Increasing and Decreasing Functions, Min and Max, Concavity

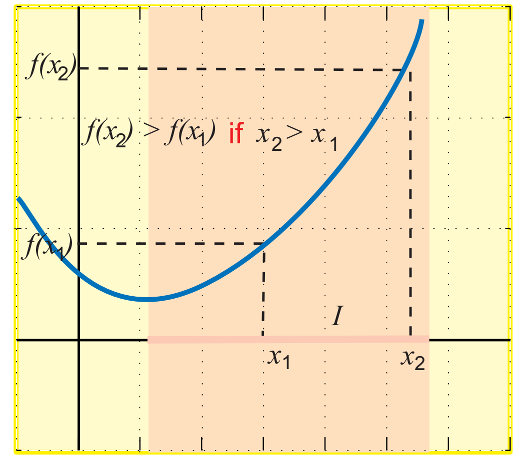
studying properties of the function using derivatives

Increasing and Decreasing Functions

characterizing function's behaviour

Definition: $(I = [,], (,), [,), (,])$

$f(x)$ is increasing on I if for each pair $x_1, x_2 \in I$



$$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$$

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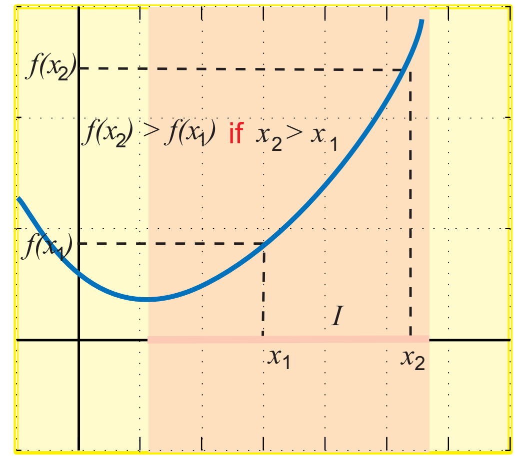

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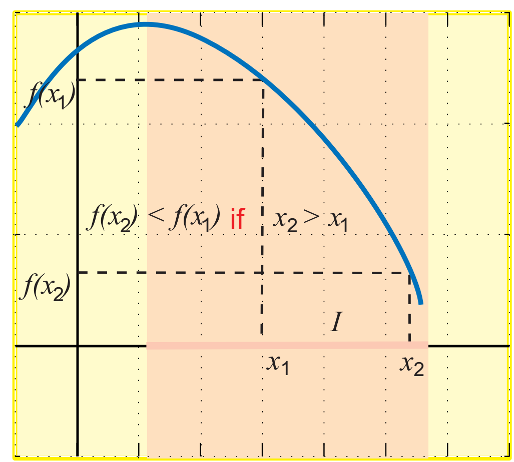
Definition:

$f(x)$ is decreasing on I if for each pair $x_1, x_2 \in I$

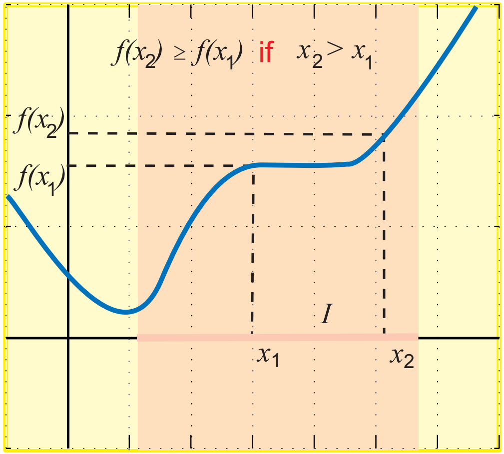


$$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$$

Increasing/decreasing = strict monotonicity

Definition: $(I = [,], (,), [,), (,])$

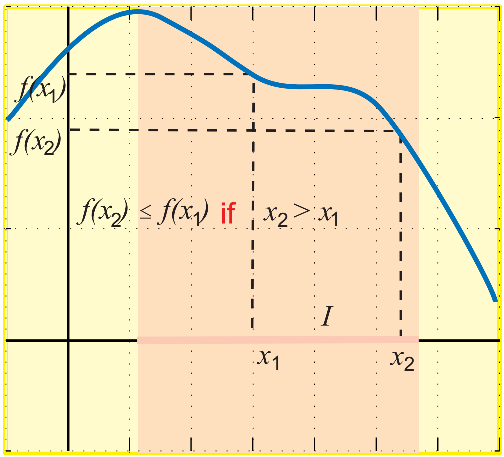
$f(x)$ is non-decreasing on I if for each pair $x_1, x_2 \in I$



$$x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$$

Definition:

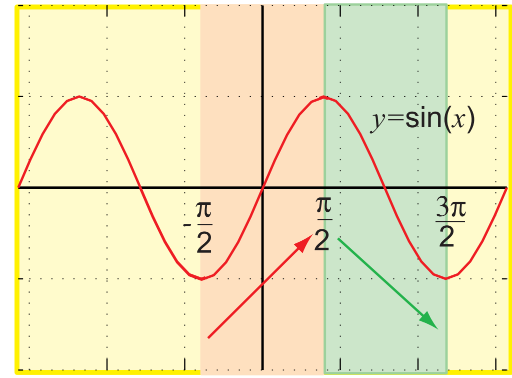
$f(x)$ is non-increasing on I if for each pair $x_1, x_2 \in I$



$$x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$$

Non-decreasing/increasing = non-strict monotonicit

Example 1. Function $\sin(x)$ is strictly monotonic on each interval



$$[-\pi/2 + k\pi, \pi/2 + k\pi], \ k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

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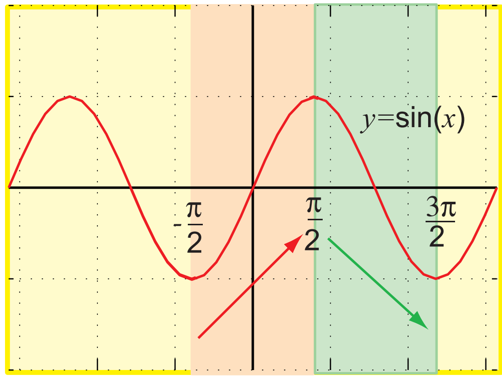

$$[-\pi/2 + k\pi, \pi/2 + k\pi], \ \ k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

It is increasing on

$$[-\pi/2 + k\pi, \pi/2 + k\pi], \ \ k = 0, \pm 2, \pm 4, \pm 6, \ldots$$

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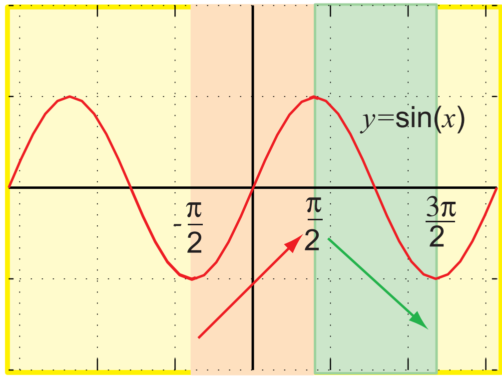

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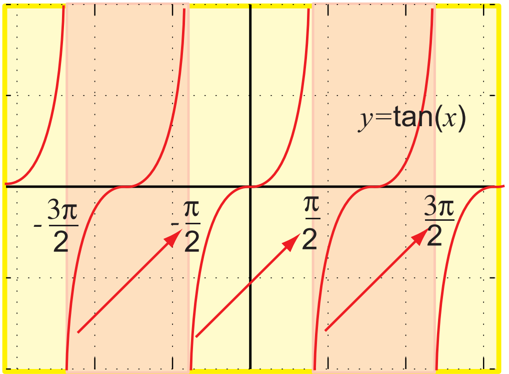
It is decreasing on

$$[-\pi/2 + k\pi, \pi/2 + k\pi], \;\; k = \pm 1, \pm 3, \pm 5, \ldots$$



Example 2. Function $\tan(x)$ is increasing on each interval

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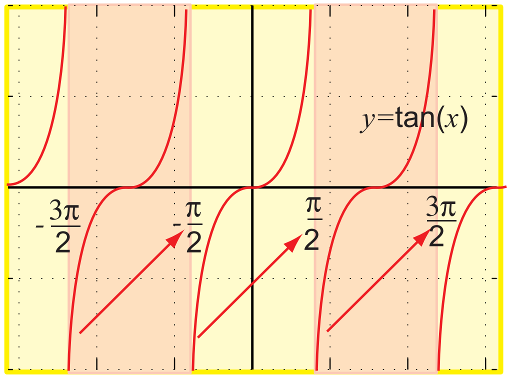

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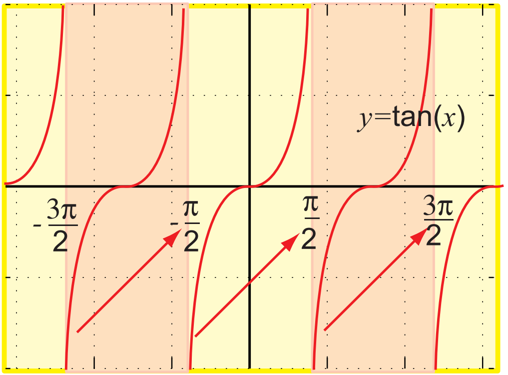

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Indeed, for $x_1 = \pi/4$ and $x_2 = 3\pi/4$,

$$x_2 > x_1 \quad \text{but} \quad \tan(x_2) = -1 < \tan(x_1) = 1$$

Derivative and monotonicity

What derivative can tell about the function?

Theorem A. If $f(x)$ is increasing on I, and $f'(x)$ exists, then $f'(x) \geq 0$ on I.

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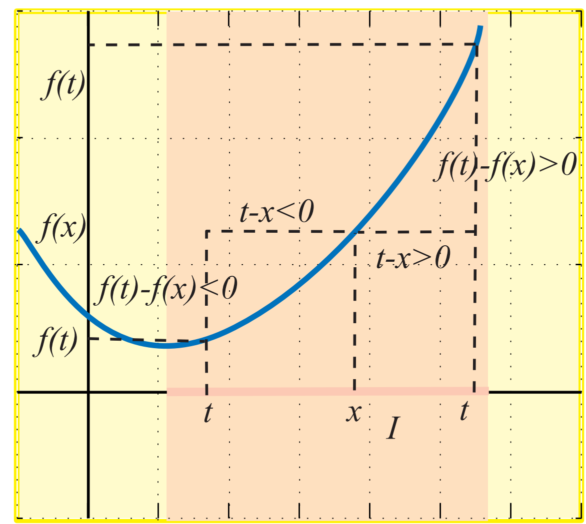
Proof. Since $f(x)$ is increasing,

$$\frac{f(t) - f(x)}{t - x} > 0, \quad \forall x, t \in I$$



Therefore,

$$f(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \geq 0$$

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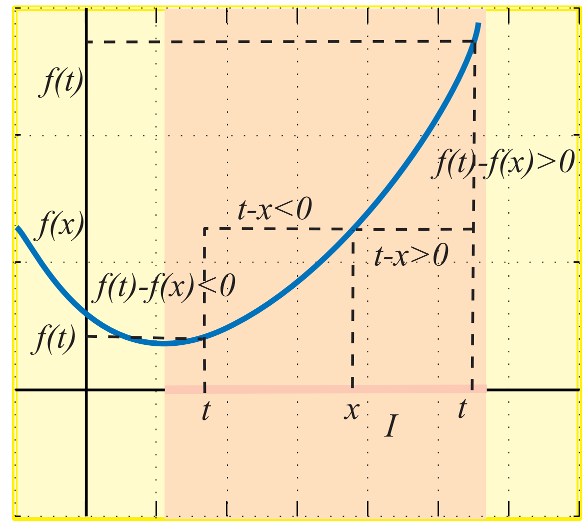

Therefore,

$$f(x) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \geq 0$$

Note that "$\geq$" can not be replaced with "$>$"! ($f(x) = x^3$ is increasing everywhere but $f'(0) = 0$).

Theorem B. If $f'(x) > 0$ on I, $f(x)$ increases.

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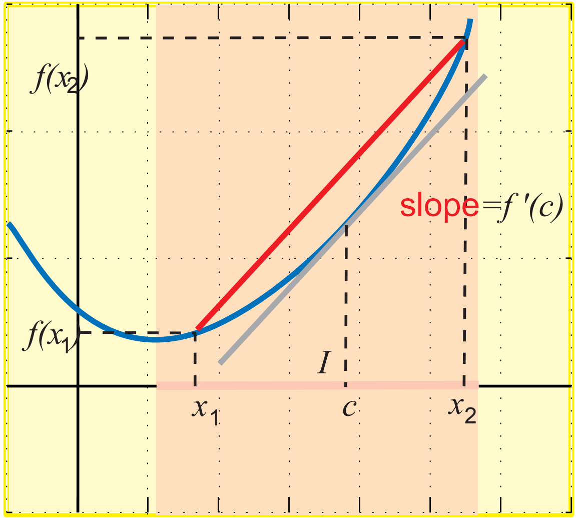

Requires Lagrange's theorem: $\forall x_1, x_2 \in I$, there exists value c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

If $f'(c) > 0$ on I, $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

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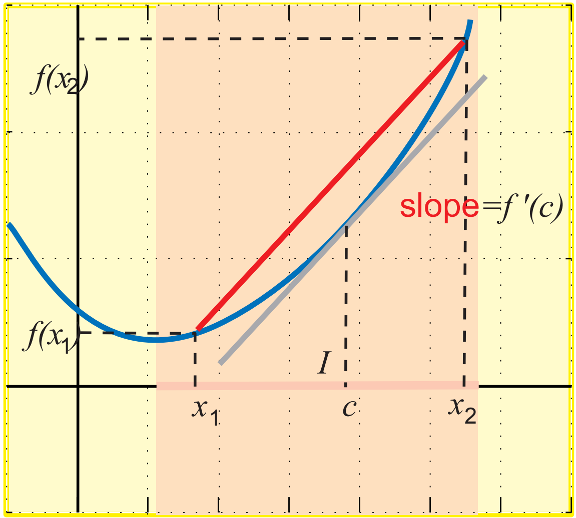

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

If $f'(c) > 0$ on I, $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

Note: $f(x)$ is increasing $\Rightarrow f'(x) \geq 0$

But $f'(x) \geq 0 \not\Rightarrow f(x)$ is increasing.

Theorem A. If $f(x)$ is decreasing on I, and $f'(x)$ exists, then $f'(x) \leq 0$ on I.

Theorem B. If $f'(x) < 0$ on I, $f(x)$ decreases.

EXAMPLES

$f'(x) > 0 (f'(x) < 0) \Rightarrow$ is increasing (decreasing)

EXAMPLE 3. Find where $f(x) = x^2 - 5x + 1$ is increasing and where it is decreasing.

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Consider

$$f'(x) = 2x - 5$$

$f'(x) > 0$ if $x > 5/2$, $f'(x) < 0$ if $x < 5/2$. By Thm. B:

$$f(x) \quad \text{is increasing for } x > \frac{5}{2}$$

$$f(x) \quad \text{is decreasing for } x < \frac{5}{2}$$

$f'(x) > 0 (f'(x) < 0) \Rightarrow$ is increasing (decreasing)

EXAMPLE 4. Find where $f(x) = (x^2 - 3x)/(x+1)$ is increasing and where it is decreasing.

$f'(x) > 0 (f'(x) < 0) \Rightarrow$ is increasing (decreasing)

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Consider

$$f'(x) = \frac{(x+3)(x-1)}{(x+1)^2}$$

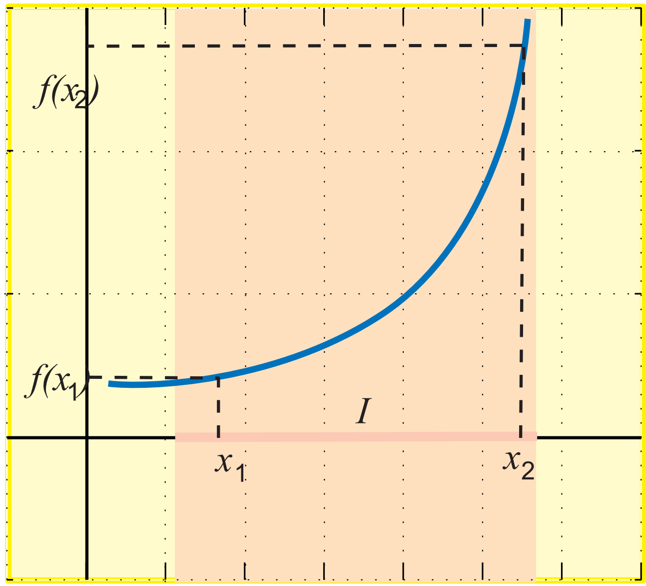
$f'(x) > 0$ for $x < -3$ and $x > 1$ (increasing)

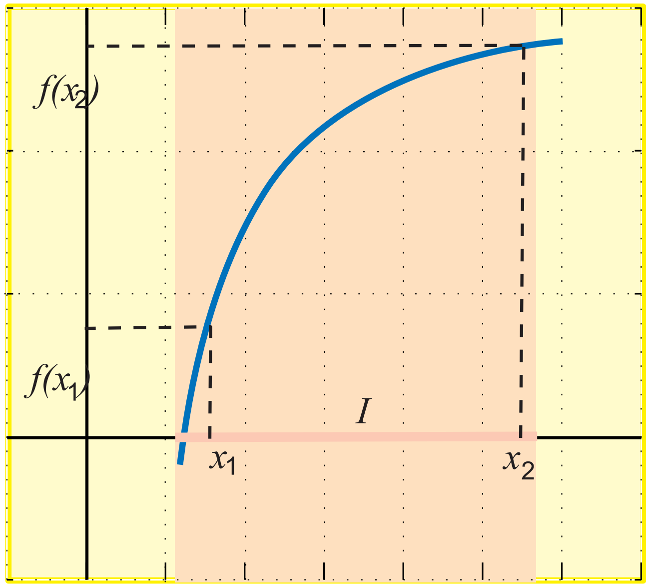
$f'(x) < 0$ for $-3 < x < -1$ and $-1 < x < 1$ (decreasing)

Concavity.

what the second derivative can tell about the function?

Two way of increasing:





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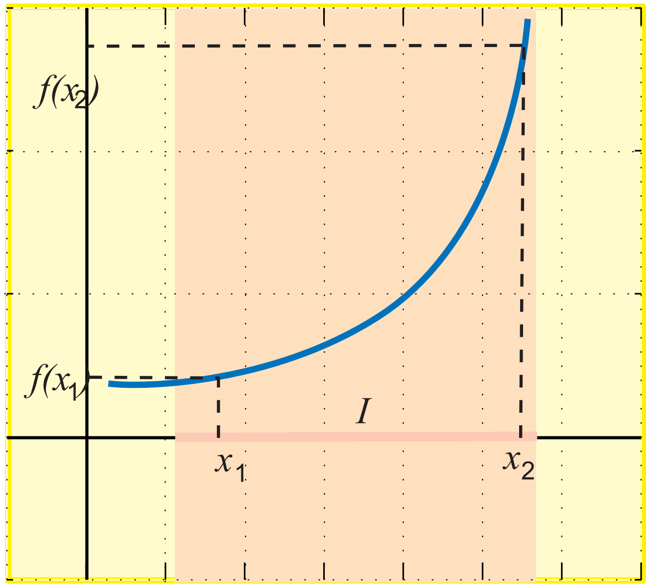
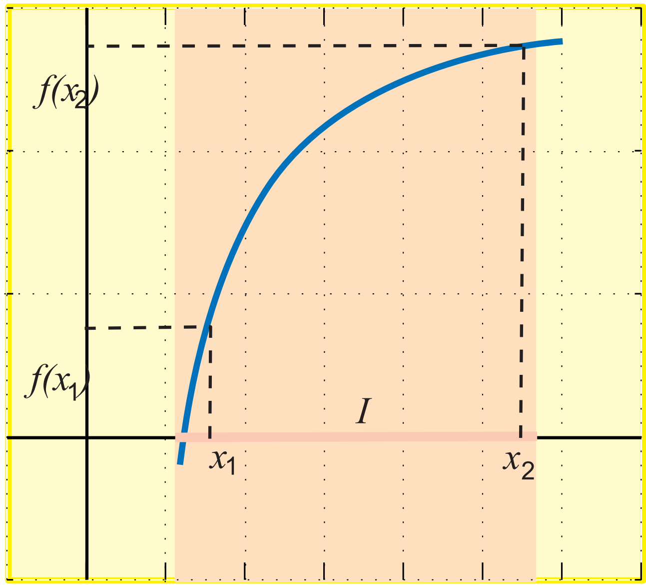
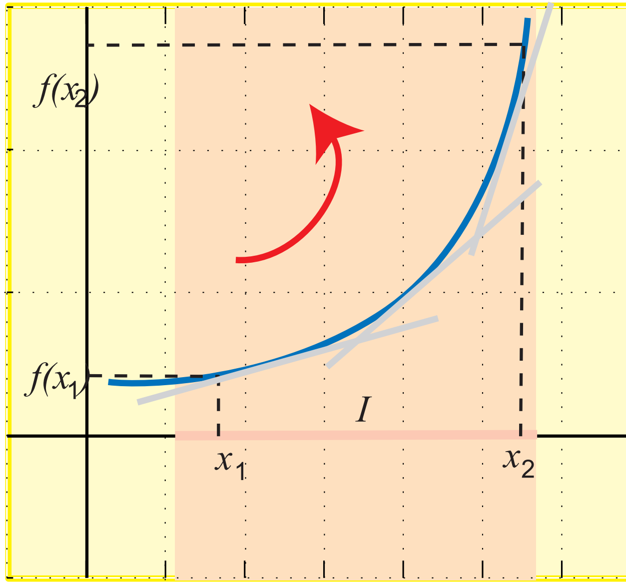




How to distinguish these two cases?

Definition $f(x)$ is concave up on I if $f'(x)$ increases on I.



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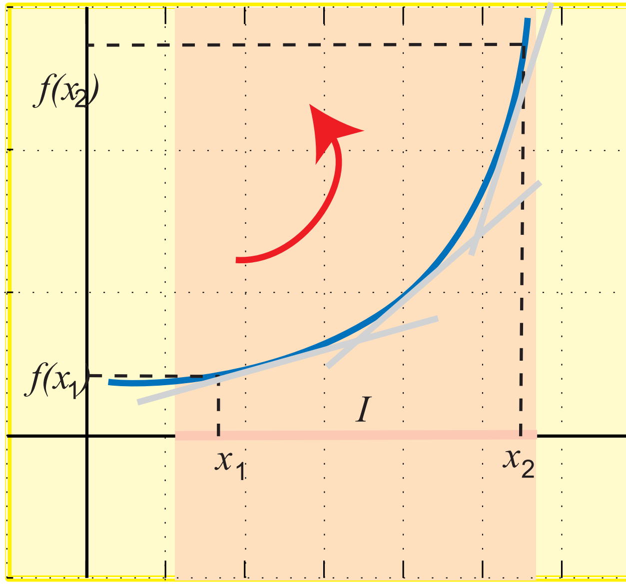
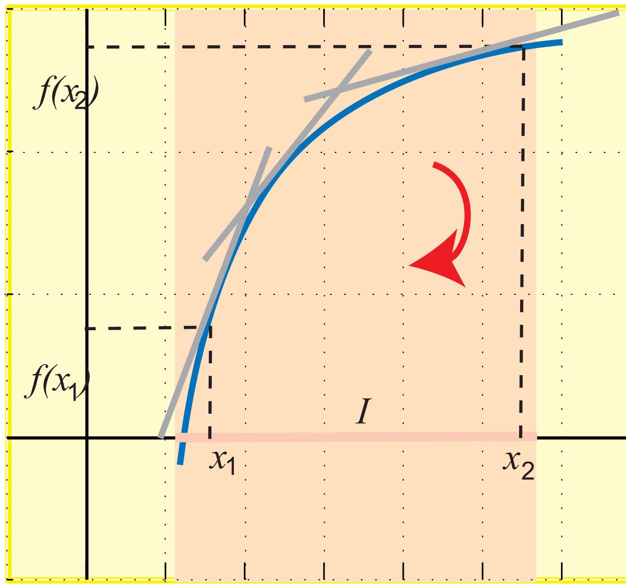


Definition $f(x)$ is concave down on I if $f'(x)$ decreases on I.



Second derivative and Concavity

$f''(x) > 0 \Rightarrow f'(x)$ is increasing = Concave up

$f''(x) < 0 \Rightarrow f'(x)$ is decreasing = Concave down

Concavity changes = Inflection point

Second derivative and Concavity

$f''(x) > 0 \Rightarrow f'(x)$ is increasing $=$ Concave up

$f''(x) < 0 \Rightarrow f'(x)$ is decreasing $=$ Concave down

Concavity changes $=$ Inflection point

Example 5. Where the graph of $f(x) = x^3 - 1$ is concave up, concave down?

Consider $f''(x) = 2x$. $f''(x) < 0$ for $x < 0$, concave down; $f''(x) > 0$ for $x > 0$, concave up.

$f''(x) > 0 (f''(x) < 0) \Rightarrow$ concave up(down)

EXAMPLE 6. Find where the graph of $f(x) = x - \sin(x)$ is concave up, concave down?

$f''(x) > 0 (f''(x) < 0) \Rightarrow$ concave up(down)

EXAMPLE 6. Find where the graph of $f(x) = x - \sin(x)$ is concave up, concave down?

$$f'(x) = 1 - \cos(x), \quad f''(x) = \sin(x)$$

$f''(x) > 0$ for $x \in [k\pi, (k+1)\pi]$, $k = 0, \pm1, \pm2, \ldots$ (concave up)

$f''(x) < 0$ for $x \in [(k-1)\pi, k\pi]$, $k = 0, \pm1, \pm2, \ldots$ (concave down)

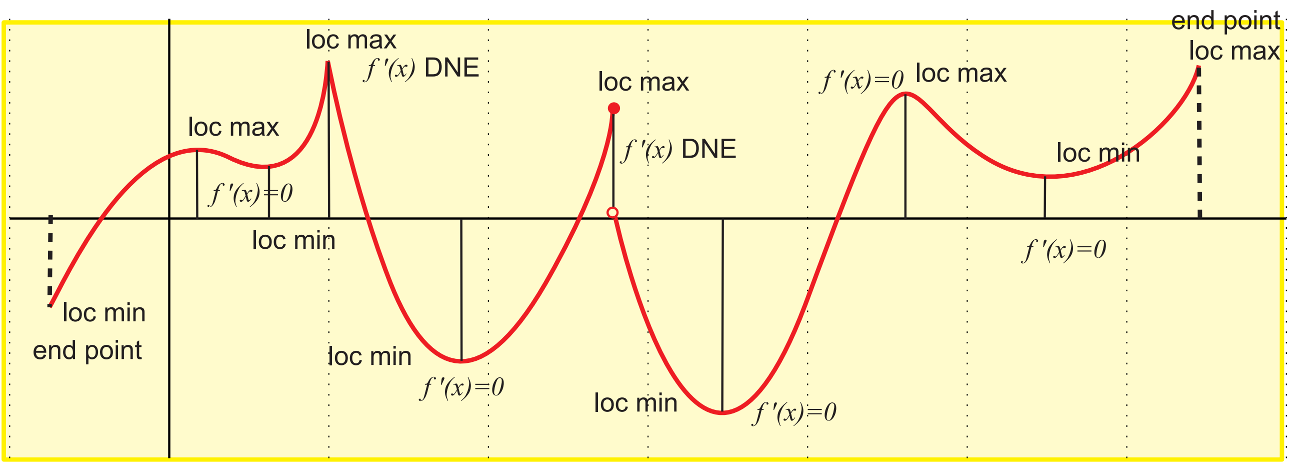
$k\pi$, $k = 0, \pm1, \pm2, \ldots$ inflection points

Minima and Maxima.

critical points, first derivative test second derivative test.

Definition. $f(c)$ is a local maximum value of $f(x)$ if there exists an interval (a, b) containing c such that $\forall x \in (a, b)$, $f(c) \geq f(x)$.

Definition. $f(c)$ is a local minimum value of $f(x)$ if there exists an interval (a, b) containing c such that $\forall x \in (a, b)$, $f(c) \leq f(x)$.



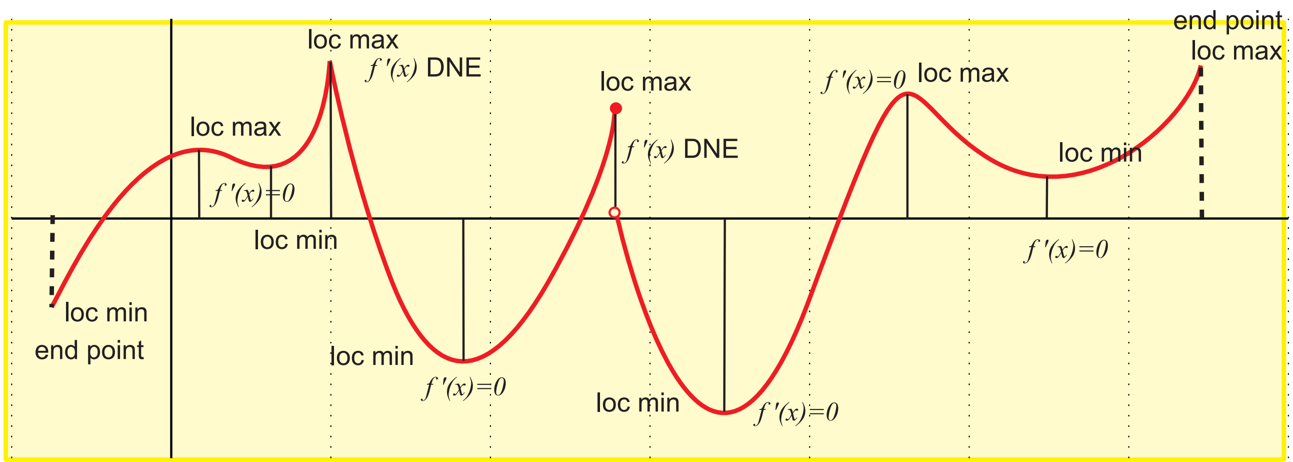
Critical Point Theorem. If $f(c)$ is a local min (max), then c is a critical point, that is

a) an end point

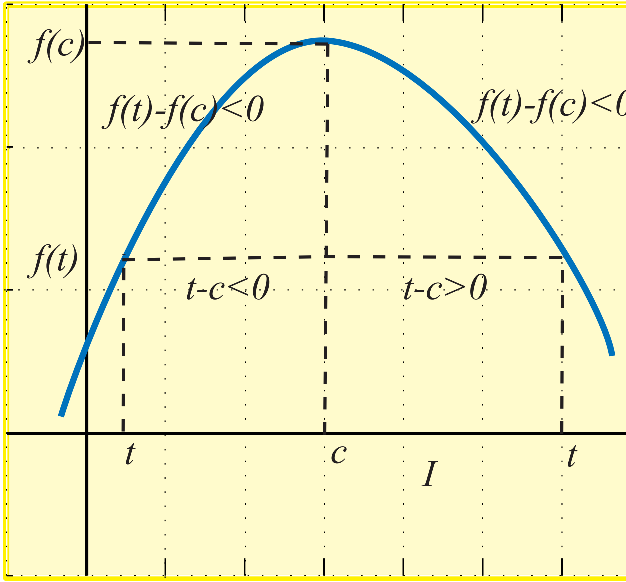
b) a stationary point, that is $f'(c) = 0$

c) a singular point, that is $f'(c)$ does not exists

(a) and c) are proved by examples.)



Proof b) If $f(c)$ is max, then



$$\frac{f(t)-f(c)}{t-c} < 0, \quad x > c,$$

$$\frac{f(t)-f(c)}{t-c} > 0, \quad x < c,$$

Or,

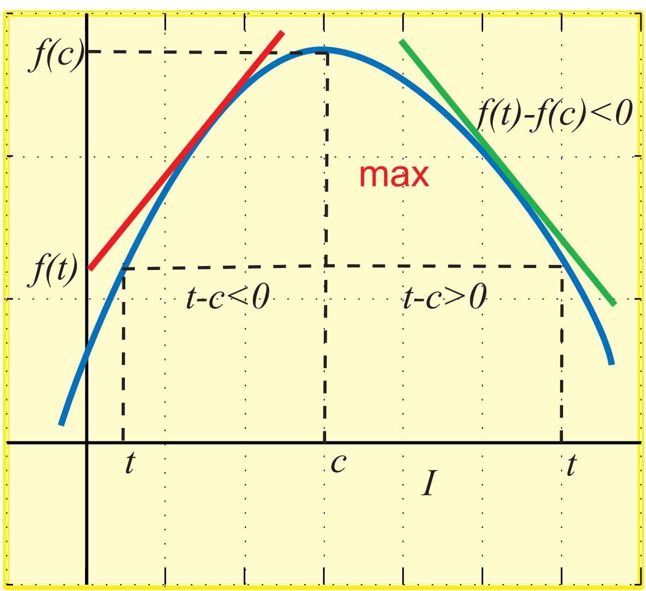
$$\lim_{t\to c+} \frac{f(t)-f(c)}{t-c} \le 0, \quad \lim_{t\to c-} \frac{f(t)-f(c)}{t-c} \ge 0,$$

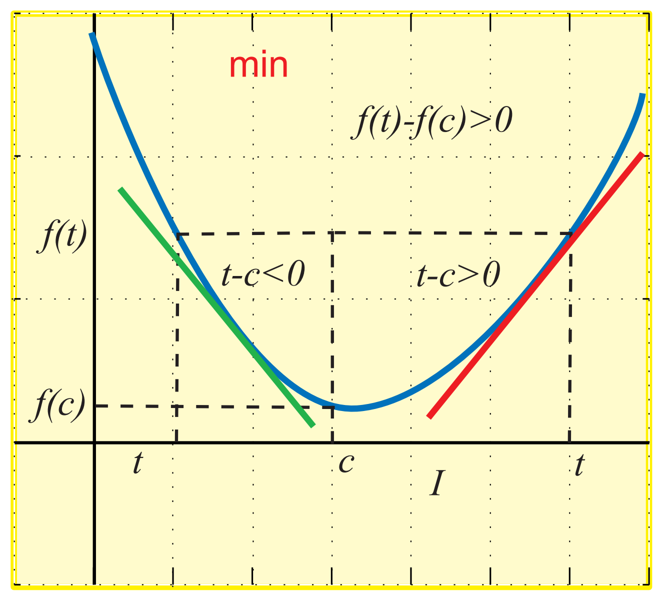
$$f'(c) = \lim_{t\to c} \frac{f(t)-f(c)}{t-c} = 0$$

First Derivative Test

$f'(x) > 0$ to the left, $f'(x) < 0$ to the right of $c \Rightarrow$ increases to the left, decreases to the right of $c \Rightarrow$ max at $x = c$.

$f'(x) < 0$ to the left, $f'(x) > 0$ to the right of $c \Rightarrow$ decreases to the left, increases to the right of $c \Rightarrow$ min at $x = c$.





Second Derivative Test

$f''(c) < 0$ and $f'(c) = 0 \Rightarrow$
$f'(x)$ is decreasing near c and passing 0 at $c \Rightarrow$
$f'(x) > 0$ to the left, $f'(x) < 0$ to the right of $c \Rightarrow$
increases to the left, decreases to the right of $c \Rightarrow$
max at $x = c$.

$f''(c) > 0$ and $f'(c) = 0 \Rightarrow$
$f'(x)$ is increasing near c and passing 0 at $c \Rightarrow$
$f'(x) < 0$ to the left, $f'(x) > 0$ to the right of $c \Rightarrow$
decreases to the left, increases to the right of $c \Rightarrow$
min at $x = c$.

EXAMPLES

EXAMPLE 7. Find where the graph of $f(x) = x\ln(x)$ is increasing, decreasing, concave up, concave down, has max, min?

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$$f'(x) = \ln(x) + 1, \quad f''(x) = \frac{1}{x}$$

$f'(x) < 0$ for $1 < x < 1/e$ (decreasing), $f'(x) > 0$ for $x > 1/e$ (increasing)

$f''(x) > 0$ for $x > 0$, (concave up), no inflection pts.

min at $x = 1/e$ (first derivative test)

EXAMPLE 8. Find where the graph of $f(x) = 1/(x^2+1)$ is increasing, decreasing, concave up, concave down, has max, min?

EXAMPLE 8. Find where the graph of $f(x) = 1/(x^2+1)$ is increasing, decreasing, concave up, concave down, has max, min?

$$f'(x) = -\frac{2x}{(x^2+1)^2}, \quad f''(x) = 2\frac{3x^2-1}{(x^2+1)^3}$$

$f'(x) > 0$ for $x < 0$ (increasing), $f'(x) < 0$ for $x > 0$ (decreasing) max at $x = 0$ (first derivative test)

$f''(x) > 0$ for $x < -1/\sqrt{3}$ and $x > 1/\sqrt{3}$, (concave up), $f''(x) < 0$ for $-1/\sqrt{3} < x < 1/\sqrt{3}$, (concave down), $-1/\sqrt{3}$, $1/\sqrt{3}$ are inflection pts.