Sophisticated Graphing. Part B.

We will now consider one more example of sketching the graph of function by studying function's derivatives.

Let us from briefly reviewing the method.

I will switch to computer screen now.

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Summary of the method

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In the first step, we need to determine function's domain and range, check if function has any symmetries, find its X and Y-intercepts.

In step two, the first derivative of the function is analyzed to determine intervals where the function is increasing, decreasing, has max and min;

Second derivative is analyzed to determine where the function is concave up and where the function is concave down.

Finally, we need to check if the function has any asymptotes

In step three, some points are plotted. Usually we plot X and Y-intercepts, points of max and min, inflection points.

In step four, the plotted points are connected with a smooth curve in such a way to maintain all the properties found in step two.

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Example 2.

Sketch the graph of function $f(x) = 2/\sqrt{x} + x$.

We start from doing precalculus analysis:

The domain of this function is $x > 0$ (since negative value of x can not sow up under the square root, also at $x = 0$ the expression is undefined since we have division by 0).

The range of the function is $y > 0$.

This function is neither odd or even,

and does not have x or y-intercepts.

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Consider first derivative of function f.

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$$f'(x) = -\frac{1}{x^{3/2}} + 1 = \frac{x^{3/2} - 1}{x^{3/2}}$$

We can make sign analysis of $f'(x)$: the multiplier in the denominator is positive for $x > 0$, the numerator is zero at $x = 1$, less than zero for $0 < x < 1$ and greater than zero for $x > 1$.

HOW DO WE INTERPRET THE INFORMATION ABOUT THE FIRST DERIVATIVE?

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If the first derivative of a function is positive on an interval, the function is increasing on this interval.

If the first derivative of a function is negative on an interval, the function is decreasing on this interval.

Moreover, the function has local minimum if the derivative changes sign from negative to positive, and

the function has local maximum if the derivative changes sign from positive to negative.

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The function is decreasing from $x = 0$ to $x = 1$, and increasing from $x = 1$ to positive infinity.

At $x = 1$ the function has local minimum.

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Consider the second derivative of function f.

$$f''(x) = \frac{3}{2\sqrt{x^5}}$$

The second derivative is positive for $x > 0$.

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We know that is the second derivative is positive, the function is concave up.

If the second derivative is negative, the function is concave down.

Points, where the concavity changes are called inflection points.

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In our example, the function is concave up all the way for $x > 0$, there is no inflection points.

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The last step in calculus analysis is to find asymptotes (if there are any).

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A naive definition of an asymptote could be the following:

If the graph follows a line, we call this line an asymptote.

Mathematically, asymptote is a line that contains a significant portion of the graph in any its neighborhood.

We use asymptotes to study behavior of the graphs in some critical situations.

As $x \to \infty$, $x \to -\infty$, or near places where the function is undefined.

A vertical asymptote usually occurs when there is division by zero or a number is being raised into an infinite power.

Horizontal asymptote $y = L$ as $x \to \infty$ means that the function approaches level $y = L$ as $x \to \infty$, or that

$$\lim_{x\to\infty} f(x) = L$$

Similarly, we can discuss horizontal asymptotes at $x \to -\infty$.

A trickier question is to check whether the function has an oblique asymptote.

As the matter of fact, these is no general way of constructing the oblique asymptote,

However, let us consider two empirical formulas that can be used to find the asymptote.

Empirical means that existence of the asymptote is guaranteed as long as formulas are giving a meaningful answer,

However, if either of both formulas fail, it does not tell us yet the there is no oblique asymptote.

The formulas are obtained from the following consideration:

If the graph follows the line $y = mx + b$ as $x \to \infty$, this means that

$$f(x) \approx mx + b$$

Solving for m in this formula, in first case by dividing by x, in second case by taking the derivative, we obtain two empirical formulas for computing the slope m.

Once the slope m is computed, one finds b from

$$b = \lim_{x\to\infty} [f(x) - mx].$$

Note that to claim that the asymptote exists we need to find both m and b, it can be that m can be computed by one of the formulas and b can not.

Oblique asymptote for $x \to -\infty$ can be found in the same manner.

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We compute m by using the second formula:

$$m = \lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\Big[-\frac{1}{x^{3/2}} + 1\Big] = 1$$

Then

$$b = \lim_{x\to\infty}(f(x) - mx) = \lim_{x\to\infty}\Big[\frac{2}{\sqrt{x}} + x - 1\cdot x\Big] = 0$$

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Next we have to plot some points to our graph.

In out example the only point that we plot is the point of minimum at $x = 1$, $f(1) = 3$.

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Finally, we draw a smooth curve so as to maintain all the properties found in step 2.