta b)$, where $|\Delta A| \leq \epsilon E$ and $|\Delta b| \leq \epsilon f$, and assume that $\epsilon \| |A^{-1}|E \| < 1$, where $\| \cdot \|$ is an absolute norm. Then

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\epsilon}{1 - \epsilon \| \, |A^{-1}|E \, \|} \frac{\| \, |A^{-1}|(E|x| + f) \|}{\|x\|},$$

and for the ∞-norm this bound is attainable to first order in ϵ.

Recall the definition of the condition number:

$$\mathrm{cond}_{E,f}(A, x) := \lim_{\epsilon \to 0} \sup \left\{ \frac{\|\Delta x\|_\infty}{\epsilon \|x\|_\infty} \; : \; (A + \Delta A)(x + \Delta x) = b + \Delta b, |\Delta A| \leq \epsilon E, |\Delta b| \leq \epsilon f \right\}$$

Use Theorem 7.4 to prove that

$$\mathrm{cond}_{E,f}(A, x) = \frac{\| \, |A^{-1}|(E|x| + f) \|_\infty}{\|x\|_\infty}$$

5. a) (7pt) (Refers to page 142) Apply Lemma 8.4 to the problem of solving an upper triangular system $Ux = b$ by substitution: $x_i = (b_i - \sum_{j=i+1}^{n} u_{ij} x_j)/u_{ii}$.

Hint. Recall Lemma 8.4.:
Lemma 8.4. If $y = (c - \sum_{i=1}^{k-1} a_i b_i)/b_k$ is evaluated in floating point arithmetic, then, no matter what the order of evaluation,

$$b_k \hat{y}(1 + \theta_k^{(0)}) = c - \sum_{i=1}^{k-1} a_i b_i (1 + \theta_k^{(i)}),$$

where $|\theta_k^{(i)}| \le \gamma_k$ for all i. If $b_k = 1$, so that there is no division, then $|\theta_k^{(i)}| \le \gamma_{k-1}$ for all i.

Recall the substitution formula:

$$x_i = (b_i - \sum_{j=i+1}^{n} u_{ij} x_j)/u_{ii}$$

Apply Lemma 8.4 to the substitution formula

b) (8pt) Use the results of part a) to prove Theorem 8.5 in the case when T is upper triangular:

Theorem 8.5 Let the triangular system $Tx = b$, where $T \in R^{n \times n}$ is nonsingular, be solved by substitution, wth any ordering. Then the computed solution $\hat{x}$ satisfies

$$(T + \Delta T)\hat{x} = b, \qquad |\Delta T| \le \gamma_n |T|.$$

6. For the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \varepsilon & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}$$

Calculate the following in ∞-norm:

a) (5pt) $\kappa(A)$. Hint:

$$A^{-1} = \begin{pmatrix} 1 & -1/\varepsilon & 1 \\ 0 & 1/\varepsilon & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

b) (5pt) $\text{cond}(A)$.

7. (10pt) On the 3rd stage of Gaussian elimination with Complete Pivoting the following matrix has been obtained:

$$A^{(k)} = \begin{pmatrix} -7 & 0.3 & 0.1 & -0.2 & 0.3 & 4 \\ 0 & 0.5 & 0.1 & 0.3 & 0.1 & -0.2 \\ 0 & 0 & 2 & 0.2 & 0.6 & 0.5 \\ 0 & 0 & 0.5 & 3 & -5 & -3 \\ 0 & 0 & 0.4 & -0.3 & -2 & 1 \\ 0 & 0 & 1 & -0.1 & 4 & 0.1 \end{pmatrix}$$

What rows are and what columns have to be permuted to calculate the next step.

8. (15pt) (Refers to page 198) Use Theorem 10.3. to prove Theorem 10.4.:

Theorem 10.4. Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite and suppose Cholesky factorization produces a computed factor $\hat{R}$ and a computer solution $\hat{x}$ to $Ax = b$. Then,

$$(A + \Delta A)\hat{x} = b, \qquad |\Delta A| \leq \gamma_{3n+1}|\hat{R}^T||\hat{R}|.$$

Hint. Use Theorem 10.3.:

Theorem 10.3. If Cholesky factorization applied to the symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$ runs to completion then the computed factor $\hat{R}$ satisfies

$$|\hat{R}^T||\hat{R}| = A + \Delta A, \qquad |\Delta A| \leq \gamma_{n+1}|\hat{R}^T||\hat{R}|.$$

Recall that if $A = R^T R$, then the linear problem $Ax = b$ is equivalent to $R^T Rx = b$ which can be solved in two steps: $R^T y = b$ and $Rx = y$.

Use proof of Theorem 9.4 as the example.