Title: Tangent Line, Velocity, Derivative and Differentiability.

Today's discussion is about the most important tool of mathematical analysis — derivative and differentiation. We will give definition of derivative and consider examples of how to evaluate slope of the tangent line and differential of the function.

Let me proceed to the first slide.

1 Slide 1

We start today's discussion with a question: what do tangent line and velocity have in common.

To answer this question we first need to recall how do we define tangent line and velocity.

2 Slide 2

Euclid's notion of tangent line as a line that touches the curve at just one point works for circles, but fails on many other interesting curves.

How to define tangent line?

3 Slide 3

One way to define tangent line is to consider the limiting position of the secant connecting points $(x, f(x))$ and $(t, f(t))$ on the graph of $y = f(x)$ as point $(t, f(t))$ moves toward $(x, f(x))$.

If the limiting position exists, it is taken for the tangent line.

If the tangent is the limiting position of the secant, slope of the tangent line is the limiting value of the slope of the secant, which yields the formula

$$\text{Slope of the tangent} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

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Example

5 Slide 5

Find slope of the tangent line to the graph of

$$f(x) = x^2 - 4$$

at point $(3, 5)$

First of all, we check that $(3, 5)$ is on the graph. Indeed, $f(3) = 5$.

Then we write the expression for the slope of the tangent line

$$\begin{array}{l}\text{Slope of} \\ \text{Tangent}\end{array} \qquad = \lim_{t \to x} \frac{(t^2 - 4) - (x^2 - 4)}{t - x}$$

Cancelling 4 in the numerator and factoring the result we calculate limit to be $2x$

$$= \lim_{t \to x} \frac{t^2 - x^2}{t - x} = \lim_{t \to x} \frac{(t - x)(t + x)}{t - x} = 2x$$

At $x = 3$,

$$\text{Slope of Tangent} = 6$$

(value of $y = 5$ is irrelevant)

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Now let us define the velocity.

Imagine an object moving along a straight line.

At each moment x location of the object is given by function $s(x)$

Over a little interval h, the displacement of the object is calculated as $s(x+h)-s(x)$, and the average velocity as $(s(x+h)-s(x))/h$.

If
$$\lim_{h\to 0}\frac{s(x+h)-s(x)}{h}$$
exists, it is called the instantaneous velocity of the object at moment x.

7 Slide 7

Example

8 Slide 8

Motion of the particle along a line is described by $s(x)=1+\cos(x)$. Find instantaneous velocity at moment $x=2$.

We start from writing down definition of instantaneous velocity:
$$\begin{array}{l}\text{Instant.}\\ \text{Velocity}\end{array} \qquad = \lim_{h\to 0}\frac{(1+\cos(x+h))-(1+\cos(x))}{h}$$
Cancelling 1 we obtain
$$= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

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$$= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$
(using $\cos(x+h)=\cos(x)\cos(h)-\sin(x)\sin(h)$)
$$= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

by re-grouping the last expression,

$$= \cos(x) \lim_{x \to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{x \to 0} \frac{\sin(h)}{h}$$

$$= -\sin(x)$$

At $x = 2$, Instantaneous Velocity $= -\sin(2)$.

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Now compare the two expressions:

$$\text{Slope of Tangent} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$\text{Instant. Velocity} = \lim_{\tau \to 0} \frac{s(x + h) - s(x)}{h}$$

The two expressions are equivalent up to the substitution

$$f = s, \qquad t - x = h$$

The question arises: Is this really the same thing?

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We arrived to the second part of our discussion: Definition of the Derivative.

12 Slide 12

The object that unites Tangent Line, Velocity, and miscellaneous Rates of Changes is called the Derivative.

Definition. Derivative of function $f(x)$ at point x (denoted by $f'(x)$ is the following limit

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

Note that with this formula we also have the second formula

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

which reduces to the first one by the substitution $h = t - x$.

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The geometrical meaning of the derivative is that derivative of a function at a point gives slope of the tangent line at that point.

Also, derivative of the position function gives the velocity.

Finally, derivative is used to calculate miscellaneous rates of change in chemistry, biology, etc.

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Examples

15 Slide 15

16 Slide 16

17 Slide 17

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19 Slide 19

We arrived to the third and the last part of our today's discussion: Differential and Differentiability.

20 Slide 20

Consider a problem of approximating function with a line.

We look at the picture at the bottom of the slide.

Consider a point moving along the line in gray

Suppose at moment x_0 the point is at height y_0

Let us calculate the increment in height corresponding to an increment h in variable x.

If the slope of the line is m, the change in y is calculated as $m \cdot h$

Finally, the height y of the point after the displacement h is calculated as the old height y_0 plus the increment $m \cdot h$.

Let us use this formula to approximate a function near the moment x by a line.

The original height is $f(x)$, the displacement when moving along a line with the slope m is $m \cdot h$, finally, the error of our approximation is $o(h)$ (reads "o-little of h").

We want this error to be small.

What do we mean by small?

We require that $o(h)$ must be a quantity of order of magnitude smaller then h, which can be expressed by the fact that

$$\lim_{h \to 0} \frac{o(h)}{h} = 0.$$

Indeed this means that $o(h)$ is weaker then h itself, when $h \to 0$.

This implies that contribution of $o(h)$ is negligibly small comparing to the contribution of $m \cdot h$ when near the moment x.

This implies that m has to be a very special number.

What is m?

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Definition:

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