Which day of the week?[1]

¶ **1.** Because the cyclic nature of dates, modular arithmetic is particularly suitable for calendrical calculations. Here we will work out one of the most such fundamental calculations, namely, to find out on which day of the week fall a particular date, like your birthday.
The days of the week repeat themselves in periods of 7, so instead of the usual names we can give each a number:

$$\text{Sunday} = 0, \text{Monday} = 1, \text{Tuesday } = 2, \text{Wednesday} = 3, \text{Thursday} = 4, \text{Friday} = 5, \text{Saturday} = 6.$$

In this way, each integer corresponds to a day-of-the-week, namely, the day determined by its remainder modulo 7. If we know the day-of-the-week of a particular date (like today, which is a Tuesday, or 2), then to find the day-of-the-week of any other date we will simply have to find the number of days that are there between today and that date, and find the remainder of that number after dividing by 7. For example, there are 151 days until July 4th. The remainder of dividing 151 by 7 is 4. Since today is a 2, July 4th is a $2 + 4 = 6$, or a Saturday.

¶ **2.** Use the same method to find the day-of-the-week of January 1st, 2009.

If the number of days in a year was a multiple of 7, then all dates would fall on the same day-of-the-week every year. However, the number of days in the year is 365, which has remainder 1 after dividing by 7, and is 366 days for leap years, which has remainder 2 after dividing by 7. This means that for an ordinary year the day-of-the-week number will increase by 1 in the following year, if this was not a leap year. So for example, today is Tuesday, February 3rd, 2009, so it will be Wednesday, February 3rd, 2010. This pattern is broken up by leap years, which occur every fourth year. To make things more difficult, the extra day of a leap year is added “in the middle”as February 29th. So for example, February 1st falls on a Sunday this year, on a Monday on 2010, on a Tuesday on 2011, and on a Wednesday on 2012, while March 1st also falls on a Sunday this year, on a Monday on 2010, on a Tuesday on 2011, but on a Thursday on 2012.
But there are still more problems to take into account. A year is actually 365.2422 days, not just 365. To account for this small error of almost $1/4$ of a day, a leap year of 366 days occurs every other 4th year as prescribed by the Julian calendar introduced by Julius Cesar. Unfortunately, .2422 is not exactly $1/4$, so calendrical errors were accumulating inexorably, and for example, seasons started to drift. To eliminate this errors, Pope Gregory XIII introduced the Gregorian calendar in 1582. In this calendar, exceptions to the 4th year rule for leap years were made in order to accommodate for the small error: as with the Julian calendar before, years whose number is divisible by 4 are leap years, but now century years like

$$1700, 1800, 1900, 2100, 2200, \cdots$$

where the number of centuries is not divisible by 4 are not longer leap years, while century years like

$$1600, 2000, 2400, \cdots$$

continue to be leap years.
The exact date on which the Gregorian calendar was adopted was October 15th, 1582. In fact, 10 days were omitted from the year 1582 in order to take into account the accumulated errors. Our calculations below will find the day-of-the-week name for dates according to the Gregorian calendar.

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Note also that not all the countries adopted the Gregorian calendar in 1582, because the Pope had no jurisdiction over non-catholic countries; for example, England and its colonies adopted it in 1752.
In the calculations that follow, the square bracket around a number denotes the whole number that is less than or equal to that number. For example, $[3] = 3$, $[1.2] = 1$, $[-3.5] = -4$. In your TI-84, this is the function called `int( )`

¶ **3.** Given a day in the form Year, Month, Day, we will compute what is called below the Month Code M, the Year Code Y, and the Day Code D. Then we find the remainder of the sum $Y + M + D$ after dividing by 7, obtaining the numerical value of the day-of-the-week for the given date.

¶ **4** (Month Code M)**.** Because the extra day of a leap year is added as February 29th, we let the month number be $m = 1$ for March, $m = 2$ for April, and so on, with January and February having $m = 11$ and $m = 12$ for the previous year, respectively.
Then the month code is

$$M = \left[\frac{1}{5}(13 \cdot m - 1)\right].$$

Example. January 6, 2007 is Day 6 of Month 11 of Year 2006. Then

$$M = \left[\frac{1}{5}(13 \cdot 11 - 1)\right] = 28.$$

¶ **5. Problem.** Find the month code of the following dates:

(a) February 3rd, 2009

(b) April 1st, 2010

(c) Your birthday

¶ **6** (Year Code Y)**.** The year number will be the actual year for all months, except for January and February dates, when it will be the actual year minus 1.
Write the year number $N = 100 \cdot c + y$. Then

$$Y = y + [y/4] - 2c + [c/4].$$

Example. January 6, 2007 has $N = 100 \cdot 20 + 6$, so $c = 20$ and $y = 6$. Therefore:
$Y = 6 + [6/4] - 2 \cdot 20 + [20/4] = -28$.

¶ **7. Problem.** Find the Year Code for the following dates

(a) February 14th, 2009

(b) April 1st, 2010

(c) Your birthday

¶ **8** (Day Code D)**.** The value of D is simply the day number of the date.

Example. January 6, 2007 has $D = 6$.

¶ **9. Problem.** Find the Day Code for the following dates

(a) July 4th, 1776

(b) March, 14th 1592

(c) Your birthday

¶ **10.** To find the day of the week W, do

$$W = M + D + Y \pmod 7$$

and use 0 for Sunday, 1 for Monday, and so on.

Example. January 6, 2007 has $W = 28 - 28 + 6 = 6 \pmod 7$, which is Saturday.

¶ **11. Problem.** What day-of-the-week was your birthday?

¶ **12.** We can write a TI-84 program to compute the day of the week. The program will ask you to input the year number `Y`, the month number `M` and the day number `D`. For example, for February 3rd, 2009, you will input `Y=2009`, `M=2` and `D=3`.

```
PROGRAM:DOW
:Prompt Y
:Prompt M
:Prompt D
:If M>2
:Then
:int((13*(M-2)-1)/5)►M
:Else
:Y-1►Y
:int((13*(M+10)-1)/5)►M
:End
:int(Y/100) ► C
:Y-100*C ► Y
:Y+int(Y/4)-2*C+int(C/4)►Y
:D+M+Y►D
:D-int(D/7)*7►D
:Disp D
```

Literature

The theoretical details of the calculations described here for finding the day of the week can be found in a number theory books. For example:

> *Invitation to Number Theory*, by Oystein Ore, The Mathematical Association of America, 1967.

A very complete book on all kinds of calendars is

> *Calendrical Calculations*, by Nachum Dershowitz and Edward Reingold, Cambrigde University Press, 2008.