¶ **1.** The picture below reproduces a famous engraving by the artist Albert Dürer (1471–1528) entitled *Melencolia I*.

In the upper right corner there is a square subdivided into 16 smaller squares, and in each there is a number between 1 and 16:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The sum of the numbers in every row, in every column, and in each diagonal is the same. It is called a magic square of order 4.
In general, a magic square of order n is a square divided into n^2 cells, and each cell contains one of the consecutive numbers from 1 to n^2. The numbers are arranged in such way that the sum of each row, of each column and of each of the two diagonals, is the same.
Magic square have captivated humans since antiquity. The earliest recorded magic square of order 3, the Lo Shu square in China, dates to 2500 BC. The earliest recorded magic square of order 4 dates to the 11th century and was found in India. One of our founding fathers, Benjamin Franklin, was a magic square enthusiast, and in his papers one find numerous magic squares of order 16.
More recently, magic squares appear profusely in the novel *The Lost Symbol*, by Dan Brown.

¶ **2.** Any square of order 1 is a magic square, but there are no magic squares of order 2 where the entries are the consecutive numbers 1, 2, 3, and 4.

¶ **3.** There are several magic squares of order 3 made with all the consecutive numbers from 1 to 9. Try to find some:

¶ **4.** If you have successfully constructed more than one magic square of order 3, you will notice that they somehow look the same. For example, the middle number is always 5. This is not accident:
First, the sum $1+2+3+4+5+6+7+8+9=45$, and since this must be equally divided into 3 rows (or 3 columns), each row, each column, and each main diagonal has sum equal to 15. Take any magic square or order 3 like

A	B	C
D	E	F
G	H	I

Each column, each row, and each of the two diagonals have the same total sum 15. If you add the middle row and the two diagonals, and from that subtract the top and bottom rows, the result is

$$(D+E+F)+(A+E+I)+(G+E+C)-(A+B+C)-(G+H+I)=15+15+15-15-15$$

or $3E=15$, or $E=5$.
Any pair of the numbers 7, 8 and 9 adds up to at least 15, and so no two of these big numbers can be in the same row or in the same column or in the same diagonal. The possibilities are:

*		
	5	*
	*	

	*	
*	5	
		*

		*
*	5	
	*	

	*	
	5	*
*		

Note that these configurations are the same up to rotation or reflection of the square. Once one of these configurations is chosen for the big numbers, the number 6 mus be placed in a corner of that diagonal that contains no high number, because $5+6=11$ is already two big, forcing the number 4 to occupy the other corner of the same diagonal, like this:

6	*	
	5	*
*		4

The note that the numbers 1 and 4 cannot be in the same row, column or diagonal. thus leaving only one possible place for it.

6	*	
1	5	*
*		4

Finally, the 3 and the 6 cannot occupy the same row or column (because $3 + 6 = 9$ will force the other number to be a 6 also). This leaves just one possibility for 3 and hence for 2. Once these numbers are put in their appropriate cells, the big numbers are filled in by doing elementary arithmetic.

6	*	2
1	5	*
*	3	4

6	7	2
1	5	9
8	3	4

¶ **5. Magic Sum.** As we have seen, in a magic square of order 3, the magic sum (the sum of all the number in any row, or in any column, or in any diagonal) is 15. It turns out that there is a formula for the magic sum of a magic square of any order, and this information is useful for checking if a particular construction is correct.

A magic square of order n has n^2 cells, arranged into n rows of n cells each, and contains all the consecutive numbers from 1 to n^2. Adding all the number in all the rows we obtain a sum

$$1 + 2 + \cdots + n^2 \quad = \quad \underline{\hspace{3cm}}$$

This total sum is evenly distributed into n rows, so each row adds up to the magic sum S, which is:

$$S \quad = \quad \underline{\hspace{1.5cm}}$$

Use this formula to find the magic sum of a magic square of order n for the values of n in this table:

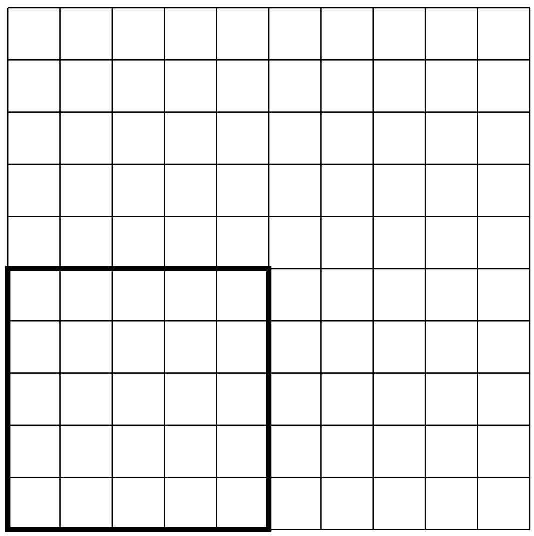
Order n	4	5	6	7	8
Magic Sum					

¶ **6.** There are two main problems regarding magic squares: How to construct a magic square of any given order? How many magic squares are there of any given order?

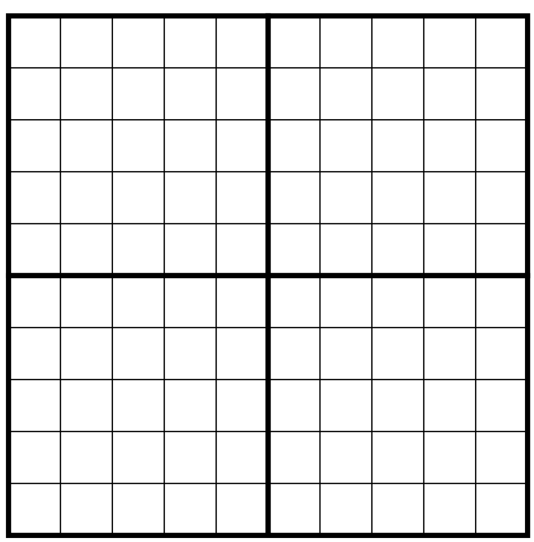
We have seen that there is one magic square of order 1, no magic square of order 2, one magic square of order 3 (up to rotation and mirror symmetry), and there are 880 magic squares of order 4 (again ignoring rotations and mirror symmetries). The number of magic squares of order 5 is $275, 305, 224$.

Constructing magic square of any given order is a good pastime. In the next pages we will explore several of the known methods.

¶ **7. Magic squares of odd order.** The following method applies to the construction of a magic square of odd order. To construct a magic square of order 5, place the original 5×5 grid and surround it by similar 5×5 grids, like this: Start by placing the number 1 in the middle of the top row. Then place the numbers 2 to 25 successively in a diagonal line that slopes upward to the right. If if happens that you run out of the original grid, and into one of the adjacent ones, reenter in the corresponding cell of the original grid. For example, after writing 1, up and right takes you out onto the fourth cell of the bottom row of the adjacent grid, and this cell corresponds to the fourth cell of the bottom row of the original grid. If it happens that you run into a cell that has already been filled, then drop down to the cell immediately below, and continue.



Here is a nice feature of this magic square: Rewrite it below in the cells of the 5×5 upper left subsquare, but subtracting 1 from each number (so it will have the numbers from 0 to 24). Then, on the upper right 5×5 subsquare, write the same numbers but in base 5 (for example, $14 = 2 \times 5 + 4$ is 24). Then place the first digits from each cell into the corresponding cell of the bottom-left subsquare, and the second digits into the cells of the bottom-right subsquare. These two subsquares have a peculiar feature, do you see it?



¶ 8. Latin squares
A Latin square of order n consists of n distincts symbols, each occuring n times, arranges in a square matrix of size n, in such a way that each row and each column contains exactly one of the n symbols. If the symbols are $0, 1, 2, \ldots, n-1$ (residues modulo n, then a special latin square can be contructed as an addition table modulo n. This is denoted $[1, 1]_n$, shown on the left for $n = 3$. More generally, if neither a nor b divides n, we construct the latin square $[a, b]_n$ by writing $0, a, 2a, 3a, \ldots$ on the left colums and $0, b, 2b, 3b, \ldots$ on the top row (reducing modulo n if necessary), and then completing each entry (i, j) with $ia + jb$ modulo n. The one on the right is denoted $[1, 2]_3$.

0	1	2
1	2	0
2	0	1

0	2	1
1	0	2
2	1	0

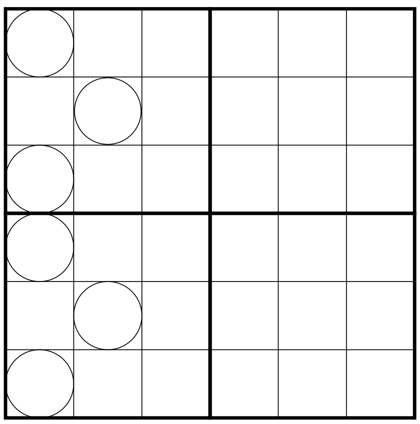
¶ 9. Construct $[2, 3]_5$

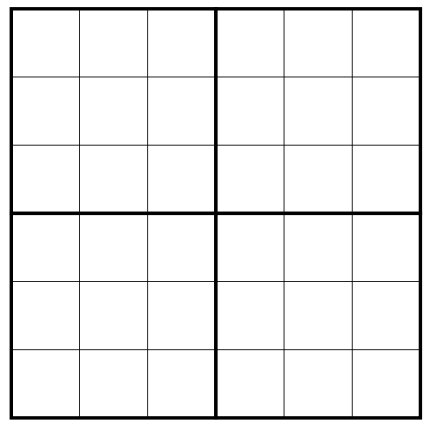
¶ 10. Eulerian squares
Two Latin squares are orthogonal if their superposition results in a square where each ordered pair of the n symbols occurs exacty once. The resulting square is called an Eulerian square. These came about after Euler posed in 1782 the following problem: is it possible to arrange 36 officers, each one of six different ranks and each belonging to one six different regiments, in a six by six square so that each row and each column contains exactly one of each rank and one of each regiment? He conjectured that the answer is NO, and this was confirmed by Tarry 118 years later.

¶ **11. Magic squares of order $2\times$ (Odd number).** Magic square of even order are more challenging to construct. One method applies to those of order $n = 2\times$ (Odd number). Here is how it works for order 6. First the 6×6 square is divided into 4 subsquares A, B, C, and D, of size 3×3 arranged as in this diagram:

A	C
D	B

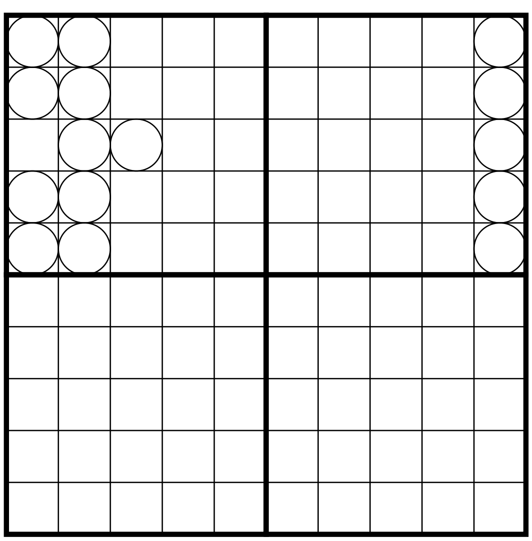
Then fill in each of the subsquares as magic squares, with numbers shifted appropriately: in this case of order 6, fill A with the numbers from 1 to 9, B with the numbers from 10 to 18, C with the numbers from 19 to 27, and D with the numbers from 28 to 36.

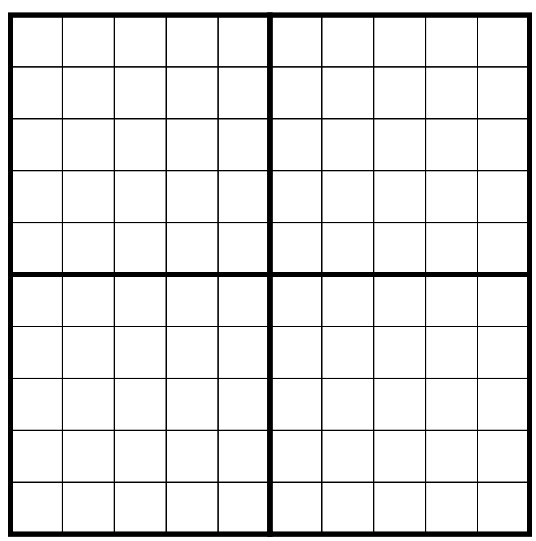




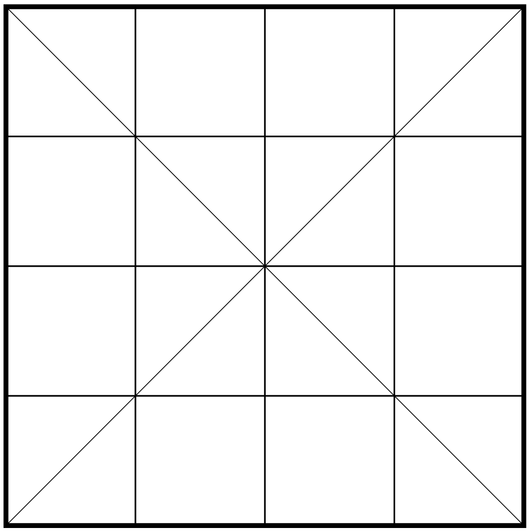
The result is not a magic square: the columns add up to the magic number, but the rows do not. To fix this, exchange the numbers in the circled cells of the subsquare A with those in the corresponding cells in subsquare D.

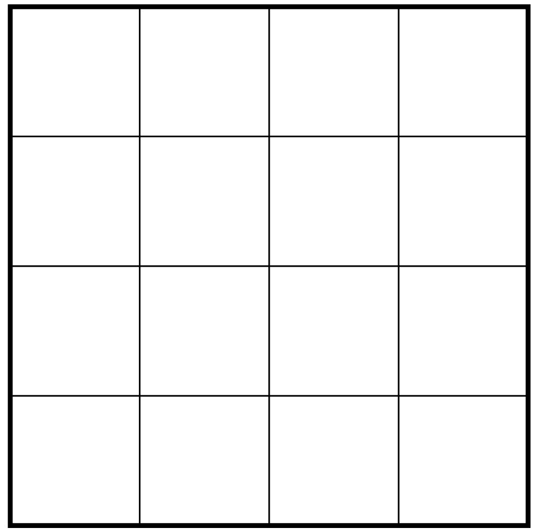
¶ **12.** Use the same technique to construct a magic square of order 10. First construct 4 magic squares of order 5, and the exchange the circle cells in A with the corresponding ones in D, and exchange the circled cells in C with the corresponding ones in B.



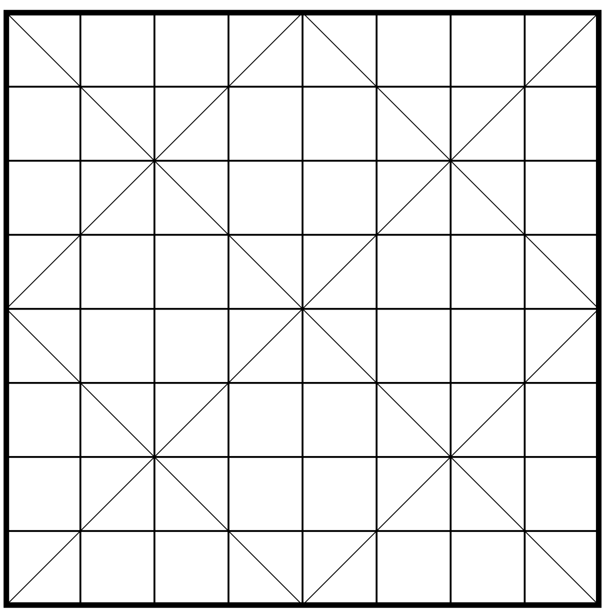


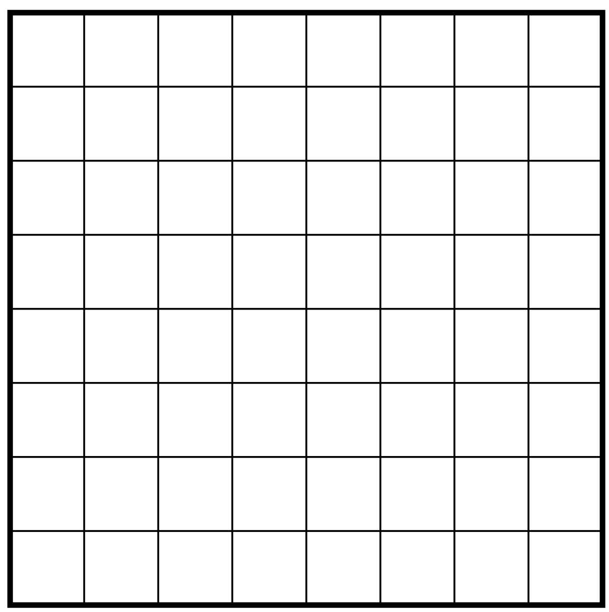
¶ **13. Magic squares of order a multiple of** 4**.** To construct a magic square of order 4, we write the numbers 1 to 16 in their natural order in rows of four, and then exchange the numbers in those diagonal cells that are opposite with respect to the center of the square.





¶ **14.** A magic square of order 8, or of order any multiple $4m$ of 4 can be constructed similarly, but taking into account more diagonals (namely, those of the 4 subsquares of order m).





¶ 15. Claude Fayette Bragdon (1866-1946) was a US architect based in Rochester, NY. He was fascinated with magic squares, and in particular with the so called magic line: a line from cell to cell following the natural order of the numbers in the cells. He utilized such magic lines as decorative patterns for his work.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

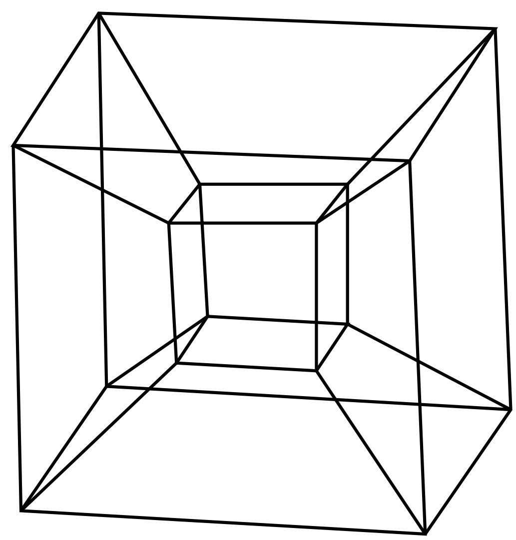
7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

¶ 16. Some magic squares have extra properties earning them the name of diabolic squares: besides the usual properties, diabolic squares are magic along broken diagonals like those containing $2, 12, 15, 5$ or $2, 3, 15, 14$ in the figure below:

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4



The figure on the right is a hypercube or tesseract: a cube of dimension 4. It has a total of 24 two-dimensional faces. Place each of the numbers of the magic square on the left on each of the 16 corners (zero-dimensional faces) of the tesseract, so that all four numbers in any of its faces add up to the magic sum.

Literature

[1] W. W. Rouse Ball, *Mathematical Recreations and Essays*, McMillan, New York, 1962.

[2] Martin Gardner, *The Second Scientific American Book of Mathematical Puzzles and Diversions*, Chicago, 1987.