Fundamental Theorem of Calculus. Part I

Connection between integration and differentiation.

Today we will discuss relationship between two major concepts of Calculus: integration and differentiation. We will show that these operations are inverse to each other. We will do so by defining a structure that allows to recover the function from its derivative using definite integral.

Let me proceed to the first slide.

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Motivation: The problem of finding antiderivatives.

Consider the following question:

Given a function, we can find its derivative, or differentiate it.

Now, given the derivative, can we find the function back? Can we antidifferenitate it?

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Definition. An antiderivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

In other words, given the function $f(x)$, you want to tell whose derivative it is.

Example 1. Find an antiderivative of 1.

The answer: An antiderivative of 1 is x.

Check by differentiation.

Example 2. Find an antiderivative of $\frac{1}{1+x^2}$.

The answer: An antiderivative of $\dfrac{1}{1+x^2}$ is $\arctan x$.

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How do you know?

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Some antiderivatives can be found by reading differentiation formulas backwards.

Indeed, according to the formulas from a calculus book, $\dfrac{1}{1+x^2}$ is the derivative of $\arctan x$

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However, no calculus book has formulas for

$$\sin(x^2), \qquad e^{(x^2)},$$
$$\sqrt{x+\sin^2(1-x^3)}$$

The question arises: do these functions have antiderivatives?

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An observation:

No matter what object's velocity $v(t)$ is, its position function $d(t)$ is always an antiderivative of $v(t)$, that is $d'(t) = v(t)$.

This suggests that all functions have antiderivatives.

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Suppose the speed of my car obeys $\sin(t^2)$ (do not try it on the road!). The car will move accordingly and the position of the car $F(t)$ will give the antiderivative of $\sin(t^2)$.

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A hypothesis:

Calculating antiderivatives must be similar to calculating position from velocity.

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We arrived to the second part of our discussion: naive derivation of Fundamental Theorem of Calculus.

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Let, at initial time t_0, position of the car on the road is $d(t_0)$ and velocity is $v(t_0)$.

LET ME SWITCH TO OVERHEAD TO MAKE A QUICK COMMENT:

At moment t_0, velocity of the car is $v(t_0)$. During the period of time Δt the car will travel approximately $v(t_0)\Delta t$.

Thus, the new position of the car is

$$d(t_1) \approx d(t_0) + v(t_0)\Delta t, \quad (t_1 = t_0 + \Delta t)$$

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Similarly, at time t_1 the velocity of the car is $v(t_1)$. During the period Δt the car will travel approximately $v(t_1)\Delta t$.

Position of the car after two moments of time is

$$d(t_2) \approx d(t_0) + v(t_0)\Delta t + v(t_1)\Delta t,$$
$$(t_2 = t_1 + \Delta t)$$

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Similarly, position of the car after n moments of time is

$$d(t_n) \approx d(t_0) + v(t_0)\Delta t + \ldots + v(t_{n-1})\Delta t$$

which can be written shorter using sigma notation:

$$= d(t_0) + \sum_{i=0}^{n-1} v(t_i)\Delta t$$

LET ME GO BACK TO COMPUTER SCREEN:

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Now let the number n of time steps before we reach certain moment of time t increases infinitely, which means that the size of a time step Δt decreases to 0; the expression becomes a limit:

$$d(t) - d(t_0) = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} v(t_i)\Delta t$$

Compare this limit to the definition on the definite integral:

$$\int_{t_0}^{t} v(\tau)\,\mathrm{d}\tau = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} v(t_i)\Delta t$$

Expressions in red coincide, therefore

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$$d(t) - d(t_0) = \int_{t_0}^{t} v(\tau)\,\mathrm{d}\tau$$

while $d(t)$ is being a position function corresponding to velocity $v(t)$, that is $d'(t) = v(t)$.

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Assuming $t_0 = 0$, $d(t_0) = 0$, distance can be calculated from velocity by

$$d(t) = \int_{0}^{t} v(\tau)\,\mathrm{d}\tau$$

First of all, is this a function in our regular sense?

Well, Yes. For each value t it defines a unique number $d(t)$

Then if $d'(t) = v(t)$?

Again, Yes.

This constitutes the assertion of Fundamental Theorem of Calculus.

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Fundamental Theorem of Calculus. Part I.

Let $f(x)$ be a continuous function (so, the definite integral of $f(x)$ exists). Then the function

$$F(x) = \int_{a}^{x} f(\tau)\,\mathrm{d}\tau.$$

is an antiderivative of $f(x)$, which is that $F'(x) = f(x)$.

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EXAMPLES

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Consider an example of evaluating an antiderivative of function

$$f(x) = \sin(x^2).$$

I WILL SWITCH TO OVERHEAD NOW.

According to the Fundamental Theorem of Calculus, Part I, the function

$$F(x) = \int_0^x \sin(\tau^2)\,\mathrm{d}\tau$$

is an antiderivative of $f(x) = \sin(x^2)$.

Indeed, according to the formula in the upper right corner, derivative of $F(x)$ is $\sin(x^2)$

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Example 4. Find an antiderivative of

$$f(x) = \mathrm{e}^{(x^2)}.$$

Again, we use the formula in upper right corner: substituting $\mathrm{e}^{(\tau^2)}$ for $f(\tau)$ we obtain an antiderivative of $\mathrm{e}^{(x^2)}$

$$G(x) = \int_0^x \mathrm{e}^{(\tau^2)}\,\mathrm{d}\tau$$

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We arrived to the last part of our discussion: the proof of Fundamental Theorem of Calculus. Part I.

LET ME SWITCH TO THE COMPUTER SCREEN

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We will be using two facts:

Interval Additive Property:

$$\int_a^b f(\tau)\,\mathrm{d}\tau = \int_a^c f(\tau)\,\mathrm{d}\tau + \int_c^b f(\tau)\,\mathrm{d}\tau$$

and Comparison Property: If $m \leq f(x) \leq M$ on $[a, b]$

$$m(b-a) \leq \int_a^b f(\tau)\,\mathrm{d}\tau \leq M(b-a)$$

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Let $F(x) = \displaystyle\int_a^x f(\tau)\,\mathrm{d}\tau$.

We will now show that $F'(x) = f(x)$.

By definition of derivative,

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{\int_a^{x+h} f(\tau)\,\mathrm{d}\tau - \int_a^x f(\tau)\,\mathrm{d}\tau}{h} \end{aligned}$$

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Notice, that by the Interval Additive Property, the expression in the numerator can be simplified:

$$\int_a^{x+h} f(\tau)\,\mathrm{d}\tau - \int_a^x f(\tau)\,\mathrm{d}\tau = \int_x^{x+h} f(\tau)\,\mathrm{d}\tau$$

Therefore,

$$F'(x) = \lim_{h\to 0} \frac{\int_a^{x+h} f(\tau)\,\mathrm{d}\tau - \int_a^x f(\tau)\,\mathrm{d}\tau}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(\tau)\,\mathrm{d}\tau$$

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Finally, if we define two numbers:

$$m = \min_{\tau\in[x,x+h]} f(\tau), \qquad M = \max_{\tau\in[x,x+h]} f(\tau)$$

Then the obvious inequality holds

$$mh \le \int_x^{x+h} f(\tau)\,\mathrm{d}\tau \le Mh$$

or, dividing by h,

$$m \le \frac{1}{h}\int_x^{x+h} f(\tau)\,\mathrm{d}\tau \le M$$

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Now by as we shrink the interval $[x, x+h]$ by considering limit as $h \to 0$. Both m and M converge (due to the continuity of $f(x)$) to the value of $f(x)$.

Therefore, by the Squeeze Theorem, the expression

$$m \le \frac{1}{h}\int_x^{x+h} f(\tau)\,\mathrm{d}\tau \le M$$

converges to

$$f(x) \le \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(\tau)\,\mathrm{d}\tau \le f(x)$$

which implies that

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(\tau)\,\mathrm{d}\tau = f(x)$$

The theorem is proved.