Tangent Line, Velocity, Derivative and Differentiability

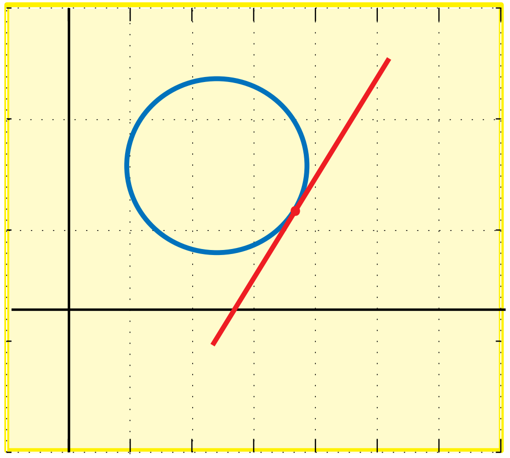
instant rate of change and approximation of the function

Motivation:

what do tangent line and velocity have in common?

Euclid: The tangent is a line that touches a curve at just one point.

Works for circles:



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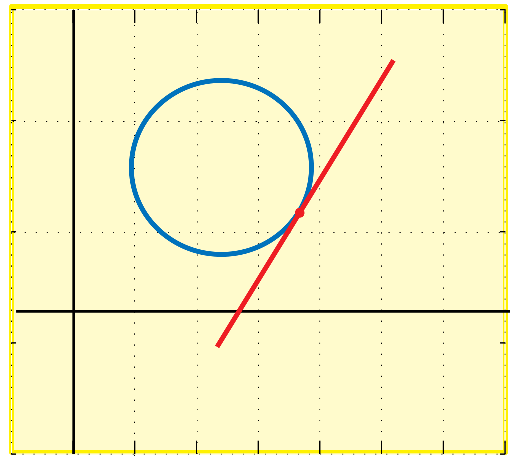
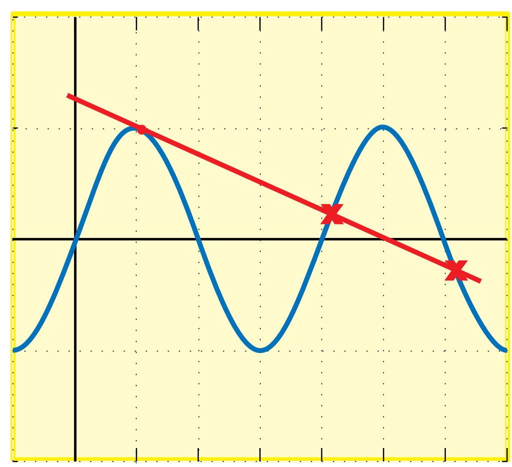


Does NOT work for $\sin(x)$!

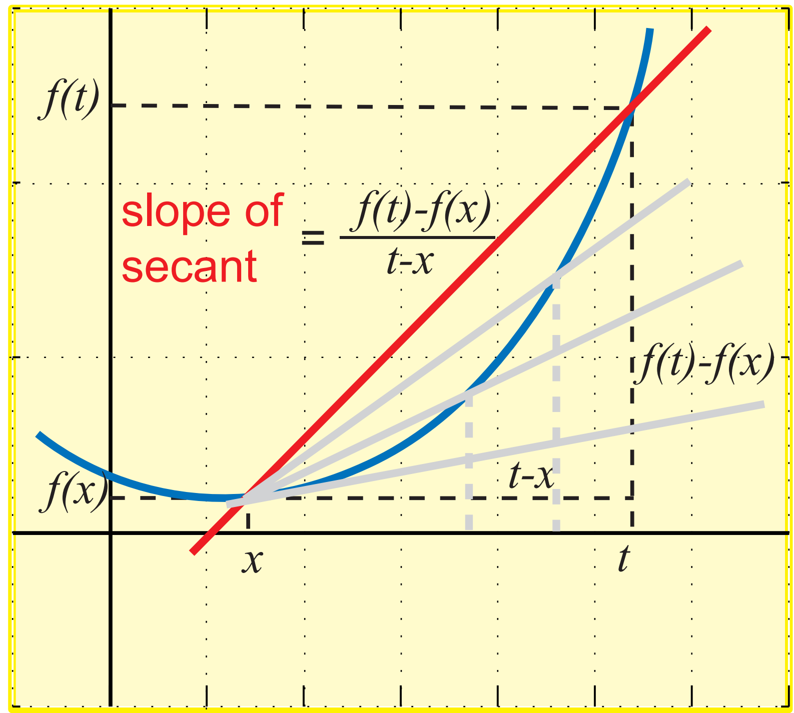


Modern: The tangent is the limiting position of the secant line (if exists).

$$\text{Tangent} = \lim_{t \to x} \text{Secant}$$

$$\begin{matrix}\text{Slope of} \\ \text{Tangent}\end{matrix} = \lim_{t \to x} \begin{matrix}\text{Slope of} \\ \text{Secant}\end{matrix}$$

$$\begin{matrix}\text{Slope of} \\ \text{Tangent}\end{matrix} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$



EXAMPLES

$$\text{Slope of Tangent} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

EXAMPLE 1. Find slope of the tangent line to the graph of $f(x) = x^2 - 4$ at point $(3, 5)$.

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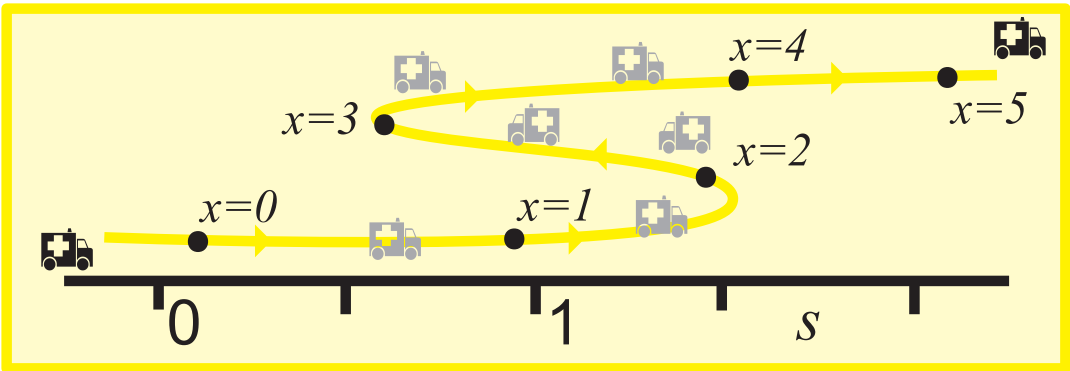
At $x = 3$, ($y = 5$ is irrelevant)

Slope of Tangent $= 6$

Velocity: How fast the position $s(x)$ is changing.



Average Velocity $=$

$$= \frac{s(x+h) - s(x)}{h}$$

Instantaneous Velocity $= \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}$

EXAMPLES

$$\text{Instant. Velocity} = \lim_{h\to 0} \frac{s(x+h)-s(x)}{h}$$

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At $x = 2$, Instantaneous Velocity $= -\sin(2)$

Compare

Slope of Tangent $= \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$

Instant. Velocity $= \lim_{\tau \to 0} \frac{s(x+h) - s(x)}{h}$

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Equivalent up to the substitution

$$f = s, \qquad t - x = h$$

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Equivalent up to the substitution

$$f = s, \qquad t - x = h$$

Is it the same thing?

Definition of the Derivative

putting things together

Slope of Tangent

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

Instant. Velocity

$$\lim_{h \to 0} \frac{s(x + h) - s(x)}{h}$$

Rate of Change

$$\lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

Slope of Tangent

Instant. Velocity

Rate of Change

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} \qquad \lim_{h \to 0} \frac{s(x+h) - s(x)}{h} \qquad \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

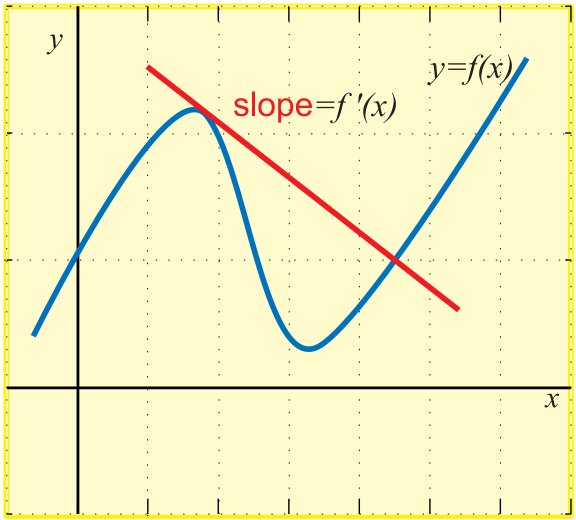
Definition: Derivative of $f(x)$ at point x is

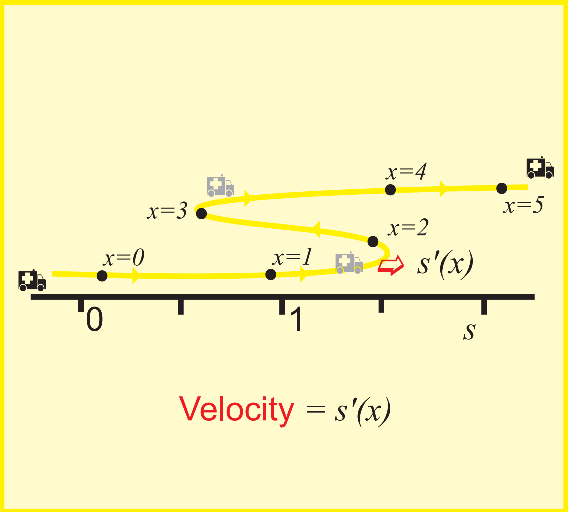
$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

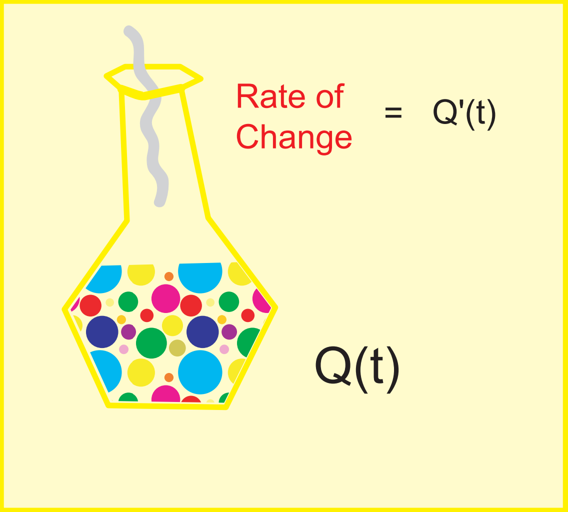
$$\left(\text{or,} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right)$$

Illustration

Slope of Tangent	Instant. Velocity	Rate of Change







EXAMPLES

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

EXAMPLE 3. Evaluate $f'(x)$ using definition of derivative, if $f(x) = x^3 + 2x$.

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$$= \lim_{t \to x} \frac{(t - x)(t^2 + tx + x^2)}{(t - x)} + \lim_{t \to x} \frac{2(t - x)}{(t - x)}$$

$$= 1 \cdot (x^2 + x^2 + x^2) + 2 = 3x^2 + 2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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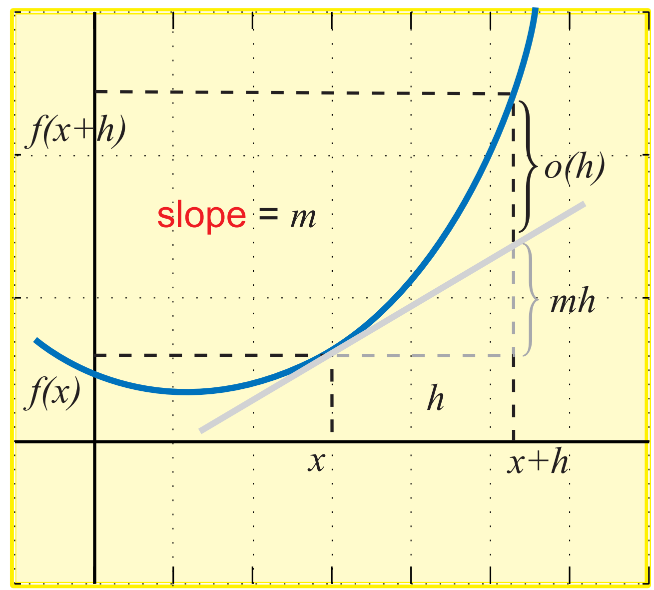
$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

The Differential

problem of linear approximation

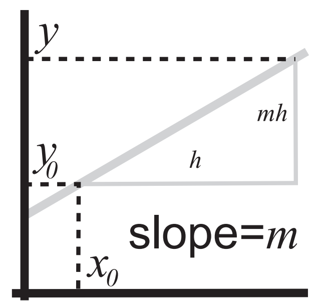
Approximating function with a line

$$f(x+h) = f(x)+m\cdot h+o(h)$$



where $$\lim_{h\to 0}\frac{o(h)}{h} = 0$$

Motion along a line:



$$y = y_0 + m\cdot h$$

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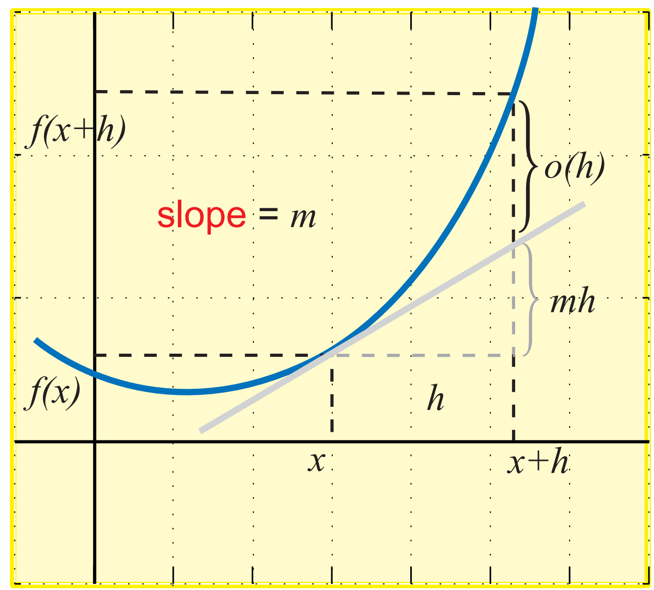

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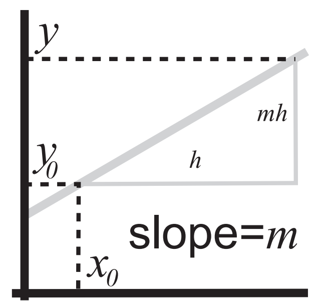

What is m ?

Definition. Function $f(x)$ is differentiable at x if there exists a number m such that

$$f(x+h) = f(x) + m \cdot h + o(h)$$

$$\text{where} \quad \lim_{h \longrightarrow 0} \frac{o(h)}{h} = 0$$

($o(h)$ “o”-little is a quantity of the order of magnitude smaller then h).

Differential:

$$\mathbf{d}f(x)\langle h \rangle = m \cdot h$$

Compute m. Assume function $f(x)$ is differentiable: $f(x+h) = f(x) + m \cdot h + o(h)$. Solving for m and rearranging,

$$\frac{f(x+h) - f(x)}{h} = m + \frac{o(h)}{h}$$

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differentiability requires $\lim_{h \to 0} \frac{o(h)}{h} = 0$ therefore,

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or $f'(x)$ exists and $f'(x) = m$.

Differentiability implies Derivative!

EXAMPLE 5. Differentiate $f(x) = \sin(x)$.

$$f(x+h) = f(x) + m \cdot h + o(h)$$

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$$\begin{aligned}\sin(x+h) &= \sin(x) + \cos(x)h \\ &\quad + [-\cos(x)h + \sin(x+h) - \sin(x)]\end{aligned}$$

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Now, $m = f'(x) = \cos(x)$ and $o(h) = [\ldots]$.

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Therefore,

$$\lim_{h \to 0} \frac{o(h)}{h} = 0$$

Evaluating differentials. Common notations:

$$\mathbf{d}f(x) = f'(x)\mathbf{d}x$$

(Compare to $\mathbf{d}f(x)\langle h\rangle = m \cdot h$, substituting $m = f'(x)$, $h = \mathbf{d}x$)

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EXAMPLE 6.

$$\mathbf{d}(\sin x) = \cos x\mathbf{d}x, \quad \mathbf{d}(\sqrt{x}) = \frac{1}{2\sqrt{x}}\mathbf{d}x,$$

$$\mathbf{d}(\arctan(x)) = \frac{1}{1+x^2}\mathbf{d}x$$