Title: Increasing and decreasing functions, min and max, concavity.

Today we will give definitions of increasing/decreasing function, minimal and maximal values of the function and concavity. We will discuss how the derivative can be used to study these properties, and solve some examples.

Let me proceed to the first slide.

1 Slide 1

In the first part of our discussion we consider increasing and decreasing functions.

2 Slide 2

Definition.

Function $f(x)$ is increasing on interval I if for each pair $x_1, x_2 \in I$ such that

$$x_2 > x_1 \;\Rightarrow\; f(x_2) > f(x_1)$$

Note that the interval I can be open, close, or neither.

Similarly

Function $f(x)$ is decreasing on interval I if for each pair $x_1, x_2 \in I$ such that

$$x_2 > x_1 \;\Rightarrow\; f(x_2) < f(x_1)$$

Function is strictly monotonic on I if it is either increasing or decreasing.

3 Slide 3

The inequalities in definitions of monotonicity can be relaxed slightly.

Namely, we define function $f(x)$ to be non-decreasing on interval I if for each pair $x_1, x_2 \in I$ such that

$$x_2 > x_1 \;\Rightarrow\; f(x_2) \geq f(x_1)$$

Also, function $f(x)$ is non-increasing on interval I if for each pair $x_1, x_2 \in I$ such that

$$x_2 > x_1 \;\Rightarrow\; f(x_2) \leq f(x_1)$$

Function is non-strictly monotonic on I if it is either non-decreasing or non-increasing.

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Example 1. Function $\sin(x)$ is strictly monotonic on each interval

$$[-\pi/2 + k\pi, \pi/2 + k\pi],\; k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

It is increasing if k is an even integer.

$$\Big([-\pi/2 + k\pi, \pi/2 + k\pi],\; k = 0, \pm 2, \pm 4, \pm 6, \ldots\Big)$$

It is decreasing if k is an odd integer.

$$\Big([-\pi/2 + k\pi, \pi/2 + k\pi],\; k = \pm 1, \pm 3, \pm 5, \ldots\Big)$$

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Example 2. Function $\tan(x)$ is increasing on each interval

$$[-\pi/2 + k\pi, \pi/2 + k\pi],\; k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Note, that you still can't say that $\tan(x)$ increases everywhere!

Indeed, if we evaluate $\tan(x)$ for $x_1 = \pi/4$ and $x_2 = 3\pi/4$, then

$$x_2 > x_1 \quad \text{but} \quad \tan(x_2) < \tan(x_1)$$

6 Slide 6

Next we will discuss how to use derivative to study monotonicity.

7 Slide 7

Theorem A.

If $f(x)$ is increasing on interval I, and $f'(x)$ exists, then $f'(x) \geq 0$ on I.

Let us prove this statement.

Since $f(x)$ is increasing on I,

$$\frac{f(t) - f(x)}{t - x} > 0,$$

For choice of t both to the left and to the right of x. Therefore,

$$f(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \geq 0$$

Note that "$\geq$" can not be replaced with "$>$"!

As an example, consider function $f(x) = x^3$ which is increasing everywhere but $f'(0) = 0$.

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Theorem B. If $f'(x) > 0$ on I, $f(x)$ increases on I

This statement can be proved by using Lagrange's theorem which guarantees that for a differentiable function, for any pair of values $\forall x_1, x_2 \in I$, there exists value c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

As long as $f'(c) > 0$, inequality $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Note that the condition $f'(x) \geq 0$ does not imply that function $f(x)$ is increasing. Only $f'(x) > 0$ is sufficient for $f(x)$ to be increasing.

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Similar statements can be proven for decreasing functions

Theorem A.

If $f(x)$ is decreasing on interval I, and $f'(x)$ exists, then $f'(x) \leq 0$ on I.

Theorem B. If $f'(x) < 0$ on I, $f(x)$ decreases on I

10 Slide 10

Examples

11 Slide 11

Example 3

12 Slide 12

Example 4

13 Slide 13

Next we discuss concavity of the function.

14 Slide 14

We start from a question.

Consider two examples of increasing functions.

How to distinguish these two cases?

15 Slide 15

Definition $f(x)$ is concave up on I if $f'(x)$ increases on I.

Definition $f(x)$ is concave down on I if $f'(x)$ decreases on I.

16 Slide 16

Let us consider a simple test to check concavity.

If $f''(x) > 0$ then, $f'(x)$ is increasing, which means that the graph is Concave up

Similarly, if $f''(x) < 0$ then, $f'(x)$ is decreasing, which means that the function is Concave down

Points, where concavity changes are called inflection points.

Example 5:

Where the graph of $f(x) = x^3 - 1$ is concave up, concave down, has inflection point?

Consider second derivative of function $f(x)$ which is $f''(x) = 2x$.

$f''(x) < 0$ for $x < 0$ which means that the function is concave down for $x < 0$ and $f''(x) > 0$ for $x > 0$ which means that the function is concave up.

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Example 6. (OVERHEAD)

18 Slide 18

We arrive to the last part of today's talk: Minima and Maxima, critical points, first and second derivative tests.

19 Slide 19

Definition. $f(c)$ is a local maximum value of $f(x)$ if there exists an interval (a, b) containing c such that $\forall x \in (a, b)$, $f(c) \geq f(x)$.

Definition. $f(c)$ is a local minimum value of $f(x)$ if there exists an interval (a, b) containing c such that $\forall x \in (a, b)$, $f(c) \leq f(x)$.

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Critical Point Theorem. If $f(c)$ is a local min (max), then c is a critical point, that is
a) an end point
b) a stationary point, that is $f'(c) = 0$
c) a singular point, that is $f'(c)$ does not exists

a) and c) are proved on examples.

Let us prove b)

21 Slide 21

Proof of the statement b)

If $f(c)$ is max, then $f(t) - f(c) < 0$ to the right of c, $t - c > 0$ to the right of c, and

$$\frac{f(t) - f(c)}{t - c} < 0, \quad x > c.$$

Also, $f(t) - f(c) < 0$ to the left of c, $t - c < 0$ to the left of c and

$$\frac{f(t) - f(c)}{t - c} > 0, \quad x < c,$$

Which implies that

$$\lim_{t \to c+} \frac{f(t) - f(c)}{t - c} \leq 0, \quad \lim_{t \to c-} \frac{f(t) - f(c)}{t - c} \geq 0,$$

$$f'(c) = \lim_{t \to c} \frac{f(t) - f(c)}{t - c} = 0$$

The case when $f(c)$ is min cab be proved similar

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First Derivative Test

if $f'(x) > 0$ to the left of c, and $f'(x) < 0$ to the right of c, this means that

$f(x)$ increases to the left of c and decreases to the right of c, this means that

$f(x)$ can not be bigger than $f(c)$ from the left and can not be bigger than $f(c)$ from the right from c, which means that $f(x)$ has maximal value at $x = c$.

Similarly, you can prove that

if $f'(x) < 0$ to the left of c, and $f'(x) > 0$ to the right of c, this means that

$f(x)$ decreases to the left of c and increases to the right of c, this means that

$f(x)$ can not be smaller than $f(c)$ from the left and can not be smaller than $f(c)$ from the right from c, which means that $f(x)$ has minimum value at $x = c$.

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Second Derivative Test

if $f''(c) < 0$ and $f'(c) = 0$ it implies that

$f'(x)$ is decreasing near c and passing through 0 at c which implies that

$f'(x) > 0$ to the left and $f'(x) < 0$ to the right of c which implies that

$f(x)$ increases to the left, decreases to the right of c which implies that

$f(x)$ has max at $x = c$.

Similarly, you can prove that

if $f''(c) > 0$ and $f'(c) = 0$ it implies that

$f'(x)$ is increasing near c and passing through 0 at c which implies that

$f'(x) < 0$ to the left and $f'(x) > 0$ to the right of c which implies that

$f(x)$ decreases to the left, increases to the right of c which implies that

$f(x)$ has min at $x = c$.

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Examples

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Example 7.

26 Slide 26

Example 8.