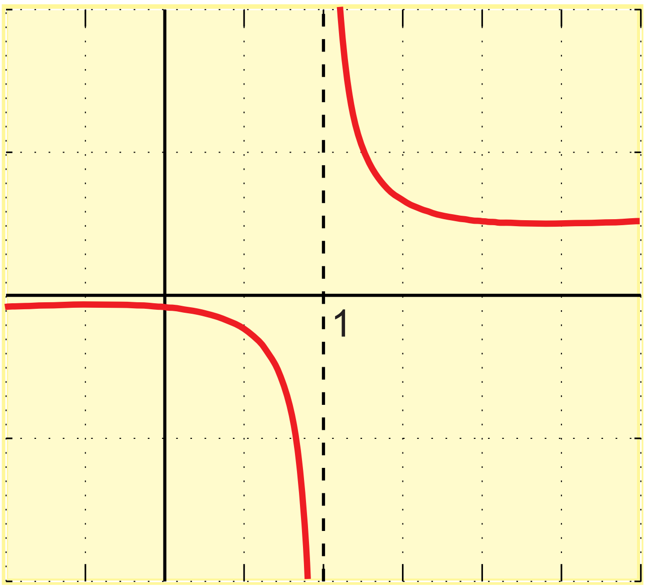
Limit of a function at a point

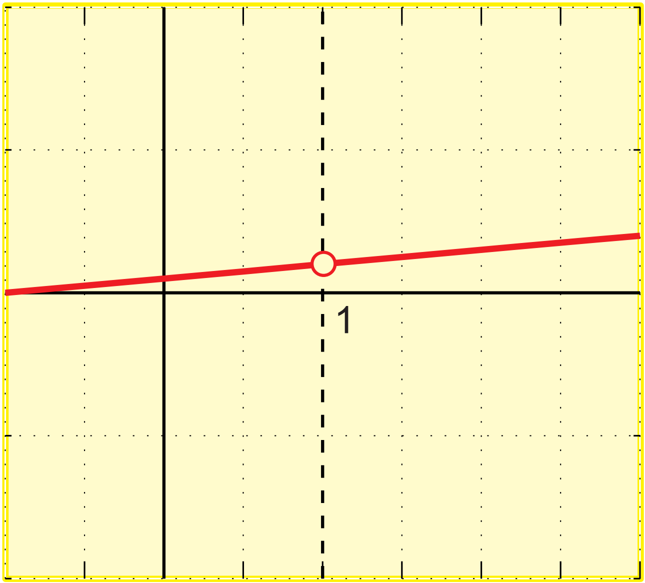
ε–δ language

Motivation:

Studying functions when they are not defined

The following functions are undefined at $x = 1$:





$$f(x) = \frac{x^2 + 1}{x - 1}$$

$$f(x) = \frac{x^2 - 1}{x - 1}$$

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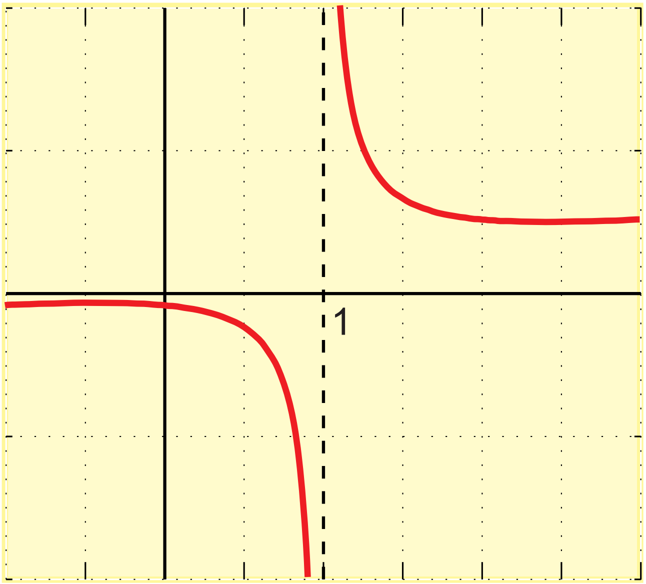
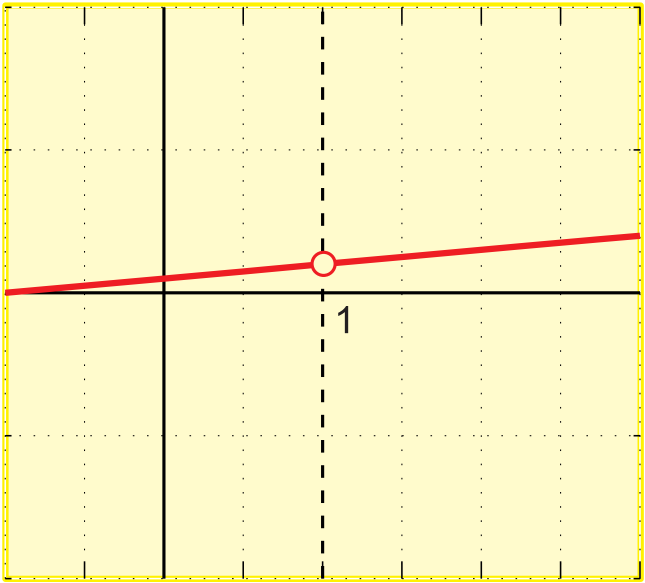


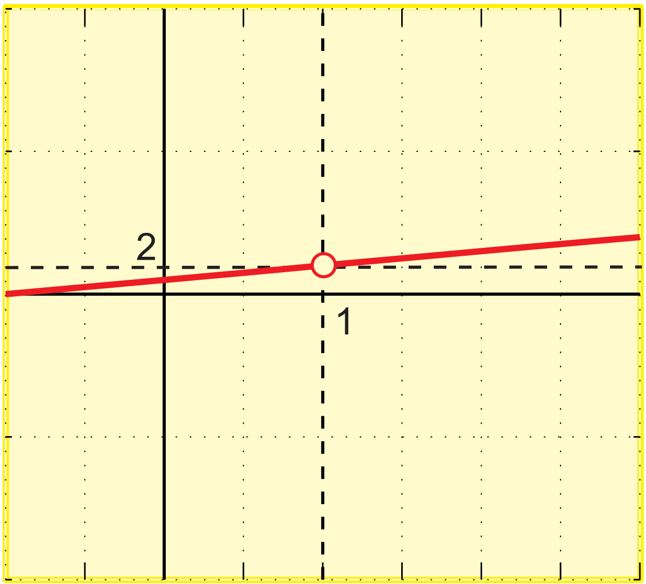
$$f(x) = \frac{x^2 + 1}{x - 1}$$



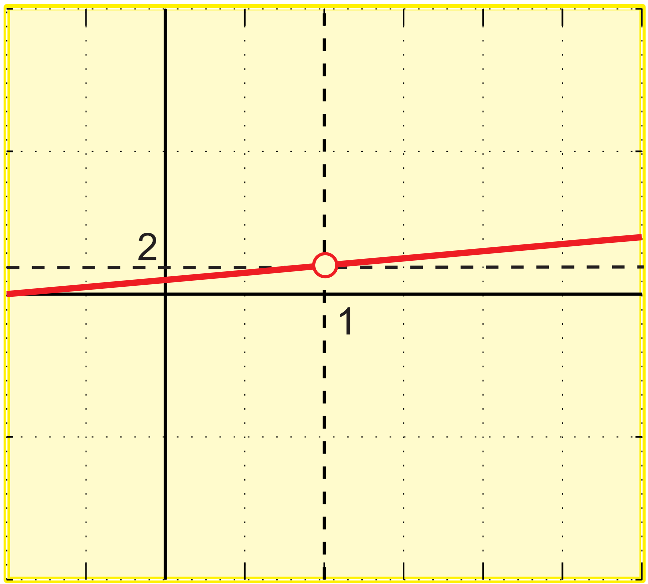
$$f(x) = \frac{x^2 - 1}{x - 1}$$

The difference can be big!

Want to distinguish the following situation:



Want to distinguish the following situation:



As x is near 1, value of $f(x) = \dfrac{x^2 - 1}{x - 1}$ is near 2.

HOW EXACTLY NEAR?

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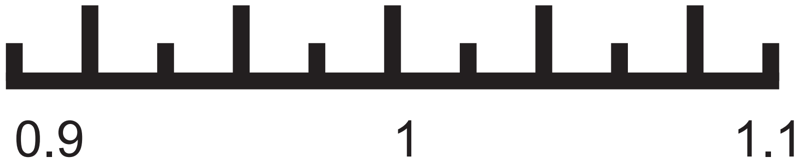
This near?



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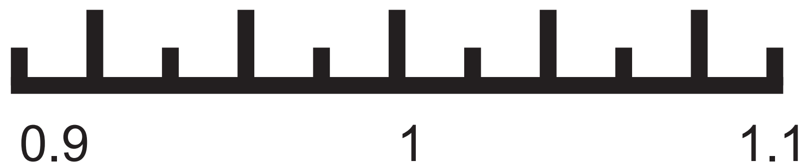

This near?

HOW EXACTLY NEAR?

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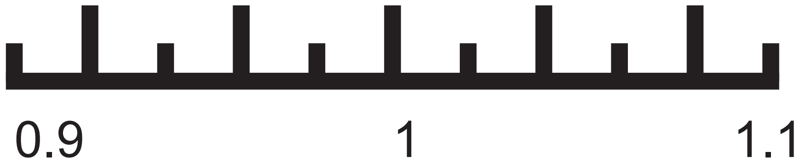
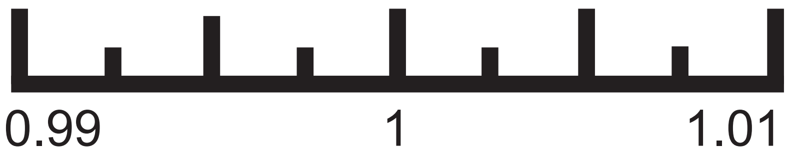
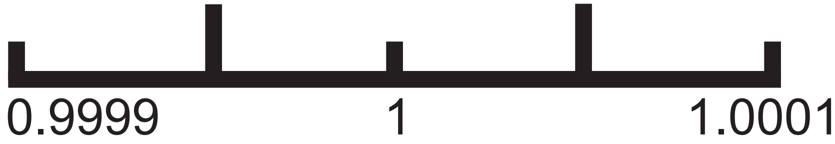


This near?



Or, this near?



!!..!!!..!!

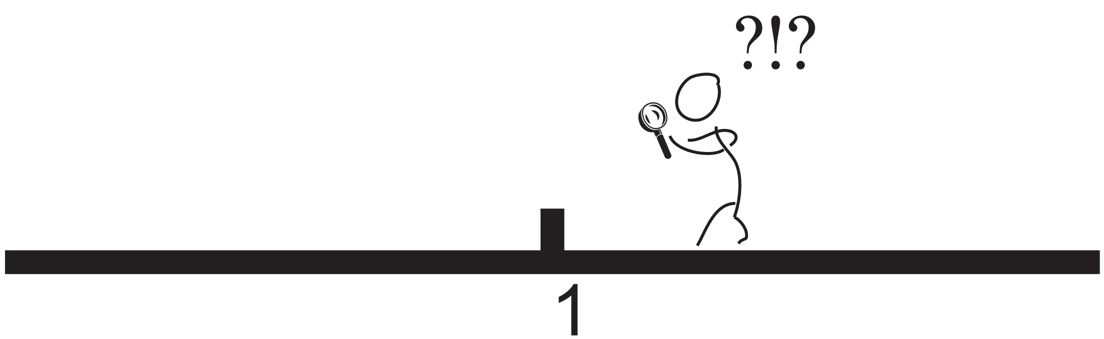
INFINITELY NEAR!

??..???..??

INFINITELY NEAR?

??..???..??

INFINITELY NEAR?

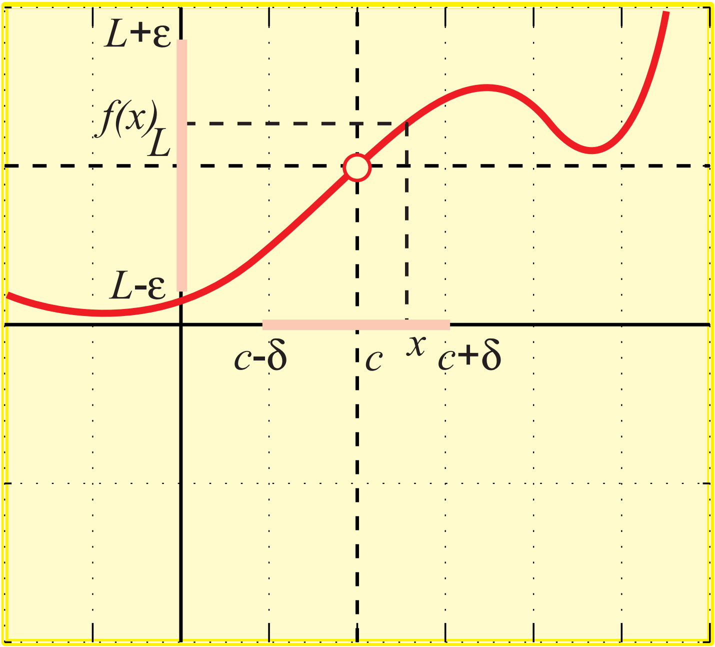


ε–δ language. Definition of Limit

Working out the infinity

Definition of Limit

DEFINITION. The number L is the limit of function $f(x)$ as x approaches c if and only if for any positive number ε there exists a positive number δ (depending on ε) such that as long as x is not equal to c but differs from c by less then δ, it implies that $f(x)$ differs from L by less then ε.



Limit in math symbols.

DEFINITION.

$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Legend: $\forall$ — for any, ε — "epsilon", $\exists$ — exists, δ — "delta", $/$ — such that, $\Rightarrow$ — implies, $\Leftrightarrow$ — if and only if, $\to$ — approaches.

$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

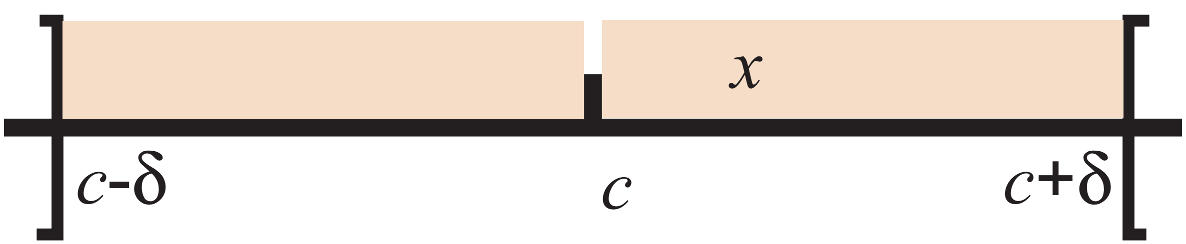
$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

The inequality in red requires that

$$-\delta < x - c < \delta, \qquad x - c \neq 0$$

or,

$$c - \delta < x < c + \delta, \qquad x \neq c$$



$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

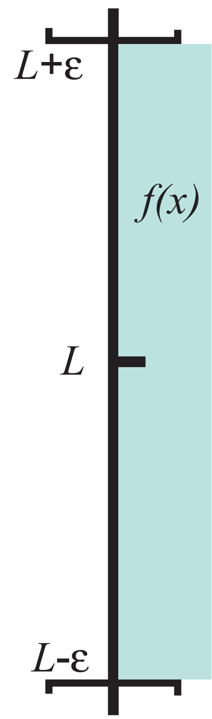
$$\forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

The inequality in green requires that

$$-\varepsilon < f(x) - L < \varepsilon,$$

or,

$$L - \varepsilon < f(x) < L + \varepsilon,$$



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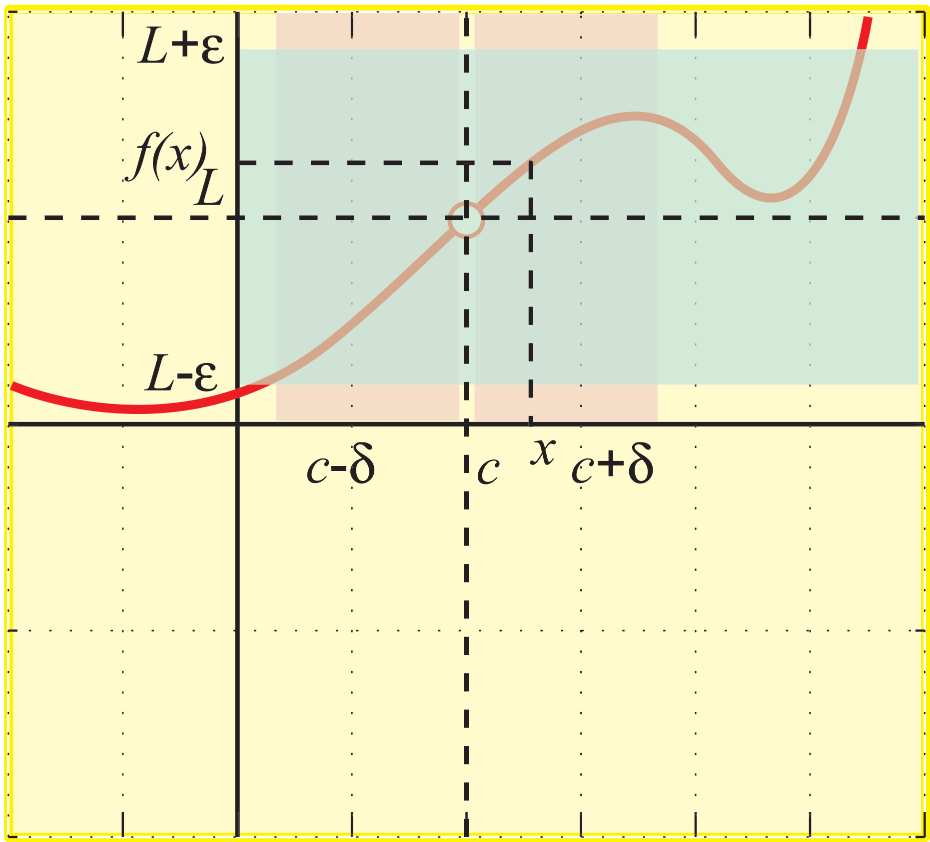
As x is in δ-corridor,
$f(x)$ is in ε-corridor:

(A narrower δ-corridor guarantees it better)



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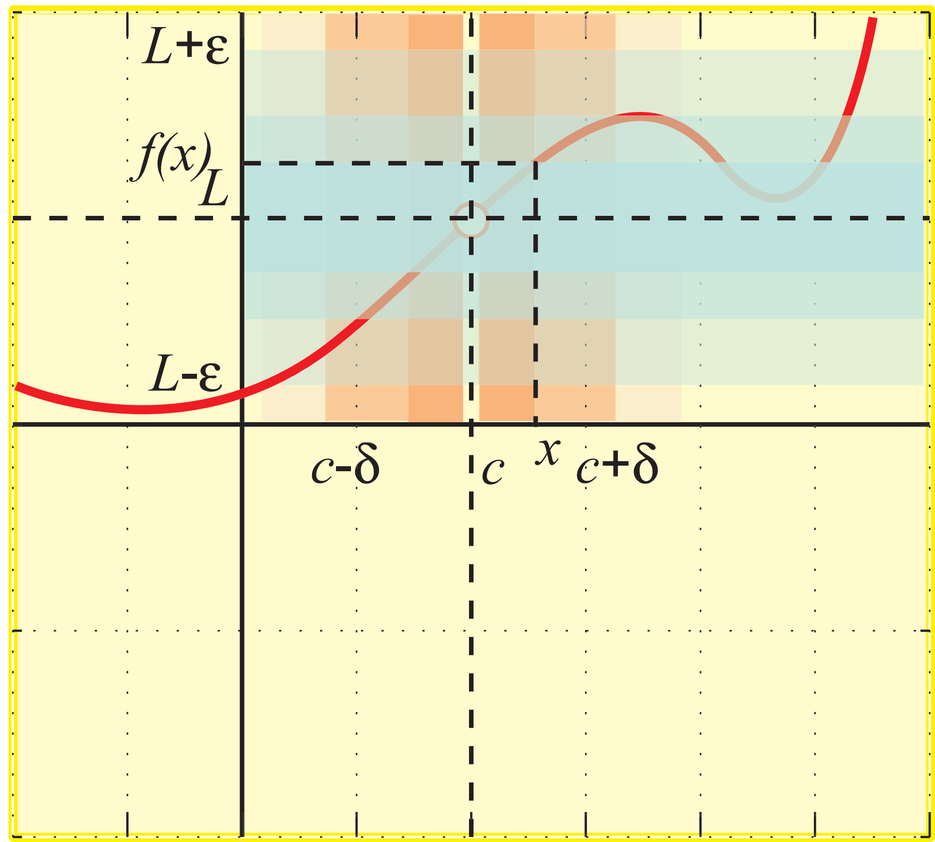
For every choice of ε there must exist δ:

(it is highly desirable to have a formula for computing δ from ε)



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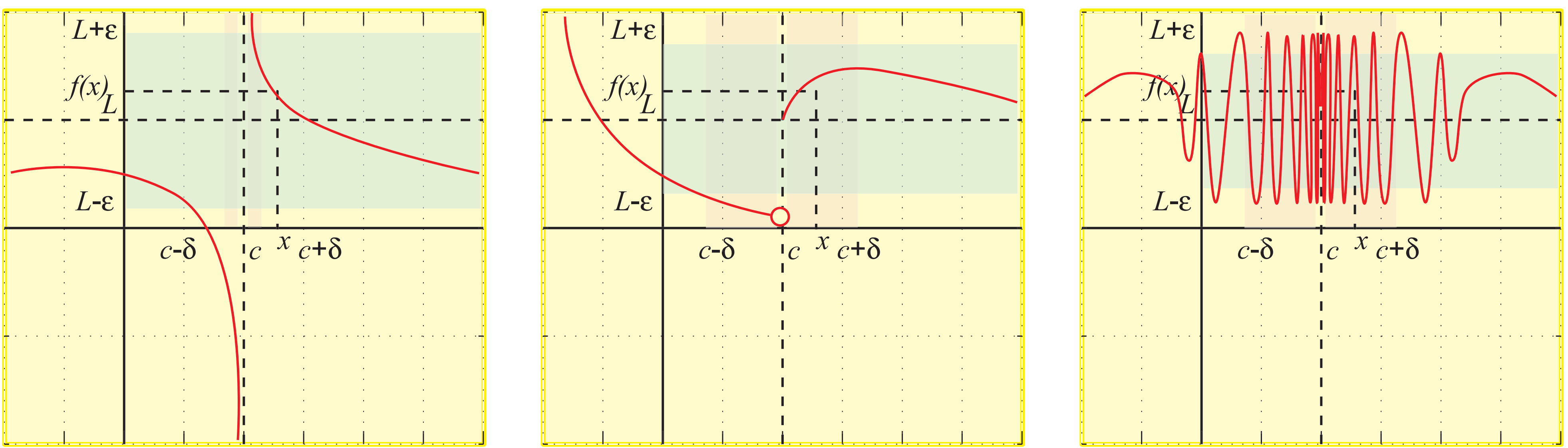
No limit situations:



EXAMPLES

$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

EXAMPLE 1. Prove by ε–δ argument

$$\lim_{x \to 2} (7x + 1) = 15 \quad \Leftrightarrow$$

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$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

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$$\lim_{x \to 2} (7x + 1) = 15 \quad \Leftrightarrow$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x-2| < \delta \Rightarrow |(7x+1)-15| < \varepsilon$$

$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

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$$\lim_{x \to 2} (7x + 1) = 15 \quad \Leftrightarrow$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x-2| < \delta \Rightarrow |(7x+1)-15| < \varepsilon$$

(By a smart choice of δ guarantee that

$$|(7x + 1) - 15| < \varepsilon$$)

The desired inequality

$$|(7x + 1) - 15|$$

The desired inequality

$$|(7x+1)-15| = |7x-14| = |7(x-2)|$$

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In particular, one can pick

$$\delta = \varepsilon/7. \qquad \text{(answer)}$$

$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

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EXAMPLE 2. Prove by ε–δ argument

$$\lim_{x \to 5} \left(\frac{x^2 - 25}{x - 5} \right) = 10 \quad \Leftrightarrow$$

$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

EXAMPLE 2. Prove by ε–δ argument

$$\lim_{x \to 5} \left(\frac{x^2 - 25}{x - 5} \right) = 10 \quad \Leftrightarrow$$

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; / \; 0 < |x - 5| < \delta \Rightarrow \left| \frac{x^2 - 25}{x - 5} - 10 \right| <$$

(By a smart choice of δ guarantee that

$$\left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon \quad)$$

The desired inequality

$$\left|\frac{x^2 - 25}{x - 5} - 10\right|$$

The desired inequality

$$\left|\frac{x^2-25}{x-5}-10\right| = \left|\frac{(x+5)(x-5)}{x-5}-10\right|$$

$$= |x-5| < \varepsilon \qquad \text{(desirable)}$$

follows from the the assumption

$$|x-5| < \delta$$

if

$$\delta \leq \varepsilon.$$

In particular, one can pick

$$\delta = \varepsilon. \qquad \text{(answer)}$$

$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

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EXAMPLE 3. Prove by ε–δ argument

$$\lim_{x \to 3} x^2 = 9 \quad \Leftrightarrow$$

$$L = \lim_{x \to c} f(x) \quad \Leftrightarrow$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

EXAMPLE 3. Prove by ε–δ argument

$$\lim_{x \to 3} x^2 = 9 \quad \Leftrightarrow$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ / \ 0 < |x - 3| < \delta \Rightarrow |x^2 - 9| < \varepsilon$$

(By a smart choice of δ guarantee that

$$|x^2 - 9| < \varepsilon$$)

The desired inequality

$$|x^2 - 9|$$

The desired inequality

$$|x^2 - 9| = |(x-3)(x+3)|$$

$$= |x-3||x+3| < \varepsilon \qquad (\text{desirable})$$

The desired inequality

$$|x^2 - 9| = |(x-3)(x+3)|$$

$$= |x-3||x+3| < \varepsilon \qquad \text{(desirable)}$$

requires controlling both $|x-3|$ and $|x+3|$.

Note that δ controls $|x-3|$ through

$$|x-3| < \delta$$

Does δ controls $|x+3|$ as well?

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Assume $\delta < 1$:

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Assume $\delta < 1$:

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Notice that if $2 < x < 4$, then

$$5 < |x+3| < 7 \qquad (\delta \text{ controls } |x+3|!)$$

Does δ controls $|x+3|$ as well?

Assume $\delta < 1$:

$$|x-3| < 1 \quad \Leftrightarrow \quad 2 < x < 4$$

Notice that if $2 < x < 4$, then

$$5 < |x+3| < 7 \qquad (\delta \text{ controls } |x+3|!)$$

Finally, $|x^2-9| = |x-3||x+3| < |x-3|7 < \varepsilon$ if $\delta < 1$ and $\delta \leq \varepsilon/7$.

Answer: $\delta = \min\{1, \varepsilon/7\}$