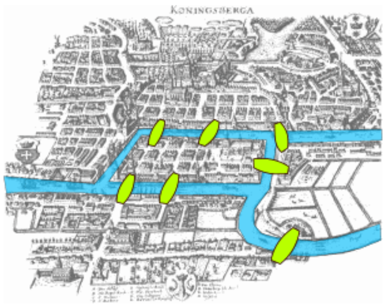
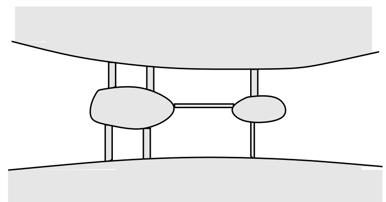
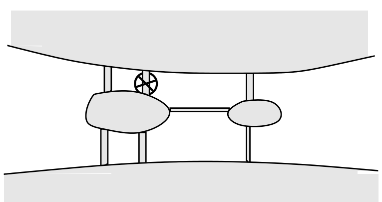
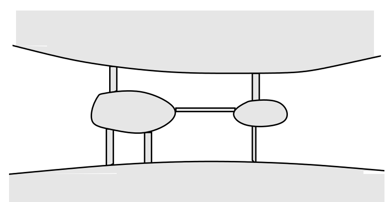
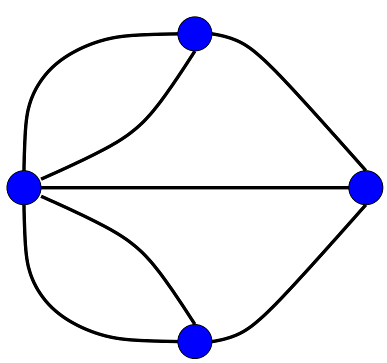
¶ **1.** Graph theory is a broad area of mathematics. Besides their theoretical significance, graphs have many practical applications: computer science, Internet, management science, scheduling.
Euler (1707–1783) is credited with having originated the areas of graph theory. He spent part of his career employed by in the city of Königsberg (now Kaliningrad). The city of Königsberg was divided by the river Pregel into four landmasses connected by bridges as shown in the left figure below. The citizens of Königsberg, as curious people and avid walkers, wondered if it was possible to cross all seven bridges without crossing any bridge more than once.

1. Can you solve the Königsberg Bridge Problem? That is, can you find a path that traverses each bridge exactly once?
2. Suppose that one of the bridges collapses, Can you now find a path across town that traverses each bridge exactly once?
3. A new bridge is going to be built to replace the old one, but in a new location so that it will be possible to take a walk across town crossing each bridge exactly once. Where should such bridge be located?

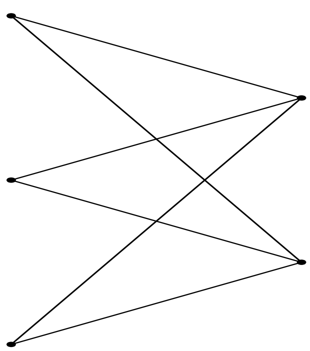
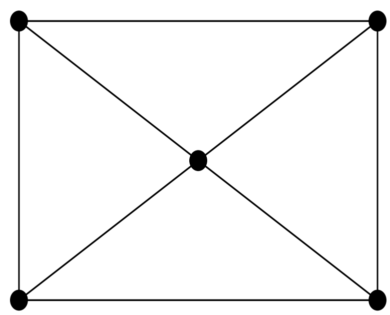
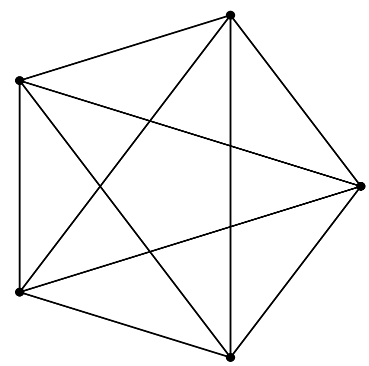



¶ **2.** Euler became intrigued by the problem, worked on it and solved it, in the following manner. He made a diagram out of the map of Königsberg, whereby each landmass was represented by a vertex, and each bridge was represented by an edge. Such diagram is called a graph or network.



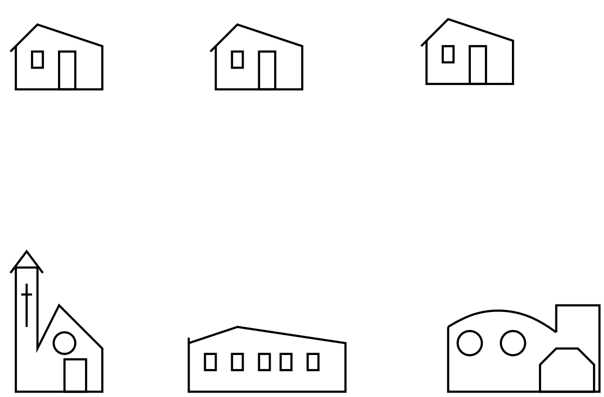
¶ **3.** A *graph* is a mathematical object consisting of a collection of points $U, V, W, \ldots$ called **vertices** or **nodes**, and a collection of arcs $a, b, c, \ldots$ called **edges**. Each edge connects two distinct vertices, or else it connects a vertex to itself. In the latter case, such edge is called a loop.

¶ **4.** We usually represent a graph by drawing a collection of points (the vertices) and a collection of arcs between some pairs of such vertices. It should be noted that different drawings may in fact represent the same graph.
A little experimentation will show that many graphs can be drawn on a piece of paper (in the plane) in such a way that edges intersect only at the vertices. A graph that can be drawn in such manner is called a **planar graph**.
Show that the two graphs on the left below are planar graphs. What about the graph on the right?

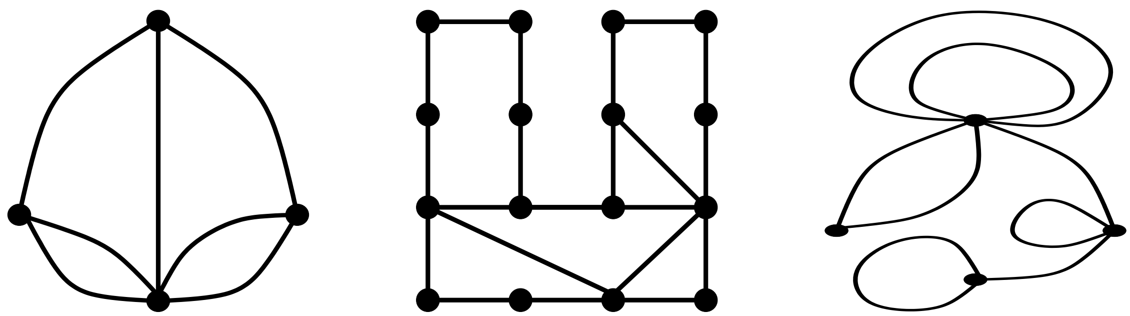




¶ **5.** There are of course non-planar graphs. Many problems in recreational mathematics are based on this property of graphs. Perhaps the best know of all is the following, know as the three utilities problem:

> There are three houses in a county, and there is a church, a school, and a supermarket. The owners of the houses want to build roads from their properties to each the church, school, and supermarket, and want to do that in a way as to avoid crossings. Is that possible?



¶ **6.** A concept that we need to introduce in order to solve the Königsberg Bridge Problem is that of **degree of a node** V. This is the number of edges that are incident to that vertex V, loops at V being counted twice because they have both ends at V. If the degree of V is odd, then V is called an odd vertex, otherwise it is called an even vertex. The **total degree** of a graph is the sum of the degrees of all its nodes.
For the graphs below, determine the degrees of their nodes, and then their total degree:

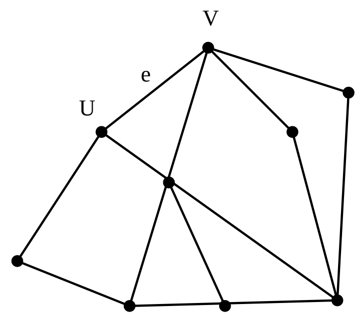


¶ **7.**

1. Explain why the total degree of a graph is always an even number. (Hint. What is the relation of this number to the number of edges of the graph?)

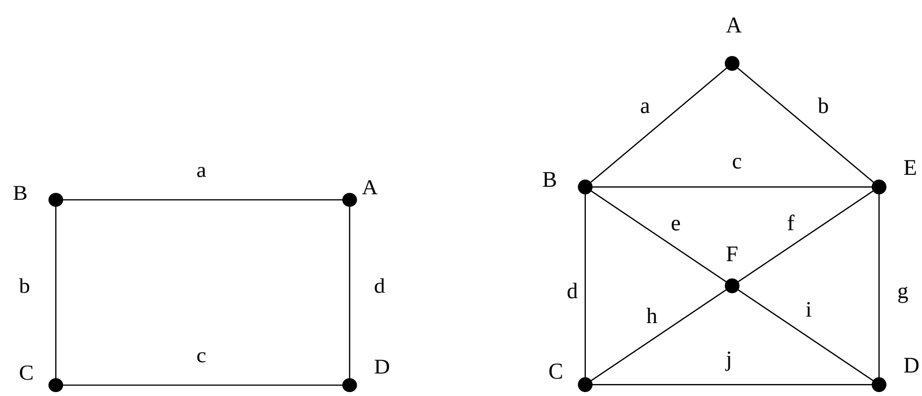
2. Explain why there is always an even number of nodes of odd degree.

¶ **8.** Other terminology that is used in graph theory and that will be relevant to solving the Königsberg Bridge Problem, is the following. A **path** in a graph is a sequence of nodes and edges (not necessarily distinct) of the form $UaVbWc\dots kZ$, so that any consecutive node-edge-node like UeV is so that the edge e joins the nodes U and V:



A path of the form $UaVb\dots kZ$ is said to join the vertex U to the vertex Z. A graph is **connected** if any two vertices can be joined by a path. The number of edges in the path is called the length of the path. If $U = Z$, the path is called a **circuit** or **cycle**.
The length of the path $UaVbWc\dots kZ$ is the number of edges of which is made.

¶ **9.** A path that traverses every edge of a graph exactly once is called an **Euler path**. An **Euler circuit** is defined analogously: it is a circuit that traverses every edge of the graph exactly once. Every Euler circuit is an Euler path, but not conversely.



The graph on the right above has an Euler circuit, and the graph on the right has an Euler path. Find them.

¶ 10. If a graph has an Euler circuit, then the degree of each of its vertices must be even: you arrive at a vertex via an edge e and you leave that vertex via a different edge $e' \neq e$. So each time that your Euler circuit passes through a vertex, it consumes 2 edges form the degree of that vertex.

Theorem 1. *A graph with an odd vertex cannot have an Euler circuit.*

Nevertheless, a graph with an odd vertex could have an Euler path. But the same reasoning based on parity proves the following.

Theorem 2. (a) *If a graph containing an odd vertex v has an Euler path, then such path must begin or end at v.*

(b) *If a graph containing two odd vertices v and w has an Euler path, then such path must begin at v and end at w, or viceversa.*

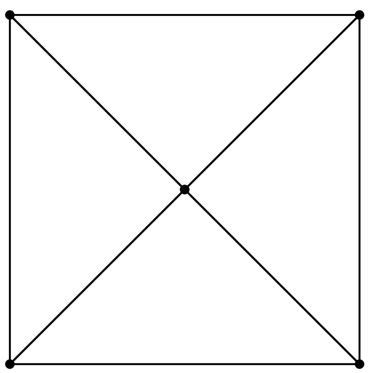
(c) *A graph with more than two odd vertices cannot have an Euler path.*

¶ 11. Euler gave in fact a necessary and sufficient condition for the existence of an Euler path or an Euler circuit.

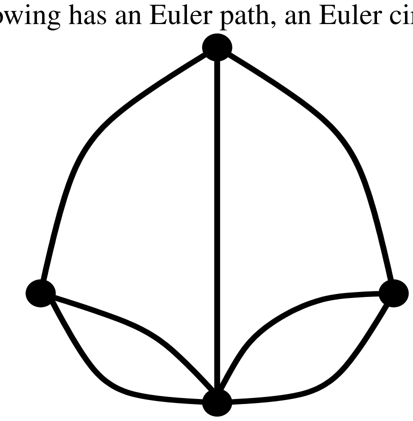
Theorem 3 (Euler)**.** (a) *A graph has an Euler circuit (or cycle) if and only if it is connected and all its vertices have even degree.*

(b) *A graph has an Euler path if and only if it is connected and it has at most two vertices of odd degree.*

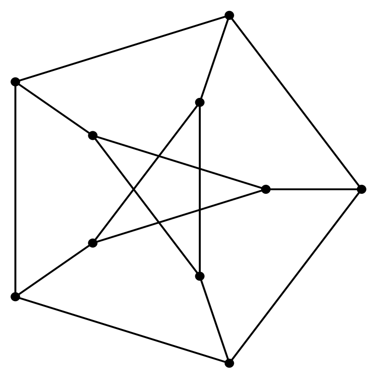
¶ 12. Determine which of the following has an Euler path, an Euler circuit, or neither.



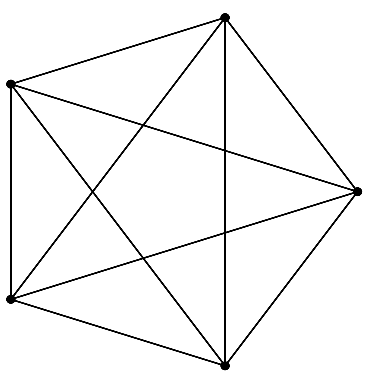
(a) CompleteLineGraph[4]



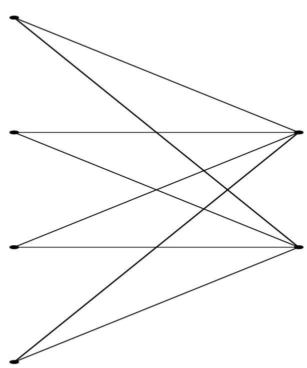
(b) Königsberg



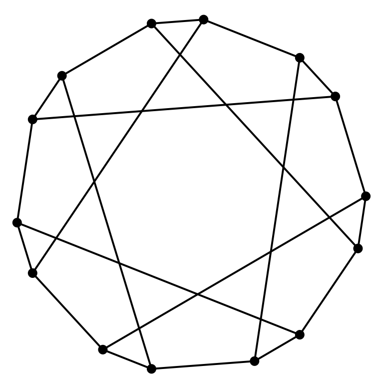
(c) PetersenGraph



(d) CompleteGraph[5]

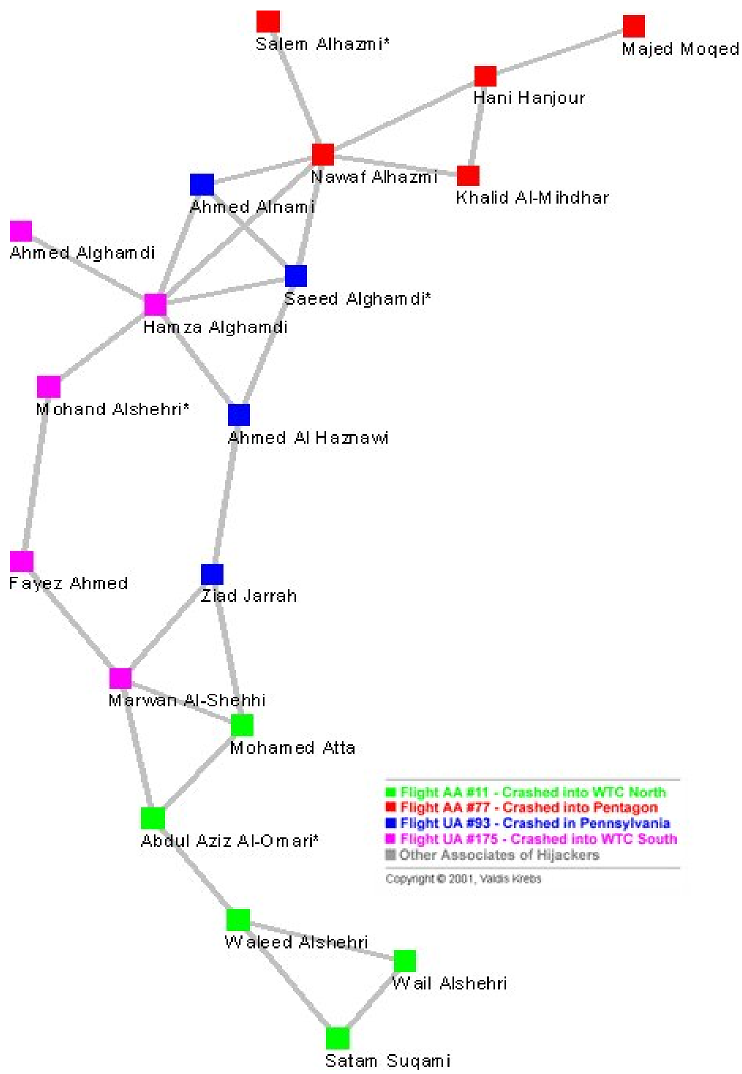


(e) CompleteGraph[4,2]



(f) CageGraph[3,6]

¶ **13.** A social network is a social structure made of individuals or organizations that are tied by one or more specific types of interdependence, like friendship, financial exchange, dislike, knowledge, beliefs, employment, education, or others.
Social network analysis views social networks in terms of graph theory, with nodes or vertices representing the individuals, and edges representing the ties between those individuals.
One of the most spectacular uses of this new branch of mathematics was to analyze the group of terrorist involved in the devastating 9/11 attacks. Using publicly available data, Valdis Krebs, a management consultant, created the following network for the group of Al Quaeda terrorist involved:



Looking at this network you cannot help to observe how distant some of the hijackers in the same team were from each other in the network. A key strategy of Al Quaeda seems to have been to keep members of the same team as disconnected as possible.

¶ **14.** A larger network including other associates of the hijackers contains more information:

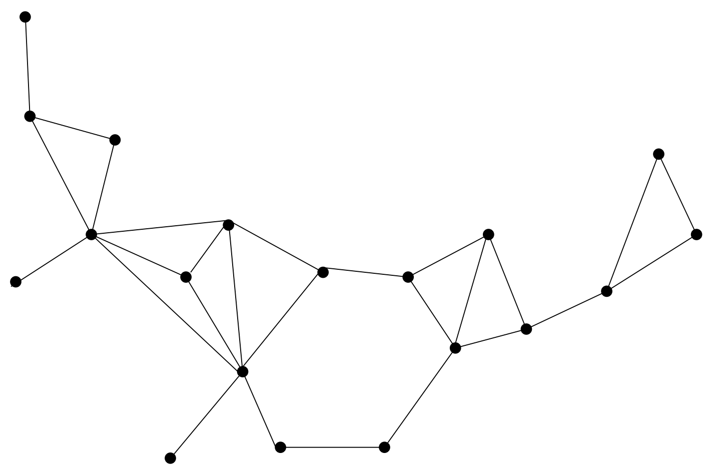
¶ **15.** The challenge for investigators in social network analysis is to extract information from any such social network. We would like to have tools for determining those individuals that play a key role in the network: leaders, facilitator, go-betweens, and so on. We would also like to have measures of the strength of cohesion, centralization, and any other measures of the network as a whole.

¶ **16.** One such measure is the **degree** of a node. An individual or entity represented by a node of high degree will likely play a leadership role, as leaders tend to have more direct connections with other individuals.

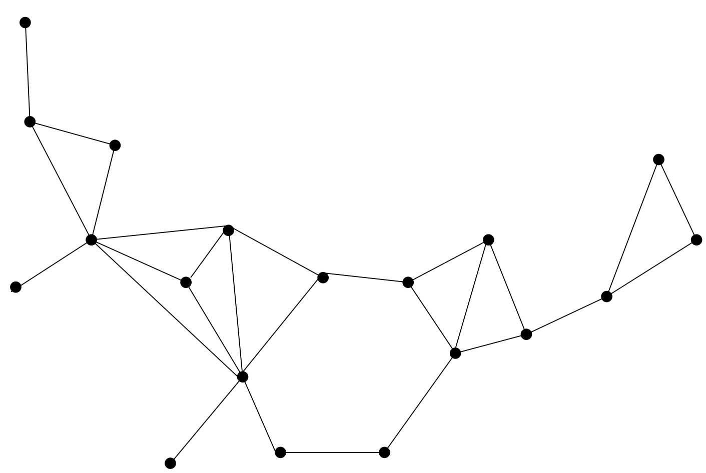
¶ **17.** Another measure is **betweenness** of a node. To define this measure, we need to introduce the concept of distance between two nodes U and V: it is the length of a shortest path from U to V. The concept of betweenness gives each node a score that reflects its role as stepping-stone along shortest paths between any other two nodes. If two nodes U and V have lots of shortest paths between them that go through another node W, then W should have a high betweenness score with respect to U and V. More specifically, the betweenness score of W as a link between U and V is

$$\beta(W|U, V) = \frac{\text{number of shortest paths from } U \text{ to } V \text{ through } W}{\text{number of shortest paths from } U \text{ to } V}$$

The overall betweenness score of a node W is the sum of all betweenness scores $\beta(W \mid U, V)$ for all possible nodes U and V.
Compute the betweenness score for the nodes in the following graph.



¶ 18. A third measure used by Krebs is the **closeness** score of a node. This measure indicates how close a node is to other nodes in the graph. For a node W you first calculate $d(W, U)$, the length of a shortest path from U to W. The sum of the reciprocals $\dfrac{1}{d(W, U)}$ for all nodes $U \neq W$ in the graph is the closeness score of W. If all the distances $d(W, U)$ are small, then their reciprocals are large, and so W will have a large closeness score.
Compute the closeness score for the nodes in the following graph.



Literature

[1] Valdis Krebs, *Uncloaking terrorist networks*, First Monday, Vol 7, No. 4, April 2002.