Title: Limit at infinity and infinite limits.

Today we will learn how to analyze mathematically behavior of functions at infinite values of variable and at places where the function is infinite. We will extend definition of limits to new cases: limits at infinity and infinite limits, and consider examples of how to evaluate them.

Let me proceed to the first slide.

1 Slide 1

Let me motivate today's discussion with the question: how to describe infinite values of function or its variable in a way suitable for analysis?

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Indeed, if there is a precise way to say: 1) value of variable is infinite, or 2) value of the function is infinite.

Analyzing the two situations can bring an important insight on function's behavior.

In particular, it could help us define horizontal and vertical asymptotes.

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Let us define the limit at infinity, or as variable approaches infinite values.

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Infinite means not finite.

This means that for any big number M is given, the variable x is bigger than M.

Here we arrive to the definition of limit at infinity:

Definition:

The number L is the limit of function $f(x)$ as x goes to ∞ if and only if for any positive number ε there exists a positive number N such that as long as $x > n$ it implies that $|f(x) - L| < \varepsilon$

In other words, function is in ε-corridor at certain value of variable and never leaves ε-corridor after this value.

In particular, function

$$\frac{5x+1}{x-2}$$

never leaves ε-corridor around 5 starting from $x = 2 + 11/\varepsilon$

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Notice that all limit laws can be extended to the limits at infinity using the definition given above.

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Let us consider some examples of finding limits at infinity.

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Evaluate limit

$$\lim_{x\to\infty} \frac{1}{x}$$

As variable x gets larger, $1/x$ gets smaller because 1 is being divided by a larger, larger, and even larger number.

The limit is 0

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Similarly, limits

$$\lim_{x\to\infty} \frac{1}{x^n}$$

and

$$\lim_{x\to\infty} \frac{1}{\sqrt[n]{x}}$$

are both zero since their denominators are large numbers as x grows large.

By a similar argument,

$$\lim_{x\to\infty} \frac{1}{x-100} = 0, \quad \lim_{x\to\infty} \frac{1}{\sqrt[n]{x-10000}} = 0$$

Indeed, in the first one, x grows large.

At certain moment x will be greater then 100 and will keep growing.

So, the limit is 0.

The same is true for the second one.

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Example 3.

Evaluate limit

$$\lim_{x\to\infty} \frac{1}{\sqrt[n]{x^2+x-C}}$$

Consider the graph of $y = x^2 + x - C$

As $x \to \infty$, the value of the function $y \to \infty$.

The limit is 0. Value of the constant is unimportant.

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Example 4.

Let us evaluate the limit

$$\lim_{x\to\infty} \frac{1}{x^2 - x - 1}$$

without evoking graphs.

A simple argument on having a large number in denominator is not immediately clear, since we have a subtraction of two big numbers: x and x^2.

Still, it is not difficult to evaluate this limit.

First rewrite it as

$$= \lim_{x\to\infty} \frac{1}{x^2\left(1 - \frac{1}{x} - \frac{1}{x^2}\right)}$$

Then, (LET ME SWITCH TO OVERHEAD NOW): $1/x$ and $1/x^2$ is small so the value of the second multiplier is approximately 1.

The value of the first multiplier is large.

The answer is zero.

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Example 5. (STILL ON OVERHEAD)

Evaluate the limit

$$\lim_{x\to\infty} \frac{5x^3 - 2x^2 - 1}{x^3 - x + 1}$$

The standard trick is to divide both numerator and denominator by the highest exponent of x:

$$= \lim_{x\to\infty} \frac{\dfrac{5x^3}{x^3} - \dfrac{2x^2}{x^3} - \dfrac{1}{x^3}}{\dfrac{x^3}{x^3} - \dfrac{x}{x^3} + \dfrac{1}{x^3}} = \lim_{x\to\infty} \frac{5 - \dfrac{2}{x} - \dfrac{1}{x^3}}{1 - \dfrac{1}{x^2} + \dfrac{1}{x^3}}$$

$$= \frac{5-0-0}{1-0+0} = 5$$

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Example 6. (STILL ON OVERHEAD)

Evaluate the limit

$$\lim_{x\to\infty} \frac{x-2}{\sqrt{2x^2 - x + 1}}$$

Again, the suggestion is to divide both numerator and denominator by the highest degree of x

$$= \lim_{x\to\infty} \frac{\dfrac{x}{x} - \dfrac{2}{x}}{\dfrac{\sqrt{2x^2 - x + 1}}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\dfrac{x}{x} - \dfrac{2}{x^3}}{\sqrt{\dfrac{2x^2}{x^2} - \dfrac{x}{x^2} + \dfrac{1}{x^2}}}$$

in this example we distribute $1/x$ inside the square root by writing it as $1/\sqrt{x^2}$.

$$= \frac{1-0}{\sqrt{2-0+0}} = \frac{1}{\sqrt{2}}.$$

WE NOW GO BACK TO COMPUTER SCREEN

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Let us revisit an important topic of theory of limits: Indeterminacy.

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Indeterminacy is a result of bad breaking in the limit expression in a way that limit laws and substitution theorems do not apply.

In most cases indeterminacy can be eliminated by re-grouping terms.

Here we give examples of four typical indeterminacies: ∞/∞, $0/0$, $0 \cdot \infty$, $\infty - \infty$

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The last topic of our discussion is infinite limits.

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We define infinite value of function by saying that for any number M given, there exist a value of variable x such that that $f(x)$ is bigger than M

Definition:

Limit of function $f(x)$ as x approaches c is equal to positive infinity if and only if, for any positive number M there exists a positive number δ such that as soon as $0 < |x - c| < \delta$ the value of the function $f(x)$ is bigger than M.

In other words, there exists a δ-corridor inside which the function is above M.

In particular, function

$$\frac{1}{(x-2)}^2$$

is above level M as long as x is in δ-corridor with $\delta = 1/\sqrt{M}$.

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As an exercise, let us develop definitions for

$$\lim_{x \to c} f(x) = -\infty, \qquad \text{and} \lim_{x \to \infty} f(x) = \infty,$$

If being bigger then any positive number M defines positive infinity, negative infinity should denote being less then any negative number M.

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