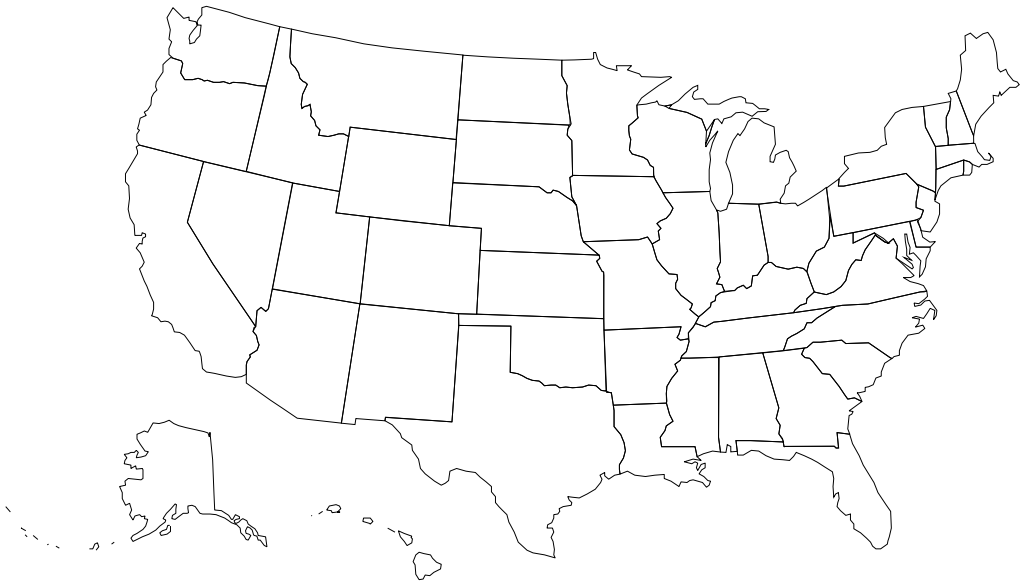
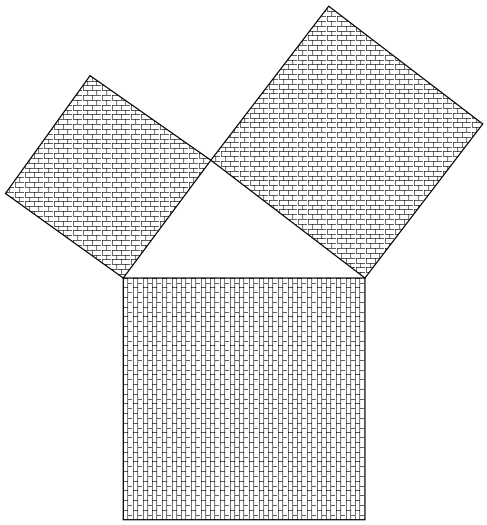
Area

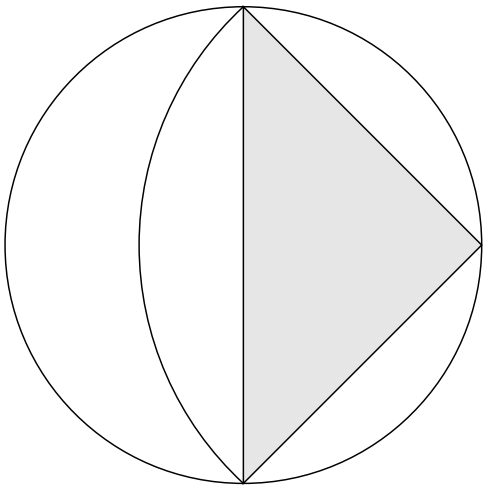
¶ 1. What is the area of the state of California? of Nevada? of Missouri?



Computing areas of planar figures, or comparing them, has been one of the first mathematical problems. Pythagoras Theorem relates states that the area of the big square equals the sum of the areas of the two smaller squares. One of the greatest feasts was accomplished by Hippocrates of Chios, who established that the area of the crescent moon shape on the left equals the area of the inscribed triangle.



Pythagoras of Samos



Hippocrates of Chios

¶ 2. Can you prove that the crescent moon and the inscribed triangle on the right figure have the same area?

Major accomplishments were obtained by Archimedes. He proved that the area of a circle is $\pi \times \text{Radius}^2$. For this he developed method of exhaustion which consists in drawing a larger polygon outside a circle and a smaller polygon inside the circle. As the number of sides of the polygon increases, it becomes a more accurate approximation of a circle. When the polygons had 96 sides each, he calculated the lengths of their sides and showed that the value of lay between $3 + 1/7$ (approximately 3.1429) and $3 + 10/71$ (approximately 3.1408).

¶ 3. In the figure below, an octagon of area A is inscribed in a unit circle, which is itself inscribed in another octagon of area B. Compute the areas A and B to obtain the estimate $A < \pi < B$

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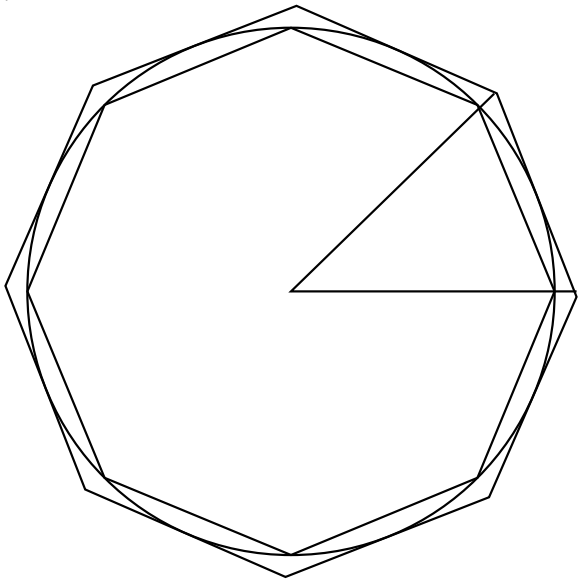
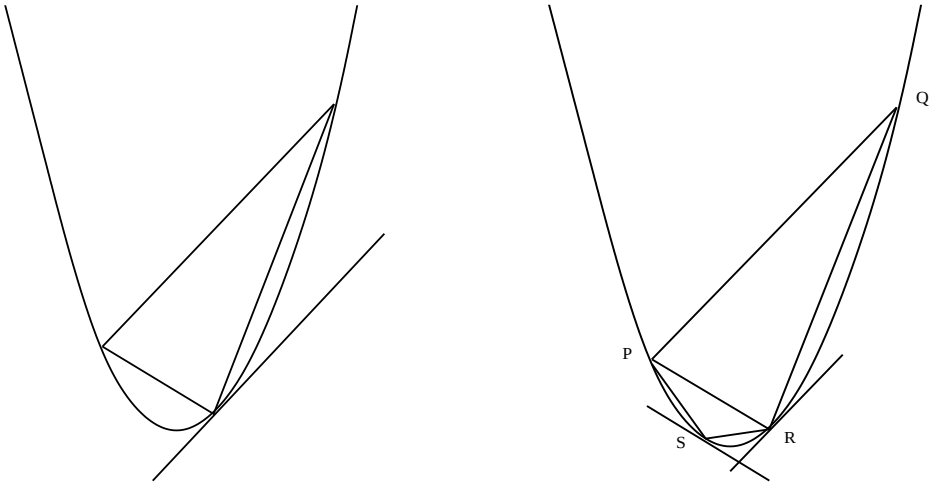


Another major accomplishment of Archimedes was his work on the parabola. There he established the following theorem: The area of a section of a parabola determined by a secant is $4/3$ of the area of the inscribed triangle determined by that secant.



¶ 4. His proof was based in the following geometric fact: The area of triangle PQR is 8 times the area of triangle PRS. Use or prove this fact to verify Archimedes theorem on the section of a parabola.

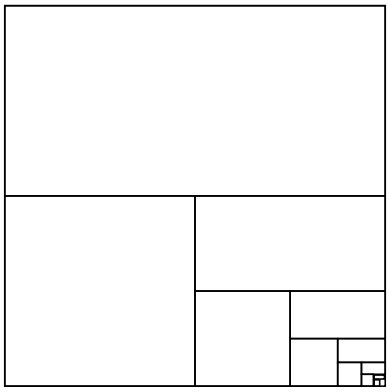
What is Area?

Area quantifies the space occupied by a two dimensional figure, much like length quantifies the expand of a 1 dimensional figure. Area is related to length in the following way:

1 The area of a rectangle is the product of the lengths of its sides.

2 If we cut a shape into several parts, the area of the original shape is the sum of the areas of the parts.

These two postulates permit us to compute the area of a variety of shapes. Any properties that we expect from area follow from these two rules. One such property is that if a shape is decomposed into several pieces and these are rearranged to form another shape, the area of the original shape and the area of the final shape are equal. Other property is that any two congruent shapes have the same area. Also, rule 2 continues to hold if we cut a shape into an infinite number of shapes, like the following unit square:



¶ 5. This unit square is divided into infinitely many rectangles and squares. Compute the sum of the areas of the parts and verify that the sum of the areas of the smallest pieces is the total area of the unit square.

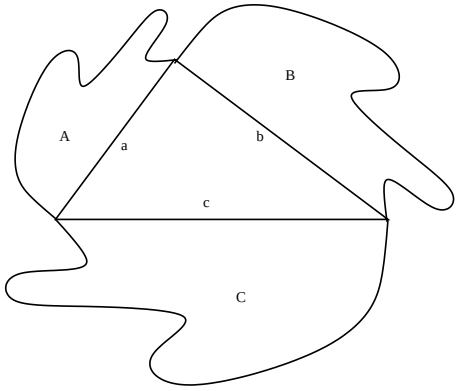
Another property is that a similarity that changes lengths by a factor L will change areas by a factor L^2.

¶ 6. The scaling property of areas has the following consequence which generalizes the Pythagorean theorem. Three similar shapes are adjacent to the sides of a right triangle, as in the figure below. Then the area of the figure adjacent to the hypothenuse is equal to the sum of the areas of the shapes adjacent to the legs of the triangle. In the figure, $A + B = C$. To verify this, do the following

1. If c is the length of the hypothenuse, and a and b are the length of the sides, then $(a/c)^2 + (b/c)^2 = 1$.

2. The areas A and C are related by $c^2A = a^2C$ and the areas B and C are related by $c^2B = b^2C$.

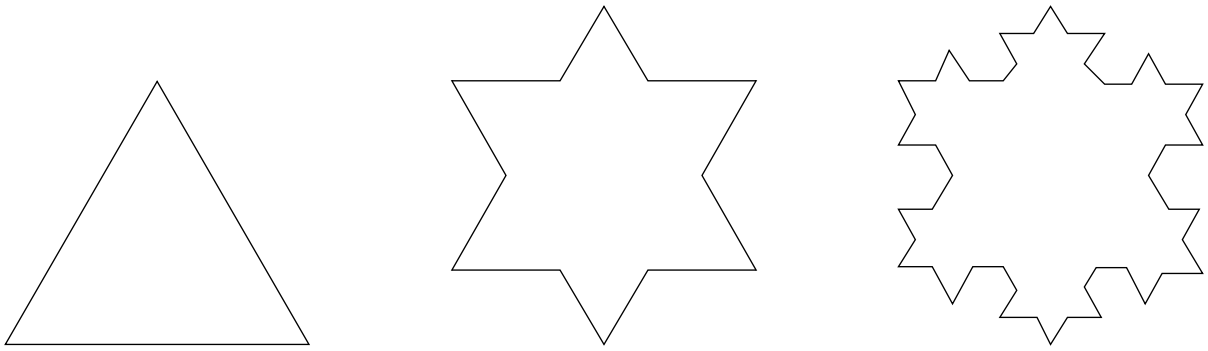
3. From 1 and 2, deduce that $A + B = C$



Planimeters

Mechanical and Electrical (Digital) Planimeters are excellent area measuring tools to use when you measure irregular shaped areas on plans or drawings. They eliminate the need for grids, charts, or calculations done by hand. Planimeters are used by professionals in these areas: Surveying, Civil Engineering, Cartography, Architecture, Medicine, Clothing, Industry, Boat & Ship Design, Office, Store & Plant Planning, Biology, and more.

Planimeters compute area of a shape in terms of the length of the boundary of that shape, using a mathematical theorem on two dimensional integrals. For this computation to be efficient, the boundary of the shape must be sufficiently regular. Here is famous example where this does not hold. It is called the Koch snowflake, and is an example of a fractal curve. The curve has infinite length but it encloses a region of finite area.



¶ 7. Find the area enclosed by the Koch curve if the original triangle was an equilateral triangle of unit length.

Literature

[1] John Bryant and Chris Sangwin, *How Round is your Circle?*, Princeton University Press, 2008.