Title: Limit of a function, ε–δ language.

Today we will discuss limits of functions: what is their meaning, why are they important, and what mathematical language is used to work with them.

1 Slide 1

As a motivation for today's discussion consider the following question: if the value of the function is undefined at some point, can we still study function's behavior near that point?

More important, is it worth studying function's behavior near points where the function is undefined?

2 Slide 2 (TWO GRAPHS)

Consider the following example.

Two functions: $(x^2+1)/(x-1)$ and $(x^2-1)/(x-1)$ are both undefined at $x=1$.

However, they behave quite differently near this point.

The fist one goes to infinity to the right of $x=1$ and to negative infinity to the left of $x=1$ while the second one does nothing of this sort.

As variable x approaches value 1, the value of the second function approaches 2.

If we want to distinguish these two cases, we have to consider what happens to the function not at $x=1$ but as x is near but not equal to 1.

The difference can be big!

3 Slide 3

Consider the second function, again.

Near value $x = 1$ the second function is approximately 2.

It does not vary wildly or reach infinite values.

It behaves nice.

How about to come up with a mathematical description of such behavior?

A suggestion could be, simply to say that:

"As x is near 1, the value of the function is near 2."

Indeed, it looks that this definition fits nicely.

But the problem arises immediately as you try to put some numbers into this definition:

4 Slide 4

HOW EXACTLY NEAR?

How many inches, centimeters, or pounds, or seconds near?

This near?

This near?

Or, may be, this near?

5 Slide 5

In fact, it should be infinitely near.

Infinitely near!

Oh, great! Infinitely near!

Now, wait a second...

6 Slide 6

What is infinitely near?

Indeed, how do we describe in mathematics the fact that x is infinitely close to 1?

7 Slide 7

Here we arrive to the second part of our discussion:

ε–δ language.

Definition of the Limit.

8 Slide 8

DEFINITION. The number L is the limit of the function $f(x)$ as x approaches c if and only if for any positive number ε there exist a positive number δ (depending on ε) such that as long as x is not equal to c but differs from c by less then δ, it implies that $f(x)$ differs from L by less then ε.

This statement can be written shorter using mathematical notations.

9 Slide 9

Attention.

I now will read the formula in blue.

DEFINITION. The number L is the limit of the function $f(x)$ as x approaches c if and only if for any positive number ε there exist a positive number δ such that as long as x is not equal to c but differs from c by less then δ, it implies that $f(x)$ differs from L by less then ε.

Let us consider this definition by parts.

10 Slide 10

The part $0 < |x - c| < \delta$ has two statements in it, one for each inequality sign.

The left inequality sign denotes that distance between x and c is less then δ.

This fact can be expressed in the inequality $-\delta < x - c < \delta$, or $c - \delta < x < c + \delta$,

The left inequality sign means that x is not equal to c, which indicates that we never consider the point c itself when study limits.

11 Slide 11

Similarly, inequality $|f(x) - L| < \varepsilon$ means that $f(x)$ differs from L by less then ε.

This fact can be expressed in the inequality $-\varepsilon < f(x) - L < \varepsilon$, or $L - \varepsilon < f(x) < L + \varepsilon$.

12 Slide 12

Now let us combine two parts together:

Inequality $L - \varepsilon < f(x) < L + \varepsilon.$ can be thought as the ε-corridor on Y-axis.

While the inequality $c - \delta < x < c + \delta$ is the δ-corridor on X-axis.

Together two parts read: as x is in δ-corridor, $f(x)$ is in ε-corridor.

Notice, that with each *delta*-corridor, you get infinitely many δ-corridors, which are narrower, that fit into the definition.

13 Slide 13

Consider the first part of the expression.

The grand challenge is that you have to construct δ for each choice of ε.

This means that we will have to consider infinitely many combinations ε–δ.

It is highly desirable then to construct a formula for computing δ from ε.

Then, building couples ε–δ will be just a matter of plugging numbers.

From the other side, if there is no such formula and we fail to find δ for just a single choice of ε, there is no limit.

14 Slide 14

Let us consider situations when there is no limit.

The definition of the limit do not apply in the first case since for each value L we can imagine, there exists a particular choice of ε (in green) such that functions leaves ε-corridor no matter how small the value of δ is.

In the second example, the function approaches two different values as it approach c from the left and from the right.

Again, there are choices of ε for which the function leaves ε-corridor either from the left or from the right of c

In the third example, the function oscillates wildly as x approaches c.

Again, for any value L, there exists a particular choice of ε such that functions leaves ε-corridor no matter how close x is to c.

15 Slide 15

Let us consider some examples of proving limits using ε-δ language.

16 Slide 16

Prove by ε-δ argument that

$$\lim_{x \to 2}(7x + 1) = 15$$

We start from writing down the definition of the limit as it applies to this example:

c is being replaced with 2,

$f(x)$ is being replaced with $(7x + 1)$,

L is being replaced with 15,

Summarizing,

$$\Leftrightarrow \quad \forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - 2| < \delta \Rightarrow |(7x + 1) - 15| < \varepsilon$$

The goal is to choose δ in a such way to guarantee $|(7x + 1) - 15| < \varepsilon$

17 Slide 17

Consider the expression $|(7x + 1) - 15|$.

It can be simplified to $|7x - 14|$.

And then to $|7(x - 2)|$, or $|7||x - 2|$, which is equal to $7|x - 2|$ since $|7| = 7$.

We want this expression to be less then ε.

The desired inequality follows from the assumption $|x - 2| < \delta$ if $\delta \leq \varepsilon/7$.

In particular, one can pick $\delta = \varepsilon/7$ as an answer.

18 Slide 18

Prove by ε-δ argument that

$$\lim_{x \to 5} \frac{x^2 - 25}{(}x - 5) = 10$$

We start from writing down the definition of the limit as it applies to this example:

c is being replaced with 5,

$f(x)$ is being replaced with $(x^2 - 25)/(x - 5)$,

L is being replaced with 10,

Summarizing,

$$\Leftrightarrow \quad \forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - 5| < \delta \Rightarrow \left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon$$

The goal is to choose δ in a such way to guarantee

$$\left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon.$$

19 Slide 19

Consider the expression

$$\left|\frac{x^2 - 25}{x - 5} - 10\right|$$

It can be rewritten as to

$$\left|\frac{(x+5)(x-5)}{x-5} - 10\right|.$$

And then simplified in into $|x + 5 - 10|$, or $|x - 5|$.

We want this expression to be less then ε.

The desired inequality follows from the assumption $|x - 5| < \delta$ if $\delta \leq \varepsilon$.

In particular, one can pick $\delta = \varepsilon$ as an answer.