

Gauss' childhood problem¹

Gauss (1777–1855) is considered to be the greatest mathematician since antiquity. He made important contributions to many fields, not only in mathematics, but in other areas of science. Already in elementary school, he made quite an impression on his teacher. This asked Gauss, then 8 years old, to add all the numbers from 1 to 100. Before the teacher could continue with the lecture of the day, Gauss had the answer: 5050. The teacher was surprised. He asked again, but now to find the sum of all the numbers up to 1,000. And Gauss answered right away: 500,500.

How did he do it? It is clear that he was not adding all the numbers from 1 to 1000 one after another, like $1 + 2 = 3$, $3 + 3 = 6$, $6 + 4 = 10$, \dots , $499,000 + 1,000 = 500,500$, as that would take quite some time (and errors will creep up quite easily). What Gauss actually did was to find a formula, an algebraic expression $s(N)$ on the variable N , so that when you plug in a value for N you get the value

$$s(N) = 1 + 2 + 3 + \dots + N.$$

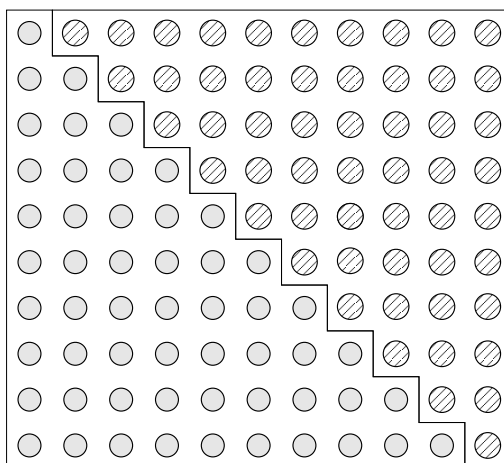
of the sum of the first N whole numbers $1, 2, \dots, N$. The algebraic formula is

$$s(N) = \frac{N(N + 1)}{2}.$$

In the next sections we will discover different ways of arriving at this formula.

1 Graphical Proof

The following picture should convince you that Gauss formula $s(N) = \frac{N(N + 1)}{2}$ for the sum $1 + 2 + \dots + N$ is correct.



The box is a rectangle of side (measured in units given by the small disks) N by $N + 1$ (in the picture, $N = 10$). It thus contains a total of $N(N + 1)$ disks. The big rectangle is divided into two halves each having the same number of small squares: $1 + 2 + \dots + N$. Hence

$$1 + 2 + \dots + N = \frac{N(N + 1)}{2}$$

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2 Educated Guess

We may suspect that there is a “polynomial” algebraic expression for $s(N)$, and try to obtain its true expression. We compute a few values by hand: $s(0) = 0$, $s(1) = 1$, $s(2) = 1 + 2 = 3$, $s(3) = 1 + 2 + 3 = 6$, and with these values we make a list:

N	0	1	2	3	4	5
$s(N)$	0	1	3	6	10	15

In the next row we write the consecutive differences, like this:

N	0	1	2	3	4	5
$s(N)$	0	1	3	6	10	15
Δ^1	1	2	3	4	5	

And in the next row we write the consecutive differences of the number in previous row:

N	0	1	2	3	4	5
$s(N)$	0	1	3	6	10	15
Δ^1	1	2	3	4	5	
Δ^2		1	1	1	1	

We continue this process: each new row is made up by the differences of consecutive terms in the previous row, and so on until we have a row with only 0's in it:

N	0	1	2	3	4	5
$s(N)$	0	1	3	6	10	15
Δ^1	1	2	3	4	5	
Δ^2		1	1	1	1	
Δ^3			0	0	0	

This indicates that if $s(N)$ is to be a polynomial expression, it must have degree 2 because the all the third differences Δ^3 are equal to 0.

To find the actual polynomial (that is, its coefficients), we need to examine the table of differences in greater detail. They are hidden in the left most numbers of each row, but not in a straightforward manner. We do a general case and consider a polynomial of degree 2 of the form $p(N) = aN^2 + bN + c$. The table of differences is

N	0	1	2	3	4
$p(N)$	c	$a + b + c$	$4a + 2b + c$	$9a + 3b + c$	$16a + 4b + c$
Δ^1	$a + b$	$3a + b$	$5a + b$	$7a + b$	$9a + b$
Δ^2		$2a$	$2a$	$2a$	$2a$
Δ^3			0	0	0

The table shows the relationship between the coefficients of $p(N)$ and the first entries of each row in the table of differences (those in a box). If we look now at our difference table for $s(N)$ we can guess that it has the form $s(N) = aN^2 + bN + c$ where the numbers a , b , and c satisfy:

$$\begin{aligned} c &= 0 \\ a + b &= 1 \\ 2a &= 1 \end{aligned}$$

from what we quickly obtain that $a = 1/2$, $b = 1/2$, and $c = 0$; or that

$$s(N) = \frac{1}{2}N^2 + \frac{1}{2}N.$$

3 Problems for Practice

The method for finding an algebraic expression for $s(N) = 1 + 2 + \dots + N$ given a few initial values $s(1) = 1$, $s(2) = 3$, works also for other sequences which have a polynomial general formula. Here are some problems for you to practice with this method.

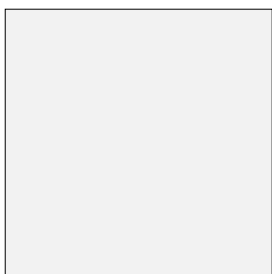
¶ 1. Find polynomial formulas $p(N)$ that give the following sets of numbers corresponding to $N = 0, 1, 2, 3, 5, \dots$:

- (i) 0, 0, 6, 24, 60, 120.
- (ii) 0, 1, 4, 10, 20, 35.
- (iii) 0, -1, 4, 27, 80, 175.

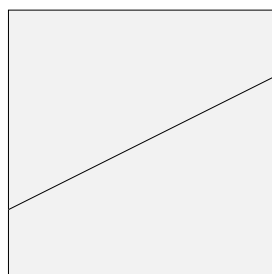
¶ 2. Find short expressions for the following (N denotes a whole number):

- (i) $1 + 3 + 5 + 7 + \dots + (2N - 1)$.
- (ii) $1^2 + 3^2 + 5^2 + \dots + (2N - 1)^2$.
- (iii) $0^3 + 1^3 + 2^3 + \dots + N^3$.

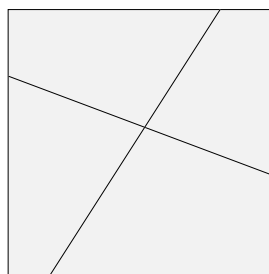
¶ 3. Find a formula for the maximal number of regions into which N lines can separate the plane.



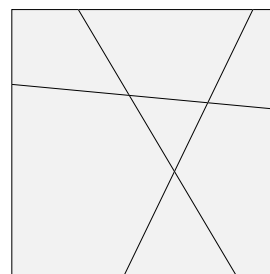
No lines, 1 region



1 line, 2 regions

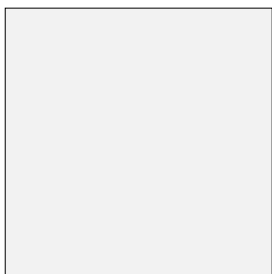


2 lines, 4 regions

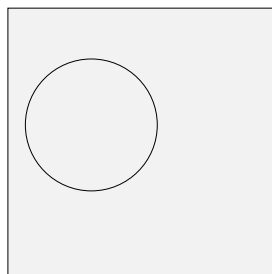


3 lines, 7 regions

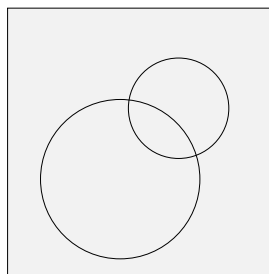
¶ 4. Find an expression $c(N)$ for the number of regions into which the plane is divided by N circles arranged so that any two intersect in two points and no three have a point in common.



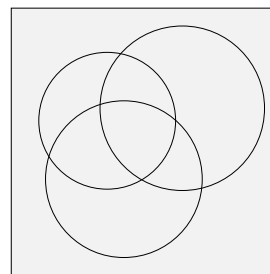
No circles, 1 region



1 circle, 2 regions



2 circles, 4 regions



3 circles, 8 regions

4 Verification by Induction

Suppose that you have done your calculation for $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$. How do you figure out the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$ with the least amount of work possible? It is true that it is not much work to add 10 numbers, but if you had to pay \$10 for each sum, that would be \$100. However, if you pay attention, you realize that once you know the answer to the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$ you only need one more sum to get the value of the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$. Indeed

$$(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) + 10 = 45 + 10 = 55$$

Proof by **Mathematical Induction** follows this pattern: We want to verify that a formula, algebraic expression, holds true for all the values of the parameter, a whole number.

- We verify that the formula holds true for the smallest possible value
- If we know that the formula holds true for the numbers $< N$, then it also holds true for the number N .
- From that we conclude that the formula holds true for all the numbers.

To illustrate how this works, we return to the Gauss' childhood problem. We have guessed that the sum of the first N whole numbers has the expression $s(N) = \frac{N(N+1)}{2}$.

(1) For the smallest possible value of N , namely, for $N = 1$, we have the sum $1 = \frac{1(1+1)}{2}$, so the expression $s(N) = \frac{N(N+1)}{2}$ is correct for $N = 1$.

(2) Given that we know that

$$1 + 2 + \dots + (N - 1) = \frac{(N - 1)(N)}{2}$$

we are supposed to verify that

$$1 + 2 + \dots + (N - 1) + N = \frac{N(N + 1)}{2}.$$

But

$$\begin{aligned} 1 + 2 + 3 + \dots + (N - 1) + N &= [1 + 2 + 3 + \dots + (N - 1)] + N && \text{(set up expression)} \\ &= \frac{(N - 1)N}{2} + N && \text{(by (1) above)} \\ &= \frac{(N - 1)N + 2N}{2} && \text{(combine fractions)} \\ &= \frac{N^2 - N + 2N}{2} && \text{(simplify numerator)} \\ &= \frac{N^2 + N}{2} \\ &= \frac{N(N + 1)}{2} && \text{(common factor)} \end{aligned}$$

5 TI-84 Programs

Your TI-84 calculator can be programmed to compute the sum $1 + 2 + 3 + \dots + N$ in a variety of ways. Each of program below illustrates a different approach to writing such a program. These programs also illustrate the variety of programming commands available in your TI-84. Below, the symbol “▶” represents the TI-84 key $\boxed{\text{STO } \blacktriangleright}$. Note also the difference between the “subtract” key $\boxed{-}$ and the “negative sign” key $\boxed{(-)}$.

PROGRAM:GAUSSA

```
:Prompt N
:0 ▶ S
:For(I,1,N,1)
:S+I ▶ S
:End
:Disp S
```

PROGRAM:GAUSSB

```
:Prompt N
:For(I,N-1,1,(-)1)
:N+I ▶ N
:End
:Disp N
```

PROGRAM:GAUSSC

```
:Prompt N
:Disp N*(N+1)/2
```

PROGRAM:GAUSSD

```
:Prompt N
:0 ▶ S
:1 ▶ I
:While I ≤ N
:S+I ▶ S
:I+1 ▶ I
:End
:Disp S
```

PROGRAM:GAUSSE

```
:Prompt N
:N ▶ S
:1 ▶ I
:Repeat I=N
:S+N-I ▶ S
:I+1 ▶ I
:End
:Disp S
```