

Mathematical Models in Electrical Engineering

- **Complex Variables**
- **Linear Algebra**
- **Partial Differential Equation**

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July 2021

CHAPTER 1

Complex Variables

Overview

Complex numbers and their algebraic operations are presented in this Chapter in Sections 1.1 and 1.2. Complex functions, set, domain, and range are discussed in Section 1.3. Section 1.4 covers limit of a function, continuity, derivative, and analytic function. Section 1.5 covers Cauchy Riemann equations and harmonic functions. Exponential and logarithmic functions are discussed and presented in Section 1.6, trigonometric and Inverse trigonometric in Sections 1.7, and hyperbolic and inverse hyperbolic in Section 1.8.

1.1 – Complex Numbers and Basic Operation

Students taking algebra are taught to set the function $y = f(x)$ equal to zero in order to obtain the x-intersections. Let us consider the following function:

$$y = f(x) = x^3 + 7x^2 + 31x + 25$$

If y is set to zero, only $x = -1$ is an acceptable solution as an x-intersection. Hence the graph of y versus x intersects the x axis only at one point, even though setting a polynomial of degree three to zero provides three roots. In this example the other two roots happen to be complex numbers and the graph of y versus x deals with real numbers.

A complex variable z can be written as $z = x + iy$ or as an ordered pair $z = (x,y)$. The real part of z is x and the imaginary part of z is y . The notation (x,y) is rarely used in engineering and instead, rectangular form $x + iy$ or polar form is utilized. The polar form of a complex number is presented in Section 1.2.

$$x = \text{Re}(z) \quad y = \text{Im}(z) \quad (1.1)$$

If $x = 0$, z is a pure imaginary number and if $y = 0$, z is a pure real number. The term i is the imaginary unit defined as:

$$i = j = \sqrt{-1} \quad = (0,1) \text{ and } i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \text{ etc.} \quad (1.2)$$

$$z = (x,y) = x(1,0) + y(0,1) = x + iy = x + yi \quad (1.3)$$

Complex number $z_1 = x_1 + iy_1 = (x_1, y_1)$ can be geometrically shown in complex plane as presented in Figure 1.1. The horizontal axis (x -axis) is referred to as the real axis and the vertical axis (y -axis) is referred to as imaginary axis.

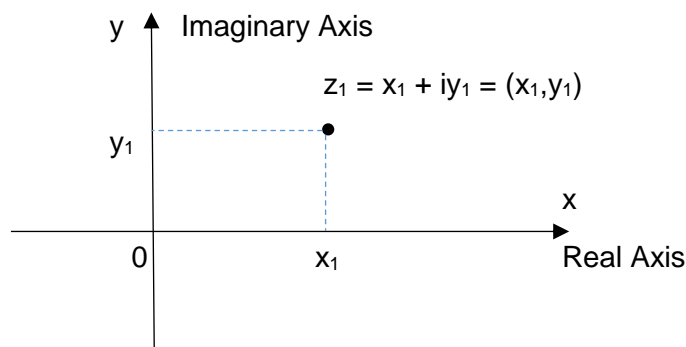


Figure 1.1 Complex Plane, Rectangular Form

Just as the Cartesian Coordinate system the real axis and the imaginary axis divide the complex plane into four quadrants

Quadrant I:	Real Part > 0	Imaginary Part > 0
Quadrant II:	Real Part < 0	Imaginary Part > 0
Quadrant III:	Real Part < 0	Imaginary Part < 0
Quadrant IV:	Real Part > 0	Imaginary Part < 0

Pure real numbers are on the horizontal (real) axis and pure imaginary numbers are on the vertical (imaginary axis). The origin "0" defines $z = (0,0) = 0 + i0$. Figure 1.2 shows the locations of some complex numbers in the complex plane.

Example 1: $z_1 = -4$ $z_2 = +5$ $z_3 = -i3$ $z_4 = +i4$
 $z_5 = 2+i2$ $z_6 = -2+i3$ $z_7 = -2-i$ $z_8 = 1-i3$

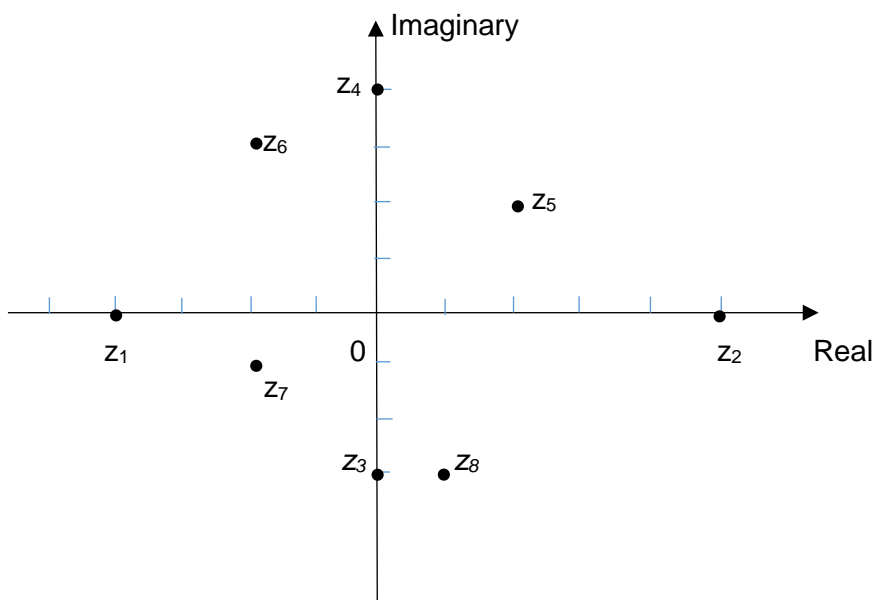


Figure 1.2 Examples of Some Complex Numbers

Complex Conjugate \bar{z} in Rectangular Form

The complex conjugate of a complex number $z = x + iy = (x,y)$ is denoted by \bar{z} and is defined as:

$$\bar{z} = x - iy = (x,-y) \quad (1.4)$$

Geometrically \bar{z} is the reflection of z with respect to real axis. Clearly $\bar{\bar{z}} = z$ and the complex conjugate of a pure real number is itself.

Example 2: $z_1 = 3 - i4 = (3,-4)$ $\bar{z}_1 = 3 + i4 = (3,4)$
 $z_2 = i2 = (0,2)$ $\bar{z}_2 = -i2 = (0,-2)$
 $z_3 = 10 = (10,0)$ $\bar{z}_3 = 10 = (10,0)$

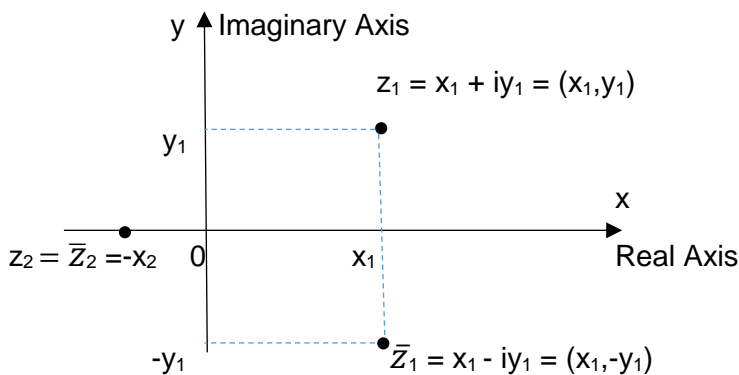


Figure 1.3 Complex Conjugate in Rectangular Form

Basic Algebraic Operations – Rectangular Form

Consider $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$

Equality

$$z_1 = z_2 \text{ implies } x_1 = x_2 \text{ and } y_1 = y_2 \quad (1.5)$$

Both the real part and the imaginary part must be equal.

Addition

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) = (x_1 + x_2, y_1 + y_2) \quad (1.6)$$

Subtraction

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2) = (x_1 - x_2, y_1 - y_2) \quad (1.7)$$

Multiplication

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 & kz &= k(x_1 + iy_1) = kx_1 +iky_1 \\ z_1 \cdot z_2 &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) & & (1.8) \end{aligned}$$

Integer Power ($n = 0, 1, 2, \dots$)

Integer power of z is given by; $z^n = \underbrace{z \cdot z \cdot \dots \cdot z}_n$. Depending on the value of n , it is probably best for this operation to be performed in polar form as discussed in Section 1.2.

Division

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = a + ib \quad z_2 \neq 0$$

The equations for a and b are to be determined. To do so both sides of the above equation are multiplied by z_2 .

$$(a + ib)(x_2 + iy_2) = x_1 + iy_1$$

Using equality, we can write:

$$\begin{aligned} ax_2 - by_2 &= x_1 \\ ay_2 + bx_2 &= y_1 \end{aligned}$$

Solving the two equations with unknown a and b results in:

$$a = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} \quad \text{and} \quad b = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \quad \text{Hence,}$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \right) \quad (1.9)$$

It should also be noted that $z^{-n} = (1/z)(1/z)\dots(1/z)$, where n is a positive integer and $z \neq 0$.

Some Complex Conjugate Application

Complex Conjugate is used in many applications. Some fundamental applications are considered here.

$$z + \bar{z} = (x + iy) + (x - iy) = 2x \quad \Rightarrow \quad x = \operatorname{Re} z = \frac{z + \bar{z}}{2} \quad (1.10)$$

$$z - \bar{z} = (x + iy) - (x - iy) = i2y \quad \Rightarrow \quad y = \operatorname{Im} z = \frac{z - \bar{z}}{2i} \quad (1.11)$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 \quad (1.12)$$

Clearly $z\bar{z}$ is real and positive and only zero if $z = 0$. This provides an alternate method of dividing complex numbers z_1 by z_2 . The numerator and the denominator of z_1/z_2 are multiplied by the complex conjugate of the denominator and the real and imaginary parts are separated.

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \right)$$

As can be observed this is the same results obtained earlier.

It can also be shown that:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \quad (1.13)$$

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2 \quad (1.14)$$

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \quad (1.15)$$

$$\overline{(z_1/z_2)} = (\bar{z}_1) / (\bar{z}_2) \quad (1.16)$$

Example 3: If $z_1 = 2 + i5$ and $z_2 = -4 + i3$, evaluate

a) $z_1 + z_2$ b) $z_1 - z_2$ c) $z_1 \cdot z_2$ d) z_1/z_2 e) \bar{z}_1 f) $1/\bar{z}_2$

- a) $z_1 + z_2 = (2 + i5) + (-4 + i3) = -2 + i8$
 b) $z_1 - z_2 = (2 + i5) - (-4 + i3) = 6 + i2$
 c) $z_1 \cdot z_2 = (2 + i5) \cdot (-4 + i3) = -8 + i6 - i20 + i^2 15 = -23 - i14$
 d) $z_1/z_2 = (2 + i5)/(-4 + i3)$

$$\frac{z_1}{z_2} = \left(\frac{2+i5}{-4+i3} \right) \left(\frac{-4-i3}{-4-i3} \right) = \frac{-8-i6-i20-i^2 15}{25} = \frac{7}{25} - i \frac{26}{25}$$

 e) $\bar{z}_1 = 2 - i5$
 f) $1/\bar{z}_2 = \left(\frac{1}{-4-i3} \right) \left(\frac{-4+i3}{-4+i3} \right) = \frac{-4+i3}{25} = -\frac{4}{25} + i \frac{3}{25}$

Fundamental Laws:

Here it assumed $z_1, z_2,$ and z_3 belong to the set S of complex numbers

Commutative Law (Addition)	$z_1 + z_2 = z_2 + z_1$
Associative Law (Addition)	$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
Commutative Law (Multiplication)	$z_1 \cdot z_2 = z_2 \cdot z_1$
Associative Law (Multiplication)	$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$
Distributive Law	$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$
Additive Identity	$z_1 + 0 = z_1$
Additive Inverse	$z_1 - z_1 = 0$
Multiplicative Inverse	$z_1 \cdot (1/z_1) = 1$

Graphical Representation of Complex Numbers – Addition and Subtraction

A complex number $z_1 = x_1 + iy_1$ represents a single point in the complex plane and can be thought of as tip of a vector $z_1 = (x_1, y_1) = x_1 + iy_1$ as shown in Figure 1.4. Similarly z_2 is shown. Addition and subtraction of complex numbers z_1 and z_2 ($z_1 + z_2$ and $z_1 - z_2$) can be interpreted as addition of two vectors z_1 and z_2 and z_1 and $-z_2$ respectively, as shown in Figure 1.4.

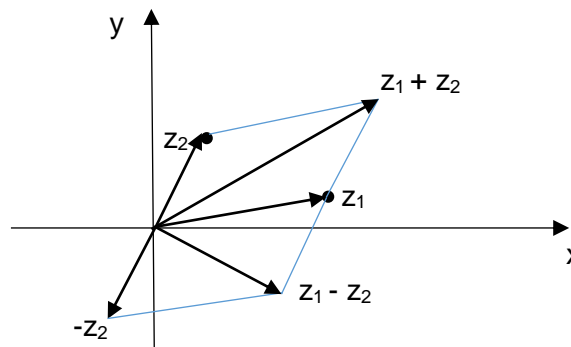


Figure 1.4 Graphical Interpretation of Addition of z_1 and z_2

1.2 – Polar Form of Complex Numbers and Basic Operations

Complex number $z = x + iy$ can be represented in the complex plane in polar form. This form simplifies many of complex number operations and is used in variety of engineering application. Figure 1.5 shows a complex number z in terms of a length (distance from the origin) and an angle (measured from positive side of the real axis in the counterclockwise sense, in radians or degrees). The length is denoted by r or $|z|$ and is called magnitude, modulus, or absolute value. The angle is denoted by θ and is called simply angle, argument ($\arg z$), or phase. The magnitude is a distance from the origin and is positive. It is zero, if complex number $z = 0$. The angle θ is referred to as the principle value and is denoted by $\text{Arg } z$, when $-\pi < \theta \leq \pi$ ($-180^\circ < \theta \leq 180^\circ$) Clearly θ in general can be represented as any integer multiple of 2π (360°) in counterclockwise sense or -2π (-360°) in clockwise sense as shown in Figure 1.6.

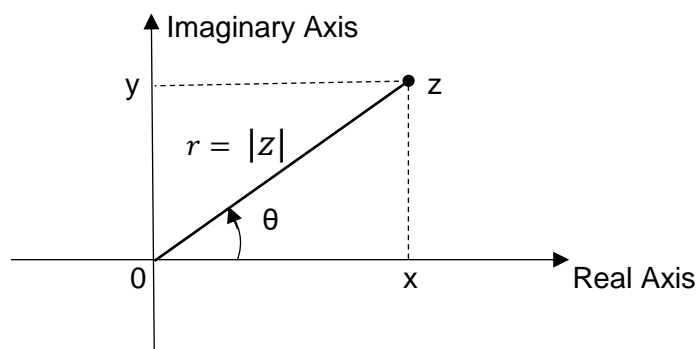


Figure 1.5 Complex Plane, Polar Form

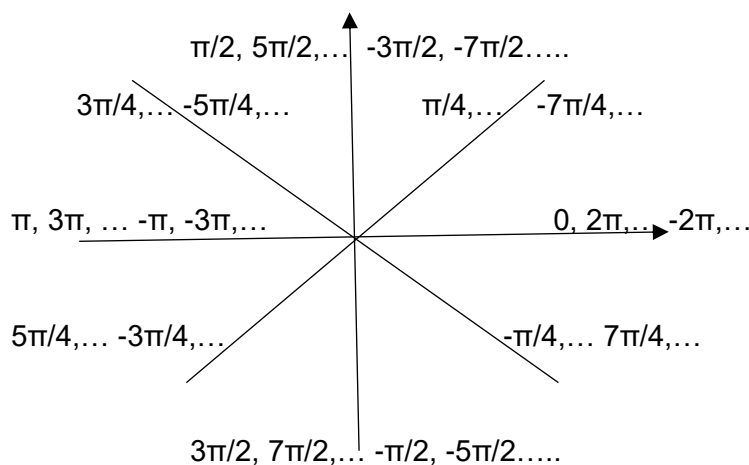


Figure 1.6 Example of Some Argument of z

Clearly $0, \pi/4, 3\pi/4, \pi, -\pi/4, -\pi/2,$ and $-3\pi/4$ are the principle values ($-\pi < \theta \leq \pi$) in this Figure.

Referring to Figure 1.4 we can write

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \quad (1.17)$$

As a result the polar form of complex number z can be represented as

$$z = r \cos \theta + i r \sin \theta = r (\cos \theta + i \sin \theta) \quad (1.18)$$

We can consider (1.17) as polar to rectangular conversion. Similarly we can write;

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \text{Arg } z = \arctan \frac{y}{x}, \text{ if } x > 0 & (1.19) \\ &= \arctan \frac{y}{x} - \pi, \text{ if } x < 0 \text{ and } y < 0 \\ &= \arctan \frac{y}{x} + \pi, \text{ if } x < 0 \text{ and } y > 0 \end{aligned}$$

The addition and subtraction of π in (1.19) has to do with the fact that the domain of \arctan is $(-\pi/2, \pi/2)$.

Considering the magnitude of a complex number z is the distance from z to the origin and its angle is measured from the positive side of real axis, it should not be necessary to use equation (1.19) to convert a complex number that is pure real or pure imaginary from rectangular to polar form and such conversion should be obvious. For example $z = 2$ has a magnitude of 2 and the angle of 0, $z = -3$ has a magnitude of 3 and the angle of π , $z = i5$ has a magnitude of 5 and the angle of $\pi/2$, and $z = -i6$ has a magnitude of 6 and the angle of $-\pi/2$. Similarly if the angle of a complex number indicates it is located on the real or imaginary axis, there should be no need to use equation (1.17) to convert from polar to rectangular form.

Example 4: Perform the following conversions:

- a) $z = -1 + i$ to polar form
 b) $z = 10 (\cos \pi/3 + i \sin \pi/3)$ to rectangular form

$$\text{a) } r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} \quad \theta = \arctan \frac{1}{-1} + \pi = (-\pi/4) + \pi = 3\pi/4$$

$$\text{b) } z = 10 (\cos \pi/3 + i \sin \pi/3) = 10 \cos \pi/3 + i 10 \sin \pi/3 = 5 + i5\sqrt{3}$$

Knowing the magnitude of a complex number z is the distance from the origin to the complex point z and using the principle of vector addition and subtraction, $|z_1|$, $|z_2|$, $|z_1 - z_2|$, and $|z_1 + z_2|$ can be shown geometrically as in Figure 1.7.

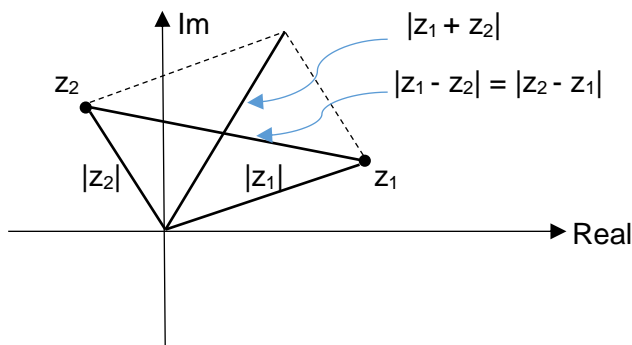


Figure 1.7 Distances $|z_1|$, $|z_2|$, $|z_1+z_2|$, and $|z_1-z_2|$

Complex Conjugate \bar{z} in Polar Form

Complex conjugate of a complex number $z = r (\cos \theta + i \sin \theta)$ in polar form is;

$$\bar{z} = r (\cos -\theta + i \sin -\theta) = r (\cos \theta - i \sin \theta) \quad (1.20)$$

Clearly the magnitude of \bar{z} the same as z and the angle \bar{z} is opposite sign of angle of z , as shown in Figure 1.8.

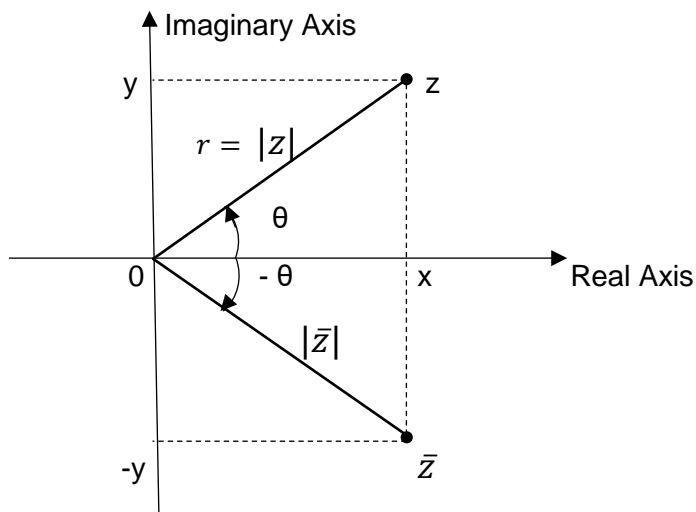


Figure 1.8 Complex Conjugate in Polar Form

Triangle Inequality

The triangle inequality states that the sum of two lengths of any two sides of a triangle is greater or equal to the length of the third side, regardless what sides are selected. Using triangle inequality and having discussed the geometrical description of magnitudes of complex numbers (Figure 1.7), we can write:

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad (1.21)$$

Using (1.21), it can be shown that:

$$|z_1 + z_2| \geq |z_1| - |z_2| \quad (1.22)$$

Equation (1.21) can be extended to:

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n| \quad (1.23)$$

Basic Algebraic Operations – Polar Form

Consider $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

Equality

$$z_1 = z_2 \text{ implies } r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2k\pi \quad (1.24)$$

Addition

Complex numbers can be added in polar form, if their arguments are the same or 180° apart. In other words, they have to be located on the same axis. Here axis is used in a general term indicating the same angle θ or $\theta \pm 180^\circ$.

$$z_1 + z_2 = r_1(\cos \theta_1 + i \sin \theta_1) + r_2(\cos \theta_1 + i \sin \theta_1) = (r_1+r_2)(\cos \theta_1 + i \sin \theta_1) \quad (1.25)$$

However, in general the complex numbers z_1 and z_2 can be converted to rectangular form, added, and converted back to polar form, if so desired.

Subtraction

Similarly complex numbers can be subtracted in polar form, if their arguments are the same or 180° apart. In other words, they have to be located on the same axis. Here axis is used in a general term indicating the same angle θ or $\theta \pm 180^\circ$.

$$z_1 - z_2 = r_1(\cos \theta_1 + i \sin \theta_1) - r_2(\cos \theta_1 + i \sin \theta_1) = (r_1-r_2)(\cos \theta_1 + i \sin \theta_1) \quad (1.26)$$

Here it is assumed $r_1 > r_2$. Again, in general the complex numbers z_1 and z_2 can be converted to rectangular form, subtracted, and converted back to polar form, if so desired.

Multiplication

$$\begin{aligned} z_1 z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \end{aligned} \quad (1.27)$$

The proof of equation (1.27) is left as an exercise.

Integer Power ($n = 0, 1, 2, \dots$)

Using equation (1.27), we can write:

$$z^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta) \quad (1.28)$$

Depending on n , this operation may produce a rather large magnitude and an argument. The argument can be written in principle form. Equation (1.28) is known as De Moivre's Formula and is simply an extension of equation (1.27).

Division

$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \quad z_2 \neq 0 \quad (1.29)$$

The proof of equation (1.29) is left as an exercise.

Using equations (1.28) and (1.29), we can write:

$$z^{-n} = (1/r^n)(\cos -n\theta + i \sin -n\theta) \quad n \geq 0 \text{ \& } z \neq 0 \quad (1.30)$$

Based on the results obtained above, the following observations are made:

$$|z_1 z_2| = |z_1| |z_2| = r_1 r_2 \quad \arg z_1 z_2 = \arg z_1 + \arg z_2 = \theta_1 + \theta_2 \quad (1.31)$$

$$|z_1/z_2| = |z_1|/|z_2| = r_1/r_2 \quad \arg z_1/z_2 = \arg z_1 - \arg z_2 = \theta_1 - \theta_2 \quad z_2 \neq 0 \quad (1.32)$$

$$z \bar{z} = (x + iy).(x - iy) = r(\cos \theta + i \sin \theta).r(\cos -\theta + i \sin -\theta) = x^2 + y^2 = r^2 = |z|^2 \quad (1.33)$$

Equations (1.31), (1.32), and (1.33) have various applications in engineering, when one is purely interested in magnitude or phase.

Example 5: If $z_1 = 5(\cos \pi/3 + i \sin \pi/3)$, $z_2 = 2(\cos 2\pi/3 + i \sin 2\pi/3)$, $z_3 = (\cos \pi/2 + i \sin \pi/2)$, $z_4 = 2(\cos -2\pi/3 + i \sin -2\pi/3)$, and $z_5 = (1 + i)$, evaluate

- a) $z_1 + z_4$ b) $z_1 - z_4$ c) $z_1 \cdot z_2 \cdot z_3$ d) z_1/z_2 e) $z_2 \bar{z}_3$
f) $(z_5)^{10}$

- a) $z_1 + z_4 = 5(\cos \pi/3 + i \sin \pi/3) + 2(\cos -2\pi/3 + i \sin -2\pi/3) = 3(\cos \pi/3 + i \sin \pi/3)$
b) $z_1 - z_4 = 5(\cos \pi/3 + i \sin \pi/3) - 2(\cos -2\pi/3 + i \sin -2\pi/3) = 7(\cos \pi/3 + i \sin \pi/3)$
c) $z_1 \cdot z_2 \cdot z_3 = 5(\cos \pi/3 + i \sin \pi/3) \cdot 2(\cos 2\pi/3 + i \sin 2\pi/3) \cdot (\cos \pi/2 + i \sin \pi/2) = 10(\cos 3\pi/2 + i \sin 3\pi/2) = 10(\cos -\pi/2 + i \sin -\pi/2)$
d) $z_1/z_2 = [5(\cos \pi/3 + i \sin \pi/3)]/[2(\cos 2\pi/3 + i \sin 2\pi/3)] = 5/2[(\cos (\pi/3 - 2\pi/3) + i \sin (\pi/3 - 2\pi/3))] = 2.5(\cos -\pi/3 + i \sin -\pi/3)$
e) $z_2 \bar{z}_3 = 2(\cos 2\pi/3 + i \sin 2\pi/3) \cdot (\cos -\pi/2 + i \sin -\pi/2) = 2(\cos \pi/6 + i \sin \pi/6)$
f) $(z_5)^{10} = (1 + i)^{10} = [\sqrt{2} (\cos \pi/4 + i \sin \pi/4)]^{10} = 32 (\cos 10\pi/4 + i \sin 10\pi/4) = 32(\cos \pi/2 + i \sin \pi/2) = i32$

Root of Complex Numbers

Equation (1.28) can be extended to obtain the nth root of a complex number $z = r(\cos \theta + i \sin \theta)$. Given $w^n = z$, then $w = [z]^{(1/n)}$ is referred to as the nth root of complex number z . We assume w to be a complex number denoted by:

$$w = R(\cos \Phi + i \sin \Phi) \quad (1.34)$$

The objective is to evaluate R and Φ . Using $w^n = z$, we can write:

$$[R(\cos \Phi + i \sin \Phi)]^n = r(\cos \theta + i \sin \theta)$$

$$R^n(\cos n\Phi + i \sin n\Phi) = r(\cos \theta + i \sin \theta) \quad (1.35)$$

Applying equation (1.24) for equality

$$R^n = r \quad \Rightarrow \quad R = \sqrt[n]{r}$$

and

$$n\Phi = \theta + 2k\pi \quad \Rightarrow \quad \Phi = \frac{(\theta + 2k\pi)}{n} \quad (1.36)$$

Hence, given $z = r(\cos \theta + i \sin \theta)$ and $w^n = z$, there are exactly n distinct roots and the n th root of the complex number z is given by

$$w = [z]^{(1/n)} = \sqrt[n]{r} \left(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right) \quad (1.37)$$

$$k = 0, 1, \dots, n-1$$

As can be noted in equation (1.37), all the roots have the same magnitude ($\sqrt[n]{r}$) and the angles of the roots are equally separated by $(2\pi/n)$. Hence all the values of w lie on a circle of radius ($\sqrt[n]{r}$) and separated by an angle $(2\pi/n)$. It is also important to note that if k is selected as $n, n+1, n+2, \dots$ the previous roots are simply repeated, indicating there are only n roots to be evaluated.

Example 6: Evaluate w , given

a) $w^4 = i16$

b) $w^3 = 27$

c) $w^2 = 1 - i$

a) $w^4 = 16 (\cos \pi/2 + i \sin \pi/2)$

$$w = \sqrt[4]{16} \left(\cos \frac{\pi/2 + 2k\pi}{4} + i \sin \frac{\pi/2 + 2k\pi}{4} \right) \quad k = 0, 1, 2, 3$$

$$k=0 \quad w_1 = 2 (\cos \pi/8 + i \sin \pi/8)$$

$$k=1 \quad w_2 = 2 (\cos 5\pi/8 + i \sin 5\pi/8)$$

$$k=2 \quad w_3 = 2 (\cos 9\pi/8 + i \sin 9\pi/8)$$

$$k=3 \quad w_4 = 2 (\cos 13\pi/8 + i \sin 13\pi/8)$$

4 Roots Shown in Figure 1.9a

b) $w^3 = 27 (\cos 0 + i \sin 0)$

$$w = \sqrt[3]{27} \left(\cos \frac{0 + 2k\pi}{3} + i \sin \frac{0 + 2k\pi}{3} \right) \quad k = 0, 1, 2$$

$$k=0 \quad w_1 = 3 (\cos 0 + i \sin 0)$$

$$k=1 \quad w_2 = 3 (\cos 2\pi/3 + i \sin 2\pi/3)$$

$$k=2 \quad w_3 = 3 (\cos 4\pi/3 + i \sin 4\pi/3)$$

3 Roots Shown in Figure 1.9b

c) $w^2 = 1 - i = \sqrt{2} (\cos -\pi/4 + i \sin -\pi/4)$

$$w = \sqrt[4]{2} \left(\cos \frac{-\pi/4 + 2k\pi}{2} + i \sin \frac{-\pi/4 + 2k\pi}{2} \right) \quad k = 0, 1$$

$$k=0 \quad w_1 = \sqrt[4]{2} (\cos -\pi/8 + i \sin -\pi/8)$$

$$k=1 \quad w_2 = \sqrt[4]{2} (\cos 7\pi/8 + i \sin 7\pi/8)$$

2 Roots Shown in Figure 1.9c

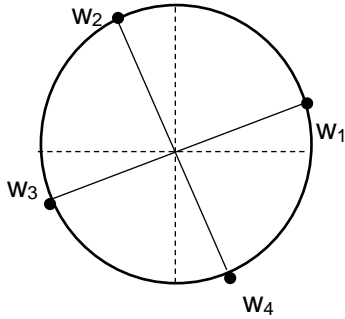


Figure 1.9a

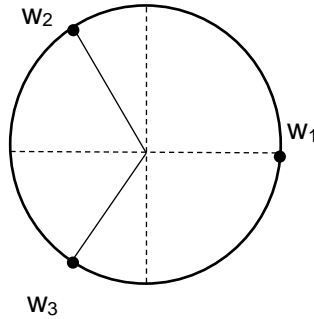


Figure 1.9b

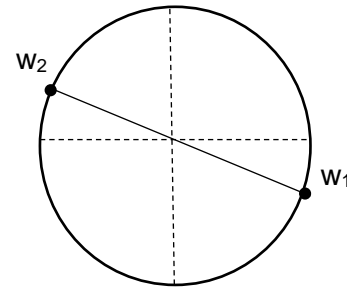


Figure 1.9c

Using equations (1.28) and (1.37), we can write the rational power of z as

$$w^n = z^m \quad \Rightarrow$$

$$w = [z]^{(m/n)} = \sqrt[n]{r^m} \left[\cos m \left(\frac{\theta + 2k\pi}{n} \right) + i \sin m \left(\frac{\theta + 2k\pi}{n} \right) \right] \quad (1.38)$$

$$k = 0, 1, \dots, n-1$$

1.3 – Complex Set, Function, Domain, and Range

Complex Set

Prior to defining a complex function, it is necessary to define a complex set. As discussed in the previous sections, a complex number z is represented as $z = x + iy = r(\cos \theta + i \sin \theta)$. A complex set D is a collection of finitely or infinitely many complex numbers. These complex numbers can be interior points, boundary points, or exterior points. A complex set can be open or closed, connected, simply connected or multiply connected, or bounded or unbounded. These terms are defined below:

Neighborhood of z_0 :

All point z that satisfy; $|z - z_0| < \epsilon$ where $\epsilon > 0$

Interior Points of a Set:

Points for which there exists at least a neighborhood of z_0 all whose points belong to that set.

Boundary Points of a Set:

Points for which every neighborhood of z_0 contains points that belong to that set as well as points that don't belong to that set

Exterior Points of a Set:

Points for which there exists at least a neighborhood of z_0 none of whose points belong to that set.

Connected Set:

It is a set that any two points of this set can be connected by a number of line segments all belonging to the set.

Simply Connected Set:

It is a connected set that every simple closed curve in the set only contains points of that set. A simple closed curve is one that does not intersect itself.

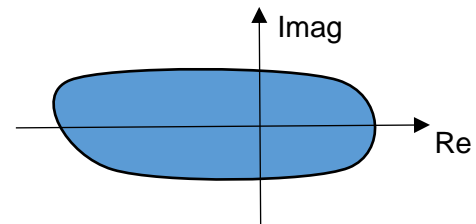


Figure 1.10 Simply Connected

Multiply Connected Set:

It is a connected set that there is at least one simple closed curve in the set with one or more points that don't belong to the set.

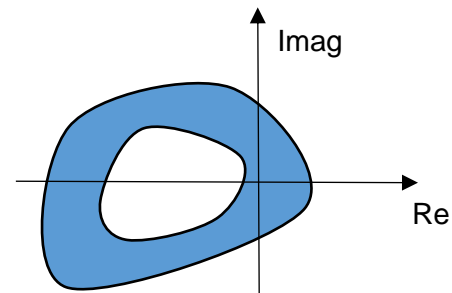


Figure 1.11 Multiply Connected

Bounded and Unbounded Sets:

If all the points, z in the set satisfy:

$$|z| < \rho,$$

Then the set is called bounded. If a set is not bounded, then it is called unbounded. For example, a set defined by all the points $|z| = 6$ is bounded and the set defined by $|z| > 10$ is unbounded.

Let us now consider some examples of sets.

Example 7: Sketch the set of complex points described by;

a) $|z| = 1$

b) $|z| > 1$

- c) $|z| < 1$
 d) $|z - (1 + i)| = 1$
 e) $1 < |z + i2| \leq 2$
 f) $-1 < \text{Im } z < 2$
 g) $0 \leq \text{Re } z \leq 1$

a) $|z| = 1$, points on a circle with center at $z = 0$ and radius 1.

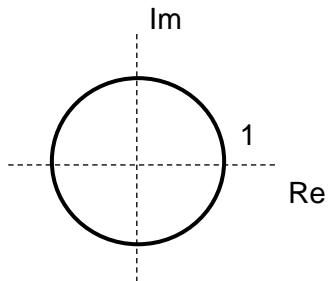


Figure 1.12

b) $|z| > 1$, exterior points to a circle with center at $z = 0$ and radius 1. Boundary point excluded, since there is no equality included with the greater than sign.

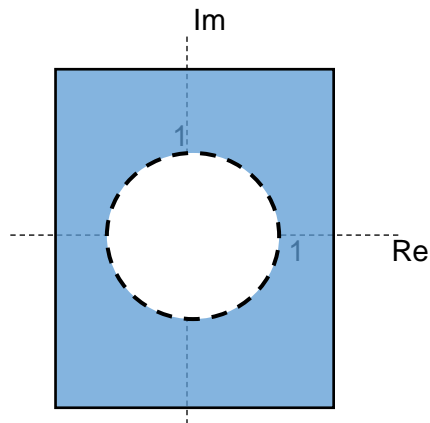


Figure 1.13

c) $|z| < 1$, interior points to a circle with center at $z = 0$ and radius 1. Boundary point excluded, since there is no equality included with the greater than sign.

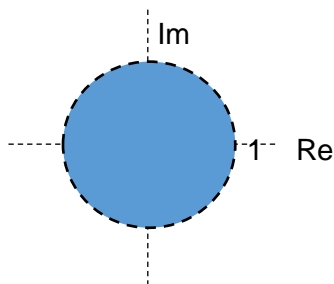


Figure 1.14

d) $|z - (1 + i)| = 1$, points on a circle with center at $z = 1 + i$ and radius 1.

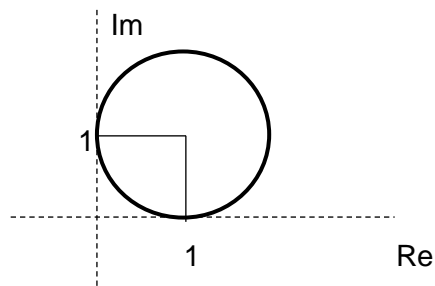


Figure 1.15

e) $1 < |z + i2| \leq 2$, interior points between two concentric circles with center at $z = -i2$ and radii 1 and 2. Boundary points of inner circle excluded, but the outer circle included.

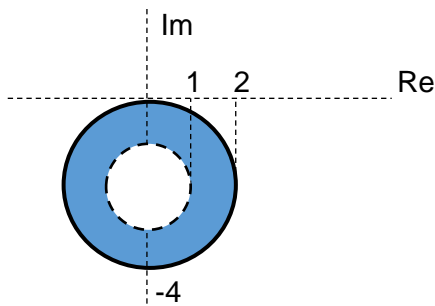


Figure 1.16

f) $-1 < \text{Im } z < 2$, set of complex points whose imaginary parts are between $y = -1$ and $y = 2$. The boundary points on horizontal lines $y = -1$ and $y = 2$ are excluded. The real parts extends from $-\infty$ to $+\infty$.

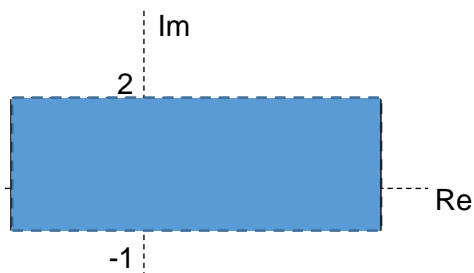


Figure 1.17

g) $0 \leq \text{Re } z \leq 1$, set of complex points whose real parts are between $x = 0$ and $x = 1$. The boundary points on vertical lines $x = 0$ and $x = 1$ are included. The imaginary parts extends from $-\infty$ to $+\infty$.

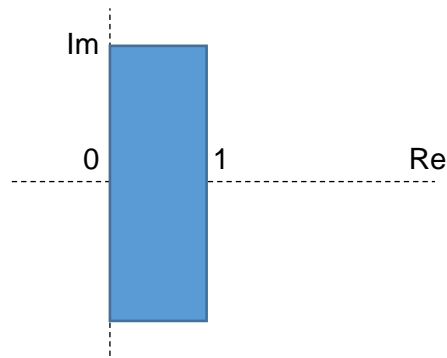


Figure 1.18

It is important to note that $|z|$ is simply the length from the complex variable z to the origin, $z = 0$. Similarly for a given complex point z_1 , $|z - z_1|$ represents the length from the complex variable z to z_1 . As can be recalled

$$|z| = \sqrt{x^2 + y^2} \quad \text{and} \quad |z - z_1| = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

This should help to better understand example problems 7a, b, c, and d. It should be clear that the magnitude $|z - z_1|$ is always greater or equal to zero and can never be negative.

Complex Function

A function f is a rule that assigns to each complex point z in a set D , a complex point w in a set R . Hence we define the complex function of a complex variable z belonging to domain D as;

$$w = f(z) = u(x,y) + iv(x,y) \quad (1.39)$$

The real part of the function $f(z)$ is $u(x,y)$ and the imaginary part of $f(z)$ is $v(x,y)$. In general $f(z)$ is complex function of x and y , where $u(x,y)$ and $v(x,y)$ are real functions of x and y . This can be interpreted as mapping a complex point z in z -plane to a complex point w in the w -plane as shown in Figure 1.19.

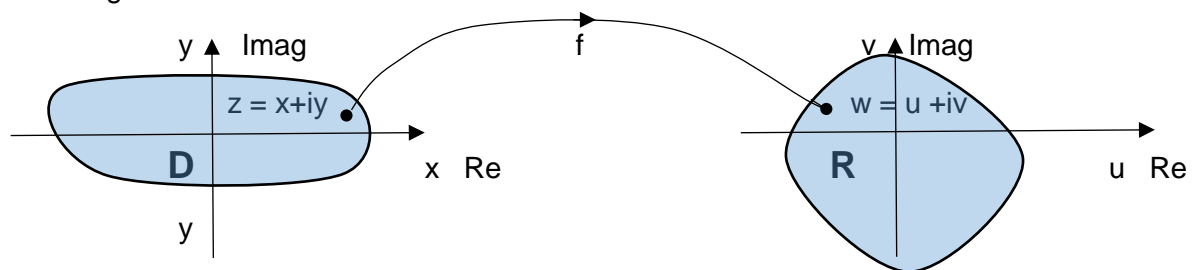


Figure 1.19 Complex Function $w = f(z) = u(x,y) + iv(x,y)$

Example 8: Represent the given functions in the form $w = f(z) = u(x,y) + iv(x,y)$,

a) $w = f(z) = z^2 + 2z + 10$

b) $w = f(z) = 1/z$

c) $w = f(z) = (z + 1)/(\bar{z} - 1)$

d) $w = f(z) = z\bar{z}$

a) $w = (x + iy)^2 + 2(x + iy) + 10 = x^2 + i2xy - y^2 + 2x + i2y + 10 =$

$w = (x^2 - y^2 + 2x + 10) + i(2xy + 2y) \Rightarrow u(x,y) = x^2 - y^2 + 2x + 10$ and $v(x,y) = 2xy + 2y$

b) $w = 1/z = 1/(x+iy) = (x-iy)/[(x+iy)(x-iy)] = [x/(x^2+y^2)] + i[-y/(x^2+y^2)] \Rightarrow u(x,y) = [x/(x^2+y^2)]$
and $v(x,y) = [-y/(x^2+y^2)]$

c) $w = (z + 1)/(\bar{z} - 1) = [x+1 + iy]/[x-1 - iy] = [x+1 + iy] [x-1 + iy]/[x-1 - iy] [x-1 + iy] =$
 $[(x^2 - 1 - y^2)]/[(x - 1)^2 + y^2] + i2xy/[(x - 1)^2 + y^2] \Rightarrow u(x,y) = [(x^2 - 1 - y^2)]/[(x - 1)^2 + y^2]$ and
 $v(x,y) = 2xy/[(x - 1)^2 + y^2]$

d) $w = z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 \Rightarrow u(x,y) = x^2 + y^2$ and $v(x,y) = 0$

Example 9: Evaluate $f(z) = 1/z$ at $z = 3 + i4$

This problem can be solved by substituting $x = 3$ and $y = 4$ in $u(x,y)$ and $v(x,y)$ of example 8b. Or one can simply substitute $z = 3 + i4$ in $f(z) = 1/z$ and perform the operation. The final result is the same.

$$f(3 + i4) = 1/(3 + i4) = (3 - i4)/[(3 + i4)(3 - i4)] = (3/25) + i(-4/25)$$

Domain and Range

Referring to the definition of a complex function $w = f(z)$ stated above and Figure 1.19, domain and range can be defined as:

Domain, D:

The set of allowable and permissible values of z in $w = f(z)$. D can include the entire z -plane or portion of it based on any given restriction.

Range, R:

The corresponding set of all values of w in $w = f(z)$.

1.4 – Limit, Continuity, Derivative, and Analytic Function

Limit

Let $w = f(z)$ and assume this function is defined in some neighborhood of z_0 but it may or may not be defined at z_0 . We write the function $f(z)$ has a limit w_0 as z approaches z_0 .

$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad (1.40)$$

and if for any real $\varepsilon > 0$, one can find a real $\delta > 0$ such that for all z satisfying $|z - z_0| < \delta$ implies $|f(z) - w_0| < \varepsilon$. Figure 1.20 shows a geometrical interpretation of this limit.

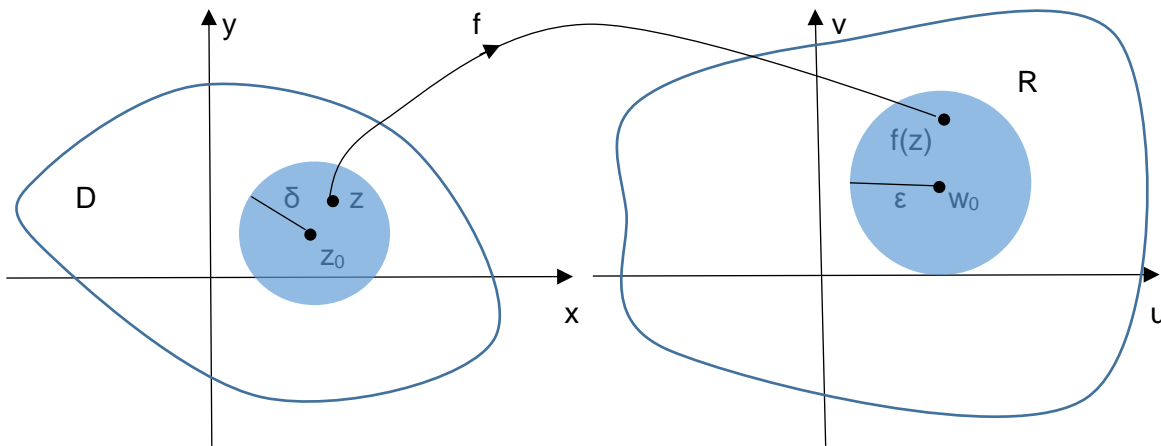


Figure 1.20 Geometrical Interpretation of Limit of a Function

Similar to real calculus, if

$$\lim_{z \rightarrow z_0} f_1(z) = L_1 \quad (1.41)$$

$$\lim_{z \rightarrow z_0} f_2(z) = L_2 \quad (1.42)$$

then

$$\lim_{z \rightarrow z_0} [f_1(z) + f_2(z)] = L_1 + L_2 \quad (1.43)$$

$$\lim_{z \rightarrow z_0} [f_1(z) \cdot f_2(z)] = L_1 \cdot L_2 \quad (1.44)$$

$$\lim_{z \rightarrow z_0} [f_1(z)/f_2(z)] = L_1/L_2 \quad L_2 \neq 0 \quad (1.45)$$

Continuity

Function $w = f(z)$ is continuous at z_0 if it is defined at z_0 and if limit of the function at z_0 is equal to the value of the function at z_0 .

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \quad (1.46)$$

Derivative

The differentiation of a complex function $w = f(z)$ is analogous to real calculus. If $w = f(z)$ is defined at z_0 , the derivative $dw/dz = df(z)/dz = f'(z)$ at z_0 is defined as

$$dw/dz = df(z)/dz = f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (1.47)$$

The increment $\Delta z = \Delta x + i\Delta y$ can approach zero on infinitely many paths, some of which are shown in Figure 1.21. Regardless of what path is taken, equation (1.47) should result in the same result for the derivative to exist.

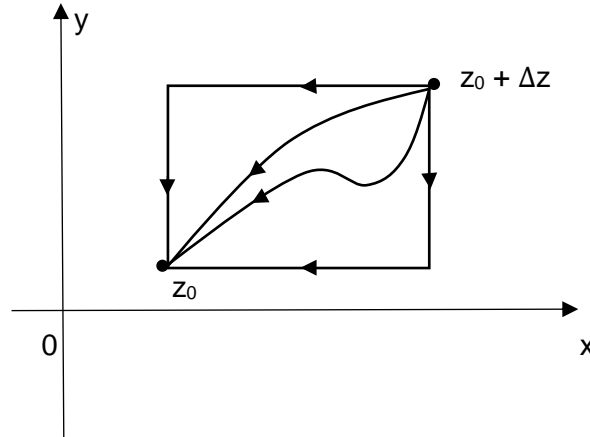


Figure 1.21 Example of Four Possible Paths of Infinitely Many $\Delta z \rightarrow 0$

Example 10: Differentiate $f(z)$

- a) $w = f(z) = 5z^2 + 2z$
- b) $w = f(z) = 1/z$
- c) $w = f(z) = \bar{z}$

$$\begin{aligned}
\text{a) } f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \rightarrow 0} \frac{5(z + \Delta z)^2 + 2(z + \Delta z) - 5z^2 - 2z}{\Delta z} \\
&= \lim_{\Delta z \rightarrow 0} \frac{5z^2 + 10z \Delta z + 5(\Delta z)^2 + 2z + 2\Delta z - 5z^2 - 2z}{\Delta z} \\
&= \lim_{\Delta z \rightarrow 0} \frac{10z \Delta z + 5(\Delta z)^2 + 2\Delta z}{\Delta z} \\
&= 10z + 2
\end{aligned}$$

$$\begin{aligned}
\text{b) } f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \rightarrow 0} \frac{1/(z + \Delta z) - 1/z}{\Delta z} \\
&= \lim_{\Delta z \rightarrow 0} \frac{z - z - \Delta z}{\Delta z(z + \Delta z)z} \\
&= -1/z^2
\end{aligned}$$

$$\begin{aligned}
\text{c) } f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} && \text{Using Equation (1.14)} \Rightarrow \\
&= \lim_{\Delta z \rightarrow 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} \\
&= \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} && \Delta z = \Delta x + i\Delta y \quad \text{and} \quad \overline{\Delta z} = \Delta x - i\Delta y
\end{aligned}$$

No cancellation can occur in the last expression. As indicated above, there are infinitely many paths for $\Delta z \rightarrow 0$. Let us consider, 1) $\Delta x \rightarrow 0$ then $\Delta y \rightarrow 0$ 2) $\Delta y \rightarrow 0$ then $\Delta x \rightarrow 0$

$$1) f'(z) = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = -1$$

$$2) f'(z) = \lim_{\substack{\Delta y \rightarrow 0 \\ \Delta x \rightarrow 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = 1$$

Hence $w = f(z) = \bar{z}$ does not have a derivative.

Certainly we are not expected to obtain $f'(z)$ using the limit definition of equation (1.47) for every given $f(z)$. For example a polynomial of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \quad (1.48)$$

is an analytic function. Furthermore a rational function of z of the form

$$f(z) = [a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0] / [b_m z^m + b_{m-1} z^{m-1} + \dots + b_1 z + b_0] \quad (1.49)$$

is analytic, except for those values of z for which the denominator is zero. The constant coefficients $a_n, a_{n-1}, \dots, a_1, a_0$ and $b_m, b_{m-1}, \dots, b_1, b_0$ are in general complex.

The definition of derivative given by equation (1.47) is identical to $f'(x)$ in real calculus. Hence the formula and the rules for differentiation are the same. Let us assume $f(z), f_1(z), f_2(z), \dots, f_n(z)$ are differentiable and C is a constant. Here are some general rules of derivative which should look familiar.

$$F(z) = C \quad F'(z) = 0 \quad (1.50)$$

$$F(z) = z^n \quad n \text{ is a integer} \quad F'(z) = n z^{(n-1)} \quad (1.51)$$

$$F(z) = f^n(z) \quad n \text{ is a integer} \quad F'(z) = n f^{n-1}(z) \cdot f'(z) \quad (1.52)$$

$$F(z) = C f(z) \quad F'(z) = C f'(z) \quad (1.53)$$

$$F(z) = f_1(z) + f_2(z) + \dots + f_n(z) \quad F'(z) = f'_1(z) + f'_2(z) + \dots + f'_n(z) \quad (1.54)$$

$$F(z) = f_1(z) \cdot f_2(z) \quad F'(z) = f'_1(z) f_2(z) + f_1(z) f'_2(z) \quad (1.55)$$

$$F(z) = f_1(z) / f_2(z) \quad F'(z) = [f'_1(z) f_2(z) - f_1(z) f'_2(z)] / [f_2(z)]^2 \quad (1.56)$$

$$F(z) = f_1(z) \circ f_2(z) = f_1(f_2(z)) \quad F'(z) = f'_1(f_2(z)) \cdot f'_2(z) \quad (1.57)$$

The derivative of other complex functions, such as exponential, logarithmic, trigonometric, and hyperbolic function will be discussed 1.6, 1.7, and 1.8 respectively.

Analytic Function

A complex function $f(z)$ is analytic at a point z_0 in domain D , if $f(z)$ is defined at z_0 and is differentiable at z_0 . This definition of analyticity can be extended to an entire domain D or a subset of domain D . Clearly $f(z) = 1/z$ is analytic everywhere except at $z = 0$ and the function $f(z) = \bar{z}$ is not analytic.

1.5 – Cauchy-Riemann Equations and Harmonic Functions

Cauchy-Riemann Equations in Rectangular Form

Theorem 1.1 Cauchy-Riemann Equations – Rectangular Form

Let $w = f(z) = u(x,y) + iv(x,y)$ be differentiable at $z = x + iy$. Then partial derivatives u_x , u_y , v_x , and v_y exist and satisfy the following equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad (1.58)$$

Furthermore if $f(z)$ is analytic in D , then u_x , u_y , v_x , and v_y exist and satisfy equation (1.58). Equation (1.58) is called Cauchy-Riemann equations or conditions.

Proof:

Since $f(z)$ is differentiable, we can write

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad (1.59)$$

The proof is based on writing equation (1.59) in terms of x and y . Next we let $\Delta z = \Delta x + i\Delta y \rightarrow 0$ on two different paths 1) $\Delta x \rightarrow 0$ then $\Delta y \rightarrow 0$ 2) $\Delta y \rightarrow 0$ then $\Delta x \rightarrow 0$. This process will result in two different equations for $f'(z)$. Since $f(z)$ is assumed to be differentiable, the two equations must be one and the same.

$$f'(z) = \lim_{\Delta x + i\Delta y \rightarrow 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \quad (1.60)$$

Now let us consider 1) $\Delta x \rightarrow 0$ then $\Delta y \rightarrow 0$. We let $\Delta x \rightarrow 0$ in equation (1.60).

$$\begin{aligned} f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - [u(x, y) + iv(x, y)]}{i\Delta y} \\ f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y+\Delta y) - u(x, y)] + i[v(x, y+\Delta y) - v(x, y)]}{i\Delta y} \quad \text{with } 1/i = -i \Rightarrow \\ f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{[v(x, y+\Delta y) - v(x, y)]}{\Delta y} - i \lim_{\Delta y \rightarrow 0} \frac{[u(x, y+\Delta y) - u(x, y)]}{\Delta y} \\ f'(z) &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = v_y - iu_y \quad (1.61) \end{aligned}$$

Next let us consider 2) $\Delta y \rightarrow 0$ then $\Delta x \rightarrow 0$. We let $\Delta y \rightarrow 0$ in equation (1.60).

$$f'(z) = \lim_{\Delta x \rightarrow 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x, y) + iv(x, y)]}{\Delta x}$$

$$f'(z) = \lim_{\Delta x \rightarrow 0} \frac{[u(x+\Delta x, y) - u(x, y)] + i[v(x+\Delta x, y) - v(x, y)]}{\Delta x}$$

$$f'(z) = \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = u_x + iv_x \quad (1.62)$$

Equations (1.61) and (1.62) must be the same for $f(z)$ to be differentiable. Hence, setting the two equations equal, we obtain,

$$\begin{aligned} \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \Rightarrow \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{or} \quad u_x = v_y \quad \text{and} \quad u_y = -v_x \end{aligned} \quad (1.63)$$

Once it is shown $f(z)$ is analytic and both conditions of equation (1.63) are satisfied, either of the two equations (1.61) and (1.62) can be used to obtain derivative of $f(z)$, if so desired.

Example 11: Check if the given $f(z)$ is analytic and if so, evaluate $f'(z)$ using equation (1.61) or (1.62)

a) $f(z) = z^3 + 2$

b) $w = f(z) = [\bar{z}]^2$

a) $f(z) = z^3 + 2 = (x + iy)^3 + 2 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + 2 = (x^3 - 3xy^2 + 2) + i(3x^2y - y^3)$

$$u_x = v_y \Rightarrow 3x^2 - 3y^2 = 3x^2 - 3y^2$$

$$u_y = -v_x \Rightarrow -6xy = -(6xy)$$

Function $f(z)$ is analytic and using equation (1.62) we have

$$f'(z) = u_x + iv_x = (3x^2 - 3y^2) + i6xy = 3[(x^2 - y^2) + i(2xy)] = 3(x + iy)^2 = 3z^2$$

b) $f(z) = [\bar{z}]^2 = (x - iy)^2 = (x^2 - y^2) + i(-2xy)$

$$u_x = v_y \Rightarrow 2x \neq -2x$$

$$u_y = -v_x \Rightarrow -2y \neq -(-2y)$$

Function $f(z)$ is not analytic.

Cauchy-Riemann Equations in Polar Form

Theorem 1.2 Cauchy-Riemann Equations – Polar Form

Let $w = f(z) = u(r, \theta) + iv(r, \theta)$ be differentiable at $z = r(\cos \theta + i \sin \theta)$. Then partial derivatives u_r , u_θ , v_r , and v_θ exist and satisfy the following equations:

$$r u_r = v_\theta \quad \text{and} \quad r v_r = -u_\theta \quad (1.64)$$

and

$$f'(z) = (\cos \theta - i \sin \theta)(u_r + iv_r) = (\cos \theta - i \sin \theta)(v_\theta - iu_\theta)/r \quad (1.65)$$

Furthermore if $f(z)$ is analytic in D , then u_r , u_θ , v_r , and v_θ exist and satisfy equation (1.64). Equation (1.64) is called Cauchy-Riemann equations or conditions in polar form.

Proof:

We use Theorem 1.1 and note,

$$u_x = u_r r_x + u_\theta \theta_x$$

$$u_y = u_r r_y + u_\theta \theta_y$$

$$v_x = v_r r_x + v_\theta \theta_x$$

$$v_y = v_r r_y + v_\theta \theta_y$$

Since $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan \frac{y}{x}$, we can evaluate r_x , r_y , θ_x , and θ_y and substitute in the above four equations. After substituting $x = r \cos \theta$, and $y = r \sin \theta$ and simplifying, we obtain

$$u_x = u_r \cos \theta - u_\theta (\sin \theta)/r \quad (1.66)$$

$$u_y = u_r \sin \theta + u_\theta (\cos \theta)/r \quad (1.67)$$

$$v_x = v_r \cos \theta - v_\theta (\sin \theta)/r \quad (1.68)$$

$$v_y = v_r \sin \theta + v_\theta (\cos \theta)/r \quad (1.69)$$

Since $u_x = v_y$ and $u_y = -v_x$, we write

$$u_r \cos \theta - u_\theta (\sin \theta)/r = v_r \sin \theta + v_\theta (\cos \theta)/r \quad (1.70)$$

$$u_r \sin \theta + u_\theta (\cos \theta)/r = -v_r \cos \theta + v_\theta (\sin \theta)/r \quad (1.71)$$

If we multiply equation (1.70) by $\cos \theta$ and equation (1.71) by $\sin \theta$ and adding them results in

$$u_r = v_\theta / r \quad \Rightarrow \quad r u_r = v_\theta \quad (1.72)$$

If we multiply equation (1.70) by $-\sin \theta$ and equation (1.71) by $\cos \theta$ and adding them results in

$$u_\theta / r = -v_r \quad \Rightarrow \quad r v_r = -u_\theta \quad (1.73)$$

This proves the first segment of Theorem 1.2. To evaluate the derivative in polar form, consider equations (1.61) and substitute v_y and u_y using equations (1.69) and (1.67).

$$f'(z) = v_y - i u_y = [v_r \sin \theta + v_\theta (\cos \theta)/r] - i [u_r \sin \theta + u_\theta (\cos \theta)/r] \quad (1.74)$$

Substituting for v_θ and u_θ using equations (1.72) and (1.73) in equation (1.74) gives

$$\begin{aligned} f'(z) &= [v_r \sin \theta + r u_r (\cos \theta)/r] - i [u_r \sin \theta - r v_r (\cos \theta)/r] \\ &= [v_r \sin \theta + u_r \cos \theta] - i [u_r \sin \theta - v_r \cos \theta] \\ &= [\cos \theta - i \sin \theta] [u_r + i v_r] \end{aligned} \quad (1.75)$$

Similarly substituting for v_r and u_r using equations (1.72) and (1.73) in equation (1.74) gives

$$\begin{aligned} f'(z) &= [(-u_\theta/r) \sin \theta + v_\theta (\cos \theta)/r] - i [(v_\theta/r) \sin \theta + u_\theta (\cos \theta)/r] \\ &= (1/r)[(-u_\theta \sin \theta + v_\theta \cos \theta) - i (v_\theta \sin \theta + u_\theta \cos \theta)] \\ &= [\cos \theta - i \sin \theta] [v_\theta - i u_\theta] / r \end{aligned} \quad (1.76)$$

Equations (1.74) and (1.75) prove the second segment of Theorem 1.2.

Harmonic Functions

If a function $\zeta(x,y)$ has first and second continuous partial derivatives in some domain D and satisfies Laplace's partial differential equation,

$$\nabla^2 \zeta = \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} = \zeta_{xx} + \zeta_{yy} = 0 \quad (1.77)$$

It is called a harmonic function.

Theorem 1.3

If $w = f(z) = u(x,y) + iv(x,y)$ is an analytic function in some domain D, then the real part (u) and the imaginary part (v) of $f(z)$ satisfy Laplace's partial differential equation or in other words are harmonic functions.

Proof:

Since $f(z)$ is analytic, we have

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Taking the partial derivative of $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y results in

$$\begin{aligned} u_{xx} = v_{yx} \quad \text{and} \quad u_{yy} = -v_{xy} \quad \text{adding the two equations} \quad \Rightarrow \\ u_{xx} + u_{yy} = 0 \end{aligned} \quad (1.78)$$

Taking the partial derivative of $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x results in

$$u_{xy} = v_{yy} \quad \text{and} \quad u_{xy} = -v_{xx} \quad \text{subtracting the two equations} \Rightarrow$$

$$v_{xx} + v_{yy} = 0 \quad (1.79)$$

Example 12: The $f(z) = u(x,y) + i v(x,y)$ is analytic. The real part of this function is given as $u(x,y) = x^3 - 3xy^2 + 5$. Evaluate the imaginary part of $f(z)$, given $f(1 + i) = 3 + i5$

$$u_x = v_y \Rightarrow v_y = 3x^2 - 3y^2 \quad v(x,y) = 3x^2y - y^3 + g(x)$$

$$v_x = -u_y \Rightarrow 6xy + g'(x) = -(-6xy) \quad g(x) = c$$

$$f(z) = (x^3 - 3xy^2 + 5) + i(3x^2y - y^3 + c)$$

$$f(1 + i) = 3 + i5 \Rightarrow (1 - 3 + 5) + i(3 - 1 + c) = 3 + i5 \Rightarrow 3 + i(2+c) = 3 + i5$$

$$g(x) = c = 3$$

$$f(z) = (x^3 - 3xy^2 + 5) + i(3x^2y - y^3 + 3)$$

Example 13: Show the real part and the imaginary part of $f(z) = z^3 + 5z^2 + 10$ are harmonic functions.

$$\begin{aligned} f(z) = z^3 + 5z^2 + 10 &= [x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3] + [5x^2 + 5(i2xy) - 5y^2] + [10] \\ &= (x^3 - 3xy^2 + 5x^2 - 5y^2 + 10) + i(3x^2y - y^3 + 10xy) \end{aligned}$$

$$u_x = 3x^2 - 3y^2 + 10x \quad u_{xx} = 6x + 10$$

$$u_y = -6xy - 10y \quad u_{yy} = -6x - 10 \quad \Rightarrow \quad u_{xx} + u_{yy} = 0$$

$$v_x = 6xy + 10y \quad v_{xx} = 6y$$

$$v_y = 3x^2 - 3y^2 + 10x \quad v_{yy} = -6y \quad \Rightarrow \quad v_{xx} + v_{yy} = 0$$

1.6 – Exponential and Logarithmic Functions

Exponential Function

Exponential function $f(z) = e^z$ is defined as

$$f(z) = e^z = e^{(x + iy)} = e^x e^{iy} \quad (1.80)$$

The term e^{iy} can be expanded using Maclaurin series as

$$\begin{aligned}
 e^{iy} &= 1 + (iy)/1! + (iy)^2/2! + (iy)^3/3! + (iy)^4/4! + (iy)^5/5! + \dots \\
 e^{iy} &= (1 - y^2/2! + y^4/4! - \dots) + i(y/1! - y^3/3! + y^5/5! + \dots) \quad \Rightarrow \\
 e^{iy} &= \cos y + i \sin y \quad (1.81)
 \end{aligned}$$

Using equation (1.81), we note that

$$e^{i2\pi} = 1, \quad e^{i\pi} = -1, \quad e^{i\pi/2} = i, \quad e^{-i2\pi} = 1, \quad e^{-i\pi} = -1, \quad e^{-i\pi/2} = -i$$

Equation (1.81) is known as Euler formula. Substituting equation (1.81) in equation (1.80) results

$$f(z) = e^z = e^x e^{iy} = e^x (\cos y + i \sin y) \quad (1.82)$$

Some Properties of e^z

Using equations (1.81) and (1.82), we observe

$$a) f(z) = e^z = e^x, \text{ if } y=0 \quad (1.83)$$

$$b) |e^z| = e^x \quad \text{and} \quad \arg e^z = y + 2n\pi \quad (n=0, \pm 1, \pm 2, \dots) \quad (1.84)$$

$$c) |e^{iy}| = 1 \quad \text{and} \quad \arg e^{iy} = y + 2n\pi \quad (n=0, \pm 1, \pm 2, \dots) \quad (1.85)$$

$$\begin{aligned}
 d) e^{z_1} \cdot e^{z_2} &= e^{x_1} (\cos y_1 + i \sin y_1) \cdot e^{x_2} (\cos y_2 + i \sin y_2) \\
 &= e^{x_1} \cdot e^{x_2} [(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i (\sin y_1 \cos y_2 + \cos y_1 \sin y_2)] \\
 &= e^{(x_1+x_2)} [\cos (y_1 + y_2) + i \sin (y_1 + y_2)] \quad \Rightarrow \\
 e^{z_1} \cdot e^{z_2} &= e^{(z_1+z_2)} \quad (1.86)
 \end{aligned}$$

$$\begin{aligned}
 e) e^{z_1} / e^{z_2} &= e^{x_1} (\cos y_1 + i \sin y_1) / e^{x_2} (\cos y_2 + i \sin y_2) \\
 &= e^{x_1} / e^{x_2} [(\cos y_1 + i \sin y_1)(\cos y_2 - i \sin y_2) / (\cos y_2 + i \sin y_2)(\cos y_2 - i \sin y_2)]
 \end{aligned}$$

$$\text{Note: } (\cos y_2 + i \sin y_2)(\cos y_2 - i \sin y_2) = \cos^2 y_2 + \sin^2 y_2 = 1$$

$$\begin{aligned}
 &= e^{(x_1 - x_2)} [(\cos y_1 \cos y_2 + \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 - \cos y_1 \sin y_2)] \\
 &= e^{(x_1 - x_2)} [\cos(y_1 - y_2) + i \sin(y_1 - y_2)] \quad \Rightarrow
 \end{aligned}$$

$$e^{z_1} / e^{z_2} = e^{(z_1 - z_2)} \quad (1.87)$$

f) $f(z) = e^z$ is periodic with period of $i2\pi$

$$e^z = e^{z+i2\pi} \quad (1.88)$$

This is simply to show using equation (1.86) and having shown $e^{i2\pi} = 1$. This is also clear by examining equation (1.84). However, the region defined by $-\pi < y \leq \pi$ is referred to as the fundamental region as is shown in Figure 1.22.

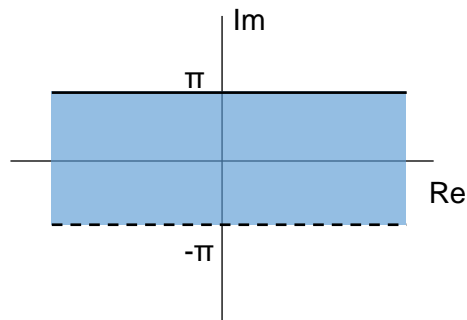


Figure 1.22 Fundamental Region of e^z

Derivative of $f(z) = e^z$

Using Cauchy-Riemann equations, we can show $f(z) = e^z$ is analytic

$$f(z) = e^z = e^x (\cos y + i \sin y) \quad u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y$$

$$u_x = v_y = e^x \cos y$$

$$u_y = -v_x = -e^x \sin y$$

To find $f'(z)$, equation (1.62) can be used

$$f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y = e^x (\cos y + i \sin y) = e^z \quad (1.89)$$

Similarly to real calculus, given $g(z)$ is analytic we can write

$$F(z) = e^{g(z)}$$

$$F'(z) = g'(z) \cdot e^{g(z)} \quad (1.90)$$

An Alternate Polar Form of z

We saw earlier that the polar form of complex number z is

$$z = r (\cos \theta + i \sin \theta)$$

Using Euler formula, we can write z as

$$z = r (\cos \theta + i \sin \theta) = r e^{i\theta} \quad (1.91)$$

As an example $z = -1 + i$ can be written $z = \sqrt{2} e^{i3\pi/4}$. Equation (1.91) is a short version of polar form of z and is routinely used in engineering problems and applications dealing with complex variables.

Example 14: Evaluate $f(z) = e^z$ at $z = -1 + i2$

$$f(z) = e^{-1 + i2} = e^{-1} (\cos 2 + i \sin 2) = 0.368(-0.42 + i 0.91) = -0.153 + i 0.334$$

Example 15: Solve $e^z = 1 - i2$ for $z = x + iy$

$$\begin{aligned} |e^z| = e^x &= |1 - i2| = \sqrt{5} & e^x \cos y &= 1 & e^x \sin y &= -2 & \Rightarrow \\ x = \ln \sqrt{5} &= (1/2)\ln 5 = 0.805 & \cos y &= 0.447 & \sin y &= -0.894 & \Rightarrow y = -1.11 \\ z &= 0.805 - i(1.11 + 2n\pi) & n &= 0, \pm 1, \pm 2, \dots \end{aligned}$$

Logarithmic Function

The natural logarithm of z represented by $\ln z$ is the inverse of exponential function exponential function.

$$w = \ln z = \log z \quad e^w = z \quad z \neq 0 \quad (1.92)$$

Represent the given z in polar form ($z = r e^{i\theta}$) and let $w = u + iv$. The objective is to compute u and v. Using equation (1.92)

$$e^w = e^{u + iv} = r e^{i\theta}$$

Equating magnitude and phase results in

$$e^u = r \quad \Rightarrow \quad u = \ln r \quad \text{and} \quad v = \theta \pm 2n\pi$$

Hence

$$w = \ln z = \ln r + i(\theta \pm 2n\pi) \quad (1.93)$$

When v is restricted to $-\pi < v \leq \pi$, the corresponding logarithm of z is referred to as the principle logarithmic function of z and is denoted by $\text{Ln } z$.

$$\text{Ln } z = \ln r + i\theta \quad (1.94)$$

Some Properties of $\ln z$

Using equations (1.81) and (1.82), we observe

$$\text{a) } \ln(z_1 z_2) = \ln z_1 + \ln z_2 \quad (1.95)$$

$$\text{b) } \ln(z_1 / z_2) = \ln z_1 - \ln z_2 \quad (1.96)$$

Derivative of $f(z) = \ln z$

Using Cauchy-Riemann equations, we can show $f(z) = \ln z$ is analytic

$$f(z) = \ln z = \ln r + i(\theta \pm 2n\pi)$$

Since derivative of a constant is zero, we can ignore $\pm 2n\pi$ and assume $\theta = \arctan(y/x)$ without addition or subtraction of π , if the real part of $z < 0$.

$$f(z) = \ln \sqrt{x^2 + y^2} + i \arctan(y/x) = (1/2) \ln(x^2 + y^2) + i \arctan(y/x)$$

$$u(x,y) = (1/2) \ln(x^2 + y^2) \quad \text{and} \quad v(x,y) = \arctan(y/x)$$

$$u_x = (1/2)(2x)/(x^2 + y^2) = x/(x^2 + y^2) \quad \text{and} \quad v_y = (1/x)/[1 + (y/x)^2] = x/(x^2 + y^2) \Rightarrow u_x = v_y$$

$$u_y = (1/2)(2y)/(x^2 + y^2) = y/(x^2 + y^2) \quad \text{and} \quad -v_x = -(-y/x^2)/[1 + (y/x)^2] = y/(x^2 + y^2) \Rightarrow u_y = -v_x$$

To find $f'(z)$, equation (1.62) can be used

$$f'(z) = u_x + iv_x = [x/(x^2 + y^2)] + i[-y/(x^2 + y^2)] = (x - iy)/(x^2 + y^2) = 1/z \quad (1.97)$$

Similarly to real calculus, given $g(z)$ is analytic we can write

$$F(z) = \ln g(z) \quad g(z) \neq 0$$

$$F'(z) = g'(z) / g(z) \quad (1.98)$$

Complex Power

Having explained exponential and logarithmic functions of complex variable z , we can now consider complex power A^B , where both A and B in general are complex.

$$A^B = e^{B \ln A} \quad (1.99)$$

Example 16: Evaluate a) $(1 - i)^{(1 + i)}$ b) 2^i

$$\begin{aligned} \text{a) } (1 - i)^{(1 + i)} &= e^{(1+i)[\ln(1-i)]} = e^{(1+i)[\ln\sqrt{2} + i(-\frac{\pi}{4} \pm 2n\pi)]} = \\ &= e^{\ln\sqrt{2} + (\frac{\pi}{4} \pm 2n\pi)} e^{i(\ln\sqrt{2} - \frac{\pi}{4} \pm 2n\pi)} = \\ &= \sqrt{2} e^{(\frac{\pi}{4} \pm 2n\pi)} [\cos(\ln\sqrt{2} - \pi/4) + i \sin(\ln\sqrt{2} - \pi/4)] \end{aligned}$$

$$\text{b) } 2^i = e^{(i)\ln 2} = e^{(i)[\ln 2 + i(0 \pm 2n\pi)]} = e^{(\pm 2n\pi)} [\cos(\ln 2) + i \sin(\ln 2)]$$

1.7 – Trigonometric and Inverse Trigonometric Functions

Trigonometric Functions

Similar to real trigonometry, Euler equation in complex form and the complex trigonometric functions $\sin z$, $\cos z$, $\tan z$, $\cot z$, $\sec z$, and $\csc z$ are defined.

$$e^{iZ} = \cos z + i \sin z \quad (1.100)$$

$$\sin z = \frac{1}{2i} (e^{iZ} - e^{-iZ}) \quad (1.101)$$

$$\cos z = \frac{1}{2} (e^{iZ} + e^{-iZ}) \quad (1.102)$$

$$\tan z = \frac{\sin Z}{\cos Z} \quad (1.103)$$

$$\cot z = \frac{\cos Z}{\sin Z} \quad (1.104)$$

$$\sec z = \frac{1}{\cos Z} \quad (1.105)$$

$$\csc z = \frac{1}{\sin Z} \quad (1.106)$$

If z is replaced by x in the above equations, we obtain the same familiar functions considered in real trigonometry. Furthermore using the above definitions we can write

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z, \quad \tan(-z) = -\tan z, \quad \cot(-z) = -\cot z \quad (1.107)$$

Since e^z and e^{iz} are analytic everywhere, $\sin z$ and $\cos z$ are also analytic everywhere. The functions $\tan z$ and $\sec z$ are analytic functions except at the points z where $\cos z$ becomes zero. The functions $\cot z$ and $\csc z$ are analytic functions except at the points z where $\sin z$ becomes zero. Since $d(e^z)/dz = e^z$ and $d(e^{iz})/dz = ie^z$, we can write the following derivatives which should look familiar.

$$f(z) = \sin z \quad f'(z) = \cos z \quad (1.108)$$

$$f(z) = \cos z \quad f'(z) = -\sin z \quad (1.109)$$

$$f(z) = \tan z \quad f'(z) = \sec^2 z \quad (1.110)$$

$$f(z) = \cot z \quad f'(z) = -\csc^2 z \quad (1.111)$$

$$f(z) = \sec z \quad f'(z) = \sec z \tan z \quad (1.112)$$

$$f(z) = \csc z \quad f'(z) = -\csc z \cot z \quad (1.113)$$

Some General Complex Trigonometric Identities

$$\cos^2 z + \sin^2 z = 1 \quad (1.114)$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \quad (1.115)$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2 \quad (1.116)$$

$$\sin 2z = 2 \sin z \cos z \quad (1.117)$$

$$\cos 2z = \cos^2 z - \sin^2 z = 1 - 2\sin^2 z = 2\cos^2 z - 1 \quad (1.118)$$

$$\begin{aligned} \cos(z) &= \cos(x + iy) = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\ &= \frac{1}{2}[e^{-y}e^{ix} + e^ye^{-ix}] = \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^y(\cos x - i\sin x)] \\ &= \frac{1}{2}[\cos x(e^{-y} + e^y) - i\sin x(e^y - e^{-y})] \end{aligned}$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y \quad (1.119)$$

Similarly,

$$\begin{aligned} \sin(z) &= \sin(x + iy) = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}(e^{i(x+iy)} - e^{-i(x+iy)}) \\ &= \frac{1}{2i}[e^{-y}e^{ix} - e^ye^{-ix}] = \frac{1}{2i}[e^{-y}(\cos x + i\sin x) - e^y(\cos x - i\sin x)] \\ &= \frac{1}{2}[\sin x(e^{-y} + e^y)] - i\frac{1}{2}[\cos x(e^{-y} - e^y)] \end{aligned}$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y \quad (1.120)$$

Using equations (1.119), (1.120), and $\cosh^2 y - \sinh^2 y = 1$, we can write

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \quad (1.121)$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \quad (1.122)$$

Equations (1.119) and (1.120) can be used to show Cauchy-Riemann equations hold and evaluate the derivatives of $\sin z$ and $\cos z$.

Example 17: Evaluate $f(z) = \sin z$ at $z = -1 + i2$

$$\begin{aligned} f(z) = \sin z &= \sin(-1 + i2) = \sin -1 \cosh 2 + i \cos -1 \sinh 2 = (-0.841)(3.762) + i(0.540)(3.627) \\ &= -3.164 + i 1.959 \end{aligned}$$

Example 18: Evaluate z , if $\cos z = 2$

$$\cos z = \cos x \cosh y - i \sin x \sinh y = 2 \quad \Rightarrow$$

$$\cos x \cosh y = 2$$

$$\sin x \sinh y = 0$$

From the second equation $\sinh y = 0 \Rightarrow y = 0$. This cannot be, since $y = 0$ implies $\cos x = 2$ in the first equation. Hence in the second equation

$$\sin x = 0 \quad \Rightarrow \quad x = n\pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

If $x = n\pi$ is substituted in the first equation $\Rightarrow \cosh y < 0$ (for n odd), which is not possible. Hence;

$$\begin{aligned} x = 2n\pi \quad (n = 0, \pm 1, \pm 2, \dots) \quad \text{using } \cos x \cosh y = 2 \quad \Rightarrow \\ \cosh y = 2 \Rightarrow y = 1.317 \end{aligned}$$

$$z = x + iy = 2n\pi + i 1.317 \quad (n = 0, \pm 1, \pm 2, \dots)$$

Inverse Trigonometric Functions

We know trigonometric functions are periodic and as a result the inverse of these function do not exist. The inverse of trigonometric functions are multiple-valued functions. If we ignore the multiple-valued condition, we can obtain equations for $\sin^{-1} z$, $\cos^{-1} z$, etc. For example let

$$w = \cos^{-1} z \quad (1.123)$$

$$z = \cos w = \frac{1}{2} (e^{iw} + e^{-iw}) \quad (1.124)$$

To obtain w , we multiply both sides of equation (1.124) by $2e^{iw}$, solve the resulting quadratic equation for e^{iw} , and finally take natural log as follows

$$e^{i2w} - 2ze^{iw} + 1 = 0$$

$$e^{iw} = z \pm \sqrt{z^2 - 1}$$

$$w = \cos^{-1} z = -i \ln(z \pm \sqrt{z^2 - 1})$$

Considering the plus sign (principle value of \ln), we obtain

$$w = \cos^{-1} z = -i \ln(z + \sqrt{z^2 - 1}) \quad (1.125)$$

Similarly,

$$w = \sin^{-1} z = -i \ln(iz + \sqrt{1 - z^2}) \quad (1.126)$$

$$w = \tan^{-1} z = \frac{i}{2} \ln \frac{i+z}{i-z} \quad (1.127)$$

It can be shown that the derivative of inverse trigonometric functions are given by.

$$f(z) = \sin^{-1} z \quad f'(z) = \frac{1}{\sqrt{1-z^2}} \quad (1.128)$$

$$f(z) = \cos^{-1} z \quad f'(z) = -\frac{1}{\sqrt{1-z^2}} \quad (1.129)$$

$$f(z) = \tan^{-1} z \quad f'(z) = \frac{1}{1+z^2} \quad (1.130)$$

The proof of equations (1.128), (1.129), and (1.130) is left as an exercise.

1.8 – Hyperbolic and Inverse Hyperbolic Functions

Hyperbolic Functions

Similar to real calculus, we define functions $\sinh z$, $\cosh z$, $\tanh z$, $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$.

$$\sinh z = \frac{1}{2} (e^z - e^{-z}) \quad (1.131)$$

$$\cosh z = \frac{1}{2} (e^z + e^{-z}) \quad (1.132)$$

$$\tanh z = \frac{\sinh z}{\cosh z} \quad (1.133)$$

$$\coth z = \frac{\cosh z}{\sinh z} \quad (1.134)$$

$$\operatorname{sech} z = \frac{1}{\cosh z} \quad (1.135)$$

$$\operatorname{csch} z = \frac{1}{\sinh z} \quad (1.136)$$

If z is replaced by x in the above equations, we obtain the same familiar functions considered in real trigonometry. Furthermore using the above definitions we can write

$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z, \quad \tanh(-z) = -\tanh z, \quad \coth(-z) = -\coth z \quad (1.137)$$

Since e^z is analytic everywhere, $\sinh z$ and $\cosh z$ are also analytic everywhere. The functions $\tanh z$ and $\operatorname{sech} z$ are analytic functions except at the points z where $\cosh z$ becomes zero. The functions $\coth z$ and $\operatorname{csch} z$ are analytic functions except at the points z where $\sinh z$ becomes zero. Since $d(e^z)/dz = e^z$, we can write the following derivatives which should also look familiar.

$$f(z) = \sinh z \quad f'(z) = \cosh z \quad (1.138)$$

$$f(z) = \cosh z \quad f'(z) = \sinh z \quad (1.139)$$

$$f(z) = \tanh z \quad f'(z) = \operatorname{sech}^2 z \quad (1.140)$$

$$f(z) = \coth z \quad f'(z) = -\operatorname{csch}^2 z \quad (1.141)$$

$$f(z) = \operatorname{sech} z \quad f'(z) = -\operatorname{sech} z \tanh z \quad (1.142)$$

$$f(z) = \operatorname{csch} z \quad f'(z) = -\operatorname{csch} z \coth z \quad (1.143)$$

Some General Complex Hyperbolic Identities

$$\cosh^2 z - \sinh^2 z = 1 \quad (1.144)$$

$$\cosh iz = \cos z \quad (1.145)$$

$$\sinh iz = i \sin z \quad (1.146)$$

$$\cos iz = \cosh z \quad (1.147)$$

$$\sin iz = i \sinh z \quad (1.148)$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \quad (1.149)$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 \quad (1.150)$$

$$\cosh(z) = \cosh x \cos y + i \sinh x \sin y \quad (1.151)$$

$$\sinh(z) = \sinh x \cos y + i \cosh x \sin y \quad (1.152)$$

The proof of above equations and some other Hyperbolic Identities can be easily shown using the definitions and is left as an exercise.

Equations (1.151) and (1.152) can be used to show Cauchy-Riemann equations hold and evaluate the derivatives of $\sinh z$ and $\cosh z$.

Example 19: Evaluate $f(z) = \sinh z$ at $z = -2 - i2$

$$\begin{aligned} f(z) = \sinh z &= \sinh(-2 - i2) = \sinh -2 \cos -2 + i \cosh -2 \sin -2 \\ &= (-3.627)(-0.416) + i (3.762)(-0.909) = 1.509 - i 3.421 \end{aligned}$$

Inverse Hyperbolic Functions

Similar to inverse trigonometric functions, we can consider the inverse hyperbolic functions. For example let

$$w = \cosh^{-1} z \tag{1.153}$$

$$z = \cosh w = \frac{1}{2} (e^W + e^{-W}) \tag{1.154}$$

To obtain w , we multiply both sides of equation (1.154) by $2e^W$, solve the resulting quadratic equation for e^W , and finally take natural log as follows

$$e^{2W} - 2ze^W + 1 = 0$$

$$e^W = z \pm \sqrt{z^2 - 1}$$

$$w = \cosh^{-1} z = \ln(z \pm \sqrt{z^2 - 1})$$

Considering the plus sign (principle value of \ln), we obtain

$$w = \cosh^{-1} z = \ln(z + \sqrt{z^2 - 1}) \tag{1.155}$$

Similarly,

$$w = \sinh^{-1} z = \ln(z + \sqrt{z^2 + 1}) \tag{1.156}$$

$$w = \tanh^{-1} z = \frac{1}{2} \ln \frac{1+z}{1-z} \tag{1.157}$$

The derivatives of inverse hyperbolic function can be obtained using implicit differentiation. This is left as an exercise.

CHAPTER 2

Integrals of Complex Functions

Overview

Integrals of complex functions are presented in this chapter. Section 1.1 addresses line integration in rectangular form and parametric form. Examples are given for writing parametric representations of different paths. Cauchy – Goursat Integral Theorem is covered in Section 2.2 and some application of this theorem is presented in Section 2.3. Cauchy Integral Formulas for derivatives of a complex function are presented in Section 2.4. Section 2.4 ends with Liouville's Theorem.

2.1 – Line Integration

Here we consider integrating a complex function $f(z)$ along a piecewise defined curve C in the complex plane from some initial complex point z_0 to z_n . Curve C is referred to as path or contour of integration.

$$\int_C^{z_n} f(z) dz \quad (2.1)$$

Let us assume $f(z)$ is a continuous function and C is a smooth curve divided into n partitions as shown in Figure 2.1.

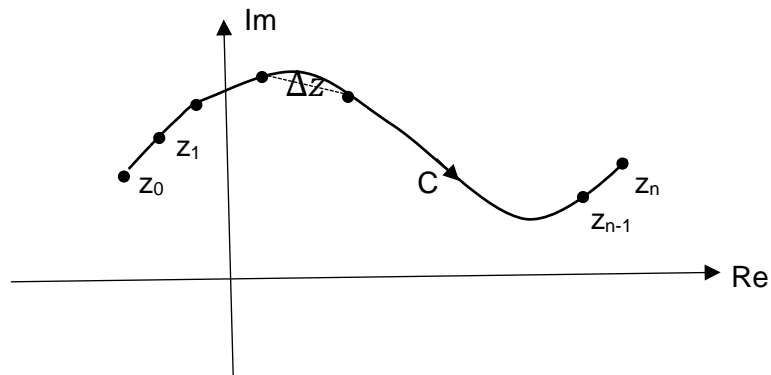


Figure 2.1 Path of Integration and Partitioning of the Path

Equation (2.1) can next be written as,

$$\int_C^{z_n} f(z) dz = \lim_{\Delta Z \rightarrow 0} \sum_{k=1}^n f(z_k) \Delta Z_k \quad (2.2)$$

As $\Delta Z \rightarrow 0$ the number of partitions approaches infinity. Equation (2.1) can be written in terms $z = x + iy$ and $\Delta Z = \Delta x + i \Delta y$.

$$\begin{aligned} \int_C^{z_n} f(z) dz &= \lim_{\Delta x + i \Delta y \rightarrow 0} \sum_{k=1}^n (u_k + i v_k) (\Delta x_k + i \Delta y_k) \\ &= \lim_{\Delta x \rightarrow 0 \text{ \& } \Delta y \rightarrow 0} \sum_{k=1}^n (u_k \Delta x_k - v_k \Delta y_k) + i (v_k \Delta x_k + u_k \Delta y_k) \\ \int_C f(z) dz &= \int_C u dx - v dy + i \int_C v dx + u dy \quad (2.3) \end{aligned}$$

If curve C is a closed curve, equation (2.3) is simply represented as

$$\oint_C f(z)dz = \oint_C udx - vdy + i \oint_C vdx + udy \quad (2.4)$$

One may presume the answer to equation (2.4) is zero, since the lower and upper limits of integration are the same. Or, the usual method of integral calculus can be used to evaluate equation (2.3) and simply ignore the path of integration. As it will be discussed, this depends on whether or not $f(z)$ is analytic in the given domain where path C is.

There are several methods of evaluating the integral of equation (2.1).

Cartesian Method of Line Integration

Equation (2.3) and the Cartesian representation of curve C can be used to evaluate the integral. We show this using a simple example.

Example 1: Evaluate $\int_C^{1+i2} \bar{z}^2 dz$ where path C is shown in Figure 2.2.

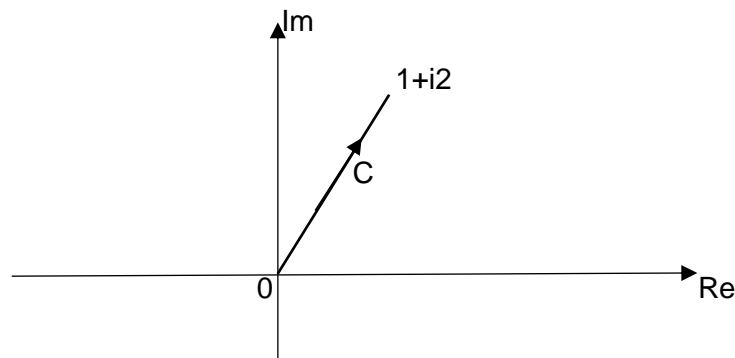


Figure 2.2 Path of Integration

$$C: \quad y = 2x \quad dy = 2dx \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 2 \quad \text{and} \\ [\bar{z}]^2 = (x - iy)^2 = (x^2 - y^2) + i(-2xy).$$

Using equation (2.3), we have

$$\int_C^{1+i2} \bar{z}^2 dz = \oint_C udx - vdy + i \oint_C vdx + udy \quad \text{integrating on C} \\ \int_C (x^2 - y^2)dx - (-2xy)dy + i \int_C (-2xy)dx + (x^2 - y^2)dy =$$

$$\int_0^1 (x^2 - 4x^2) dx - (-2x)(2x)(2dx) + i \int_0^1 (-2x)(2x) dx + (x^2 - 4x^2)(2dx) =$$

$$\int_0^1 5x^2 dx + i \int_0^1 -10x^2 dx = \frac{5}{3} - i \frac{10}{3}$$

It is of importance to note that even though the procedure shown for this example seems simple and not time consuming, as the integrand $f(z)$ and path of integration C become more complex, evaluating the integral of $f(z)$ on path C using equation method which is based on using equation (2.3) or equation (2.4) as applicable becomes very tedious.

Parametric Method of Line Integration

Equation (2.3) and the Cartesian representation of curve C can be written in parametric form in terms of parameter t . Consider the original integral form of equation (2.1).

$$\int_C^{z_n} f(z) dz$$

First we represent path C in parametric form $z(t)$, unless it is already in that form.

$$C: \quad z(t) = x(t) + iy(t) \quad \text{or} \quad z(t) = r(t) e^{i\theta(t)} \quad a \leq t \leq b \quad (2.5)$$

Next we substitute $z(t)$ in the integrand $f(z(t))$. Finally we write dz as

$$dz = \frac{dz(t)}{dt} dt \quad (2.6)$$

where $z(t)$ in equation (2.6) is the same as equation (2.5).

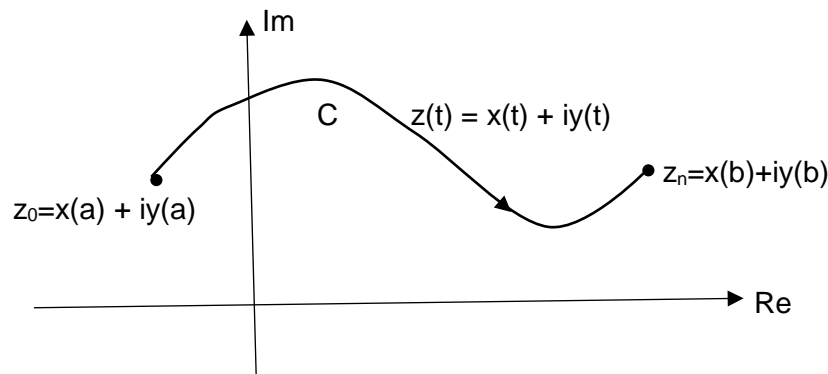


Figure 2.3 Parametric Path of Integration

The parametric form of equation (2.1) is

$$\int_C^{z_n} f(z) dz = \int_a^b f(z(t)) \frac{dz(t)}{dt} dt \quad (2.7)$$

The proof of equation (2.7) is rather simple and is considered below.

Theorem 2.1 Parametric Method of Line Integration

Let function $f(z)$ be defined on path C represented in parametric form as described by equation (2.5), then the integral of $f(z)$ on path C is given by

$$\int_C^{z_n} f(z) dz = \int_a^b f(z(t)) \frac{dz(t)}{dt} dt$$

Proof:

Substituting for $f(z(t))$ and $dz(t)/dt$ on the right hand side of equation (2.3), we can write

$$\int_C^{z_n} f(z) dz = \int_C u dx - v dy + i \int_C v dx + u dy$$

With curve C represented as $z(t) = x(t) + iy(t)$. Furthermore $dx=(dx/dt)dt$ and $dy=(dy/dt)dt$, we can write

$$\begin{aligned} \int_C^{z_n} f(z) dz &= \int_a^b \left[u(x(t), y(t)) \frac{dx(t)}{dt} dt - v(x(t), y(t)) \frac{dy(t)}{dt} dt \right] + \\ & i \int_a^b \left[v(x(t), y(t)) \frac{dx(t)}{dt} dt + u(x(t), y(t)) \frac{dy(t)}{dt} dt \right] \\ &= \int_a^b \left[u(x(t), y(t)) + i v(x(t), y(t)) \right] \left[\frac{dx(t)}{dt} + i \frac{dy(t)}{dt} \right] dt \\ &= \int_a^b f(z(t)) \frac{dz(t)}{dt} dt \end{aligned}$$

Prior to using equation (2.7) in evaluating a line integral using parametric form, it is useful to consider some examples of parametric representation of different paths.

Example 2: Write the parametric representation of the following paths

- Straight line from $z = 0$ to $z_1 = x_1 + iy_1$
- Straight line from $z_1 = x_1 + iy_1$ to $z_2 = x_2 + iy_2$
- A circle centered $z = 0$ with radius ρ in Counterclockwise direction
- An upper semicircle centered at $z = 0$ with radius ρ in Counterclockwise direction
- A circle centered at the $z_0 = x_0 + iy_0$ with radius ρ in Counterclockwise direction
- An ellipse centered at $z = 0$ whose Cartesian equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- An ellipse centered at $z = h + ik$ whose Cartesian equation is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

a) C: $z(t) = z_1 t = (x_1 + iy_1)t = tx_1 + ity_1$

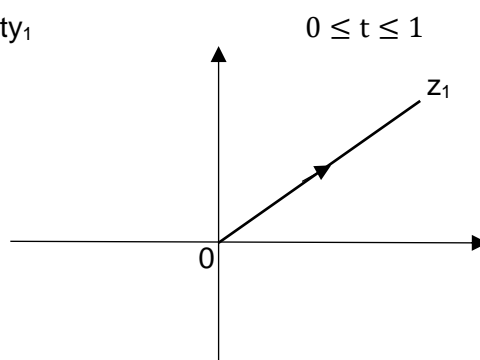


Figure 2.4

b) C: $z(t) = z_1 + (z_2 - z_1)t = (x_1 + iy_1) + [(x_2 + iy_2) - (x_1 + iy_1)]t$ $0 \leq t \leq 1$
 $= [x_1 + (x_2 - x_1)t] + i [y_1 + (y_2 - y_1)t]$

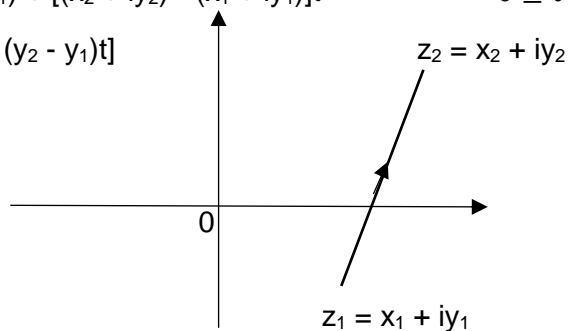


Figure 2.5

c) C: $z(t) = \rho e^{it} = \rho (\cos t + i \sin t)$

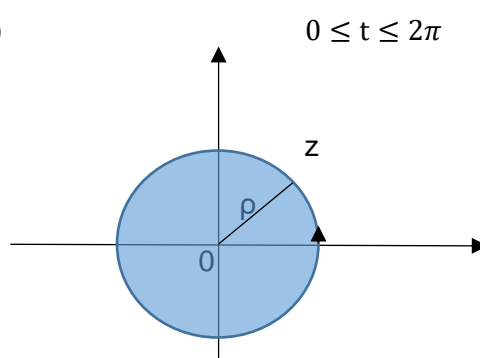


Figure 2.6

d) C: $z(t) = \rho e^{it} = \rho (\cos t + i \sin t)$ $0 \leq t \leq \pi$
 Same equation as in part (c)

e) C: $z(t) = z_0 + \rho e^{it} = z_0 + \rho (\cos t + i \sin t)$ $0 \leq t \leq 2\pi$
 $= (x_0 + \rho \cos t) + i (y_0 + \rho \sin t)$

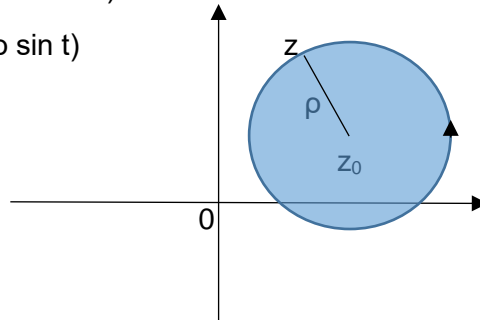


Figure 2.7

f) C: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
 $z(t) = a \cos t + i b \sin t$
 Center at $z = 0$

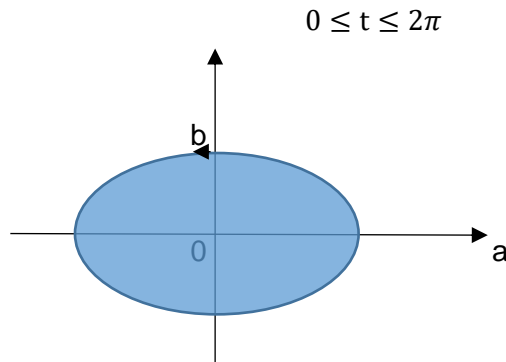


Figure 2.8

g) C: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
 $z(t) = (a \cos t + h) + i (b \sin t + k)$
 Center at $z_0 = h + ik$

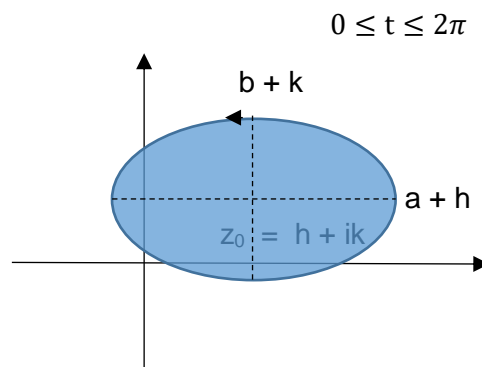


Figure 2.9

Example 3: Evaluate $\int_C^{1+i2} \bar{z}^2 dz$ where path C is shown in Figure 2.3.

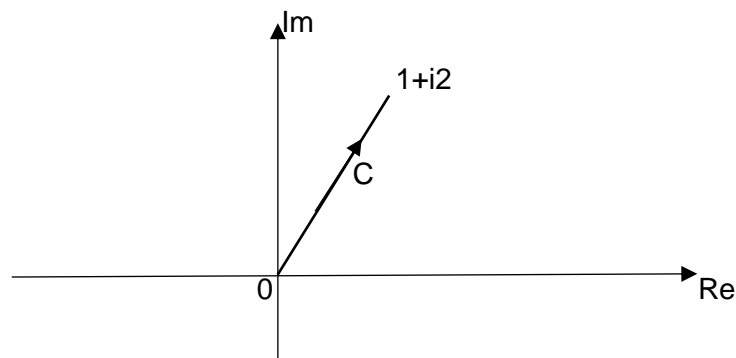


Figure 2.10 Path of Integration

$$C: \quad z(t) = (1 + i2)t = t + i2t \quad 0 \leq t \leq 1 \quad \text{and} \quad (dz/dt) = (1+i2)$$

Using equation (2.7), we have

$$\begin{aligned} \int_a^b f(z(t)) \frac{dz(t)}{dt} dt &= \int_0^1 [\overline{(t + i2t)}]^2 (1 + i2) dt = (1-i2)^2 (1+i2) \int_0^1 t^2 dt \\ &= 5(1-i2) \frac{1}{3} = \frac{5}{3} - i \frac{10}{3} \end{aligned}$$

This is the same result obtained in example 1.

If curve C is represented by $y = f(x)$, it suffices to represent the parametric form of this curve as,

$$C: \quad z(t) = t + if(t)$$

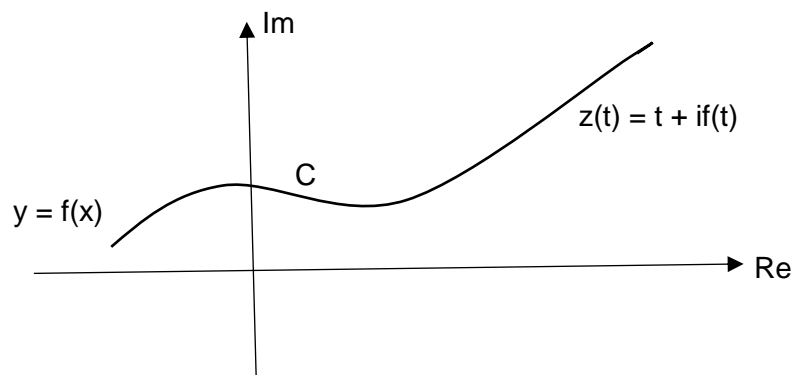


Figure 2.11 Parametric Path of Integration for $y = f(x)$

Example 4:
$$\oint_C f(z) dz = \oint_C \frac{1}{(z-z_0)^n} dz$$

In the above equation, n is an integer and C is a closed circle of radius ρ with center at z_0 in counterclockwise direction.

Using example (2e), we write
$$z(t) = z_0 + \rho e^{it} \quad 0 \leq t \leq 2\pi$$

Using equation (2.7), we have

$$\int_a^b f(z(t)) \frac{dz(t)}{dt} dt = \int_0^{2\pi} \frac{1}{(z_0 + \rho e^{it} - z_0)^n} (i\rho) e^{it} dt = i\rho^{(1-n)} \int_0^{2\pi} e^{i(1-n)t} dt =$$

$$\begin{aligned} [(i\rho^{(1-n)})/i(1-n)][e^{i(1-n)t}]_0^{2\pi} &= 0 && \text{for all } n \neq 1 && (2.8) \\ &= 2\pi i && \text{for } n = 1 && \Rightarrow \end{aligned}$$

$$\oint_C \frac{1}{(z-z_0)} dz = 2\pi i \quad (2.9)$$

Also if $z_0 = 0$ and the center of the circle is at $z = 0$, we have

$$\oint_C \frac{1}{z} dz = 2\pi i \quad (2.10)$$

Theorem 2.2 ML – Inequality

Let function $f(z)$ be a continuous function for all values of z on curve C and let $|f(z)| \leq M$. Let the length C from initial to final point of integration be L. Then

$$\left| \int_C f(z) dz \right| \leq ML \quad (2.11)$$

Proof:

Using equation (2.2), we write

$$\int_C f(z) dz = \lim_{\Delta Z \rightarrow 0} \sum_{k=1}^n f(Z_k) \Delta Z_k$$

$$\left| \lim_{\Delta Z \rightarrow 0} \sum_{k=1}^n f(Z_k) \Delta Z_k \right| \leq \lim_{\Delta Z \rightarrow 0} \sum_{k=1}^n |f(Z_k) \Delta Z_k|$$

$$\left| \lim_{\Delta Z \rightarrow 0} \sum_{k=1}^n f(Z_k) \Delta Z_k \right| \leq \lim_{\Delta Z \rightarrow 0} \sum_{k=1}^n |f(Z_k)| |\Delta Z_k|$$

$$\left| \lim_{\Delta Z \rightarrow 0} \sum_{k=1}^n f(Z_k) \Delta Z_k \right| \leq M \lim_{\Delta Z \rightarrow 0} \sum_{k=1}^n |\Delta Z_k|$$

As $\Delta Z \rightarrow 0$, we can conclude

$$\left| \int_C f(z) dz \right| \leq M \int_C |dz|$$

With $L = \int_C |dz| = \text{length of curve } C$, we now have

$$\left| \int_C f(z) dz \right| \leq ML$$

Some Properties of Line Integrals

$$\text{a) } \int_C kf(z) dz = k \int_C f(z) dz \quad \text{K is a complex constant} \quad (2.12)$$

$$\text{b) } \int_C [f_1(z) \pm f_2(z)] dz = \int_C f_1(z) dz \pm \int_C f_2(z) dz \quad (2.13)$$

$$\text{c) } \int_C^{z_3} f(z) dz = \int_{C_1}^{z_2} f(z) dz + \int_{C_2}^{z_3} f(z) dz \quad (2.14)$$

Curve C consists of partitions C_1 and C_2 . This rule can be extended to as many partitions that make up curve C .

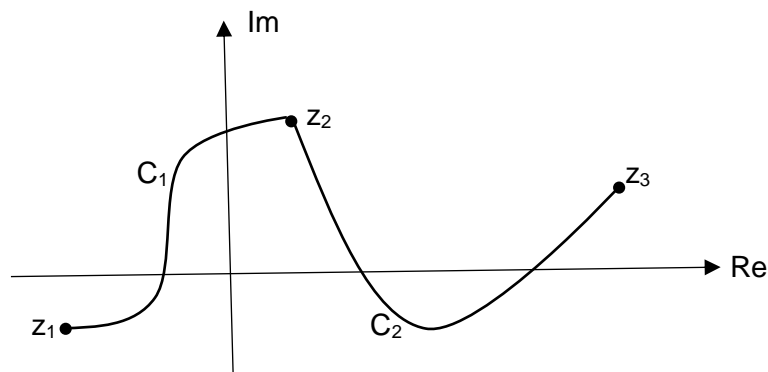


Figure 2.12 Curve C Consisting of partitions C_1 and C_2

$$\text{d) } \int_{C_1}^{z_2} f(z) dz = - \int_{C_2}^{z_1} f(z) dz \quad (2.15)$$

Curves C_1 and C_2 are exactly the same, but in opposite directions.

Example 5: Evaluate $\int_C^i z dz$, where C consists of straight horizontal line from $z = 0$ to $z = 1$ and a quarter circle from $z = 1$ to $z = i$ as shown.

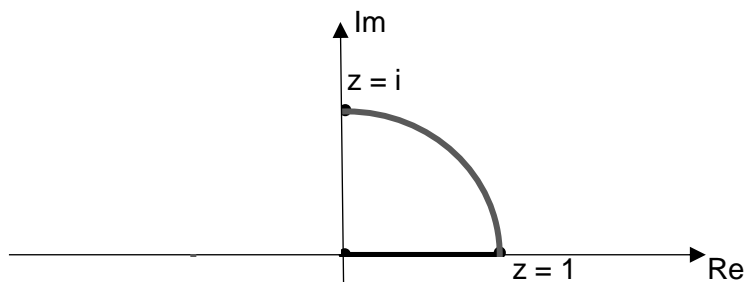


Figure 2.13

$$\begin{aligned} \text{C1: } z(t) &= t & 0 \leq t \leq 1 & & dz/dt &= 1 \\ \text{C2: } z(t) &= e^{it} & 0 \leq t \leq \pi/2 & & dz/dt &= i e^{it} \end{aligned}$$

$$\begin{aligned} \int_C^i z dz &= \int_0^1 t(1) dt + \int_0^{\pi/2} e^{it} (i e^{it}) dt = \frac{1}{2} + \left(\frac{1}{2}\right) [e^{i2t}]_0^{\pi/2} = \frac{1}{2} + \left(\frac{1}{2}\right) [e^{i\pi} - 1] \\ &= -(1/2) \end{aligned}$$

Example 6: Evaluate $\int_C^i z dz$, where C consists of straight vertical line from $z = 0$ to $z = i$.

$$\text{C: } z(t) = i t \quad 0 \leq t \leq 1 \quad dz/dt = i$$

$$\int_C^i z dz = \int_0^1 i t (i) dt = -(1/2)$$

It is not a coincidence that the results of examples 5 and 6 are the same. This happens when the integrand $f(z)$ is analytic in a simply connected domain. Simply connected set was defined in Section 1.3. The line integrals that are independent of the path of integration and its implication will be discussed in more detail in Section 2.3.

2.2 – Cauchy - Goursat Integral Theorem

Here we consider integrating a complex function $f(z)$ on a simple closed curve C . We begin our discussion assuming curve C is in a simply closed domain D and the function $f(z)$ is analytic everywhere in domain D .

The simply and multiply connected sets were defined in Section 1.3. Here we define simple and not simple closed paths and review the definitions of simply and multiply connected domains.

Simple and Not Simple Closed Paths

A closed curve C that does not intersect itself at any point is called a Simple closed path. If a closed curve C intersects itself at one or more points, it is called Not a Simple closed path. This definition is illustrated in Figure 2.14 and Figure 2.15.

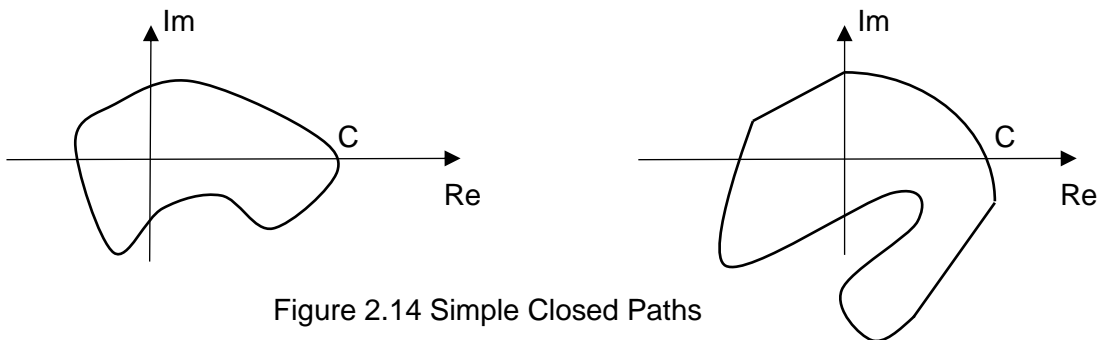


Figure 2.14 Simple Closed Paths

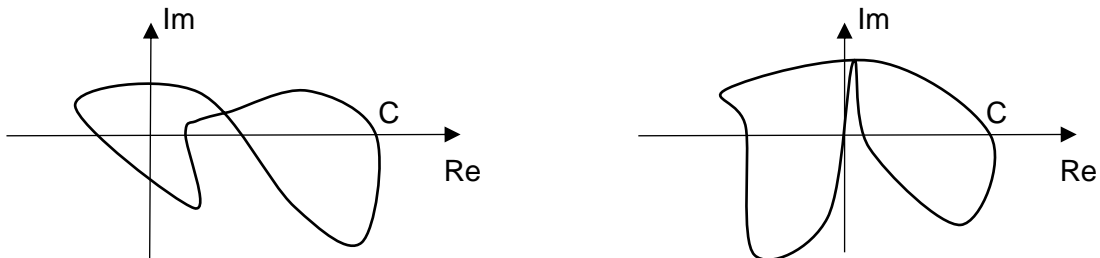


Figure 2.15 Not Simple Closed Paths

Simply and Multiply Connected Domain

A simply connected domain is a set that every simple closed curve in the set only contains points of that set. A multiply connected domain is a set that there is at least one simple closed curve in the set with one or more points that don't belong to the set. This definition is illustrated in Figure 2.16 and Figure 2.17.

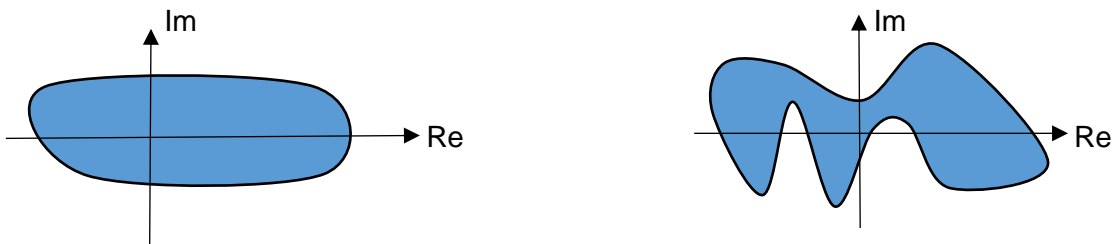


Figure 2.16 Simply Connected Domain



Figure 2.17 Not Simply Connected Domain

Theorem 2.3 Cauchy – Goursat Integral Theorem

If the complex function $f(z)$ is analytic everywhere in a simply connected domain D and C is any simple closed path C in D , then

$$\oint_C f(z) dz = 0 \quad (2.16)$$

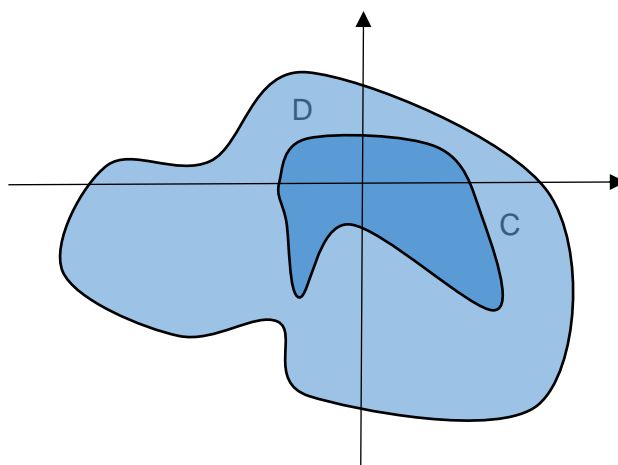


Figure 2.18 Simple Closed Curve in a Simply Connected Domain

Proof:

The proof of Cauchy – Goursat theorem is based on Green's theorem and use of Cauchy-Riemann equations. Green's theorem relates line integrals and double integrals. Here we simply state Green's theorem and refer the reader to calculus for the proof of the theorem.

Green's Theorem: Let P and Q be functions of (x,y) , defined over a region D , bounded by curve C and have continuous partial derivatives. Let C be positively oriented, piecewise smooth, and a simple closed curve, then

$$\oint_C P(x,y) dx + Q(x,y) dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \quad (2.17)$$

Since the function $f(z) = u(x,y) + i v(x,y)$ is assumed to be analytic, the integral of $f(z)$ on a closed path C can be written as

$$\oint_C f(z) dz = \oint_C u dx - v dy + i \oint_C v dx + u dy$$

Applying Green's theorem to this integral results in

$$\oint_C f(z) dz = \iint_D \left[\left(-\frac{\partial v}{\partial x} \right) - \left(\frac{\partial u}{\partial y} \right) \right] dx dy + i \iint_D \left[\left(\frac{\partial u}{\partial x} \right) - \left(\frac{\partial v}{\partial y} \right) \right] dx dy \quad (2.18)$$

Using Cauchy-Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in equation (2.18)

$$\oint_C f(z) dz = 0$$

Example 7: Evaluate $\oint_C f(z) dz$ for the given $f(z)$ and the closed path C in CCW direction

a) $f(z) = z^2 + 4z + e^z + 10$

C: $|z - i2| = 10$

b) $f(z) = \frac{z+5}{(z-5)(z^2+9)}$

C:

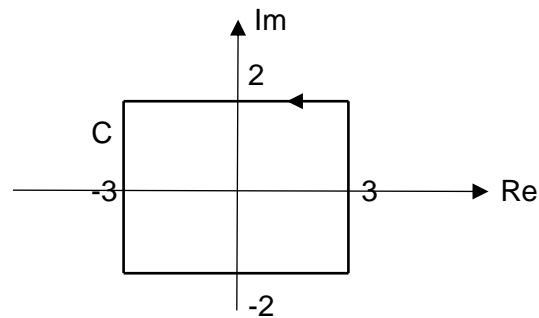


Figure 2.19 Path C – Example 7b

a) $\oint_C f(z) dz = 0$, since $f(z) = z^2 + 4z + e^z + 10$ is analytic everywhere in the region enclosed by C .

b) $\oint_C f(z) dz = 0$, even though $f(z) = \frac{z+5}{(z-5)(z^2+9)}$ is not analytic at $z = 5$ and $z = \pm i3$.
 However $f(z)$ is analytic everywhere in the region enclosed by C and the points $z = 5$ and $z = \pm i3$ are all outside this region.

It is important to revisit example 4 of this Chapter. In example 4 we considered the following integral

$$\oint_C f(z) dz = \oint_C \frac{1}{(z-z_0)^n} dz \quad \begin{array}{l} C: z(t) = |z - z_0| = \rho \text{ or } z(t) = z_0 + \rho e^{it} \\ \text{CCW direction} \end{array}$$

We saw

$$\begin{aligned} \oint \frac{1}{(z-z_0)^n} dz &= 0 \quad \text{for all } n \neq 1 \\ &= 2\pi i \quad \text{for } n = 1 \end{aligned}$$

Clearly the results for $n > 1$, does not follow from Cauchy – Goursat Theorem. For example $f(z) = \frac{1}{(z-z_0)^2}$ is not analytic at $z = z_0$. Hence the condition for $f(z)$ to be analytic in D is sufficient rather than necessary for Cauchy – Goursat Integral Theorem.

2.3 – Some Applications of Cauchy - Goursat Theorem

In this section we consider some basic applications of Cauchy – Goursat Integral Theorem. We consider a simply connected domain and integration independence of path. We also apply Cauchy – Goursat Integral Theorem to multiply connected domains.

Simply Connected Domain

We saw, if $f(z)$ is analytic everywhere in a simply connected domain D and C is any simple closed path C in D , then

$$\oint_C f(z) dz = 0$$

We partition the counterclockwise closed curve C into curves C_1 and C_2 at points z_1 and z_2 as shown at in Figure 2.20. Using Cauchy – Goursat Integral Theorem

$$\begin{aligned} \int_{C_1}^{z_2} f(z) dz + \int_{C_2}^{z_1} f(z) dz &= 0 \Rightarrow \\ \int_{C_1}^{z_2} f(z) dz &= - \int_{C_2}^{z_1} f(z) dz \Rightarrow \end{aligned}$$

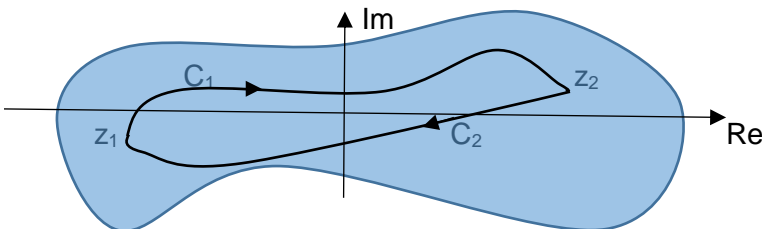


Figure 2.20 Closed Path C – Partitioned

Considering curve C_3 to be the same as C_2 , but in opposite direction as shown in Figure 2.21, we have

$$\int_{C_1}^{z_2} f(z) dz = \int_{C_3}^{z_2} f(z) dz \quad (2.19)$$

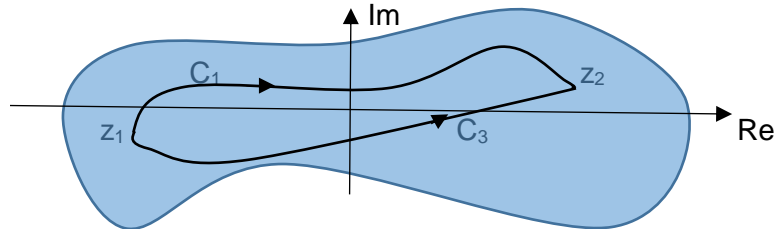


Figure 2.21 Closed Path C – Partitioned

Independence of the Path of Integration

Based on the above discussion and equation (2.19), we can state the following Theorem.

Theorem 2.4

If $f(z)$ is an analytic function everywhere in a simply connected domain D , then the integral

$$\int_C f(z) dz$$

is independent of the path of integration C .

Theorem 2.5 Existence of Antiderivative of $f(z)$

If $f(z)$ is an analytic function everywhere in a simply connected domain D , then

$$\int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1) \quad (2.20)$$

where $F(z)$ defined as antiderivative of $f(z)$ is an analytic function and $F'(z) = f(z)$ in D .

Proof:

Let us define $F(z)$ as

$$F(z) = \int_{z_1}^z f(z) dz \quad (2.21)$$

Using equation (2.3), we have

$$F(z) = \int_{z_1}^z f(z) dz = \int u dx - v dy + i \int v dx + u dy \quad (2.22)$$

The limits of the integration are from x_1 to x and y_1 to y . The real part and imaginary parts of $F(z)$ are

$$\operatorname{Re} F(z) = \int u dx - v dy \quad \operatorname{Im} F(z) = \int v dx + u dy$$

Using Cauchy-Riemann equations, we can write

$$\begin{aligned} [\operatorname{Re} F(z)]_x &= [\operatorname{Im} F(z)]_y & \text{and} & & [\operatorname{Re} F(z)]_y &= -[\operatorname{Im} F(z)]_x & \Rightarrow \\ u(x,y) &= u(x,y) & \text{and} & & -v(x,y) &= -v(x,y) \end{aligned}$$

Hence Cauchy-Riemann equations hold for $F(z)$ and as a result $F(z)$ is an analytic function. Using equation (1.62), we can obtain the derivative of $F(z)$ as

$$\begin{aligned} F'(z) &= [\operatorname{Re} F(z)]_x + i [\operatorname{Im} F(z)]_x = u(x,y) + iv(x,y) & \Rightarrow \\ F'(z) &= f(z) \end{aligned} \quad (2.23)$$

Now let us consider the integral, $\int_C^{z_2} f(z) dz$. Using $F'(z) = f(z)$, we can write.

$$\int_{z_1}^{z_2} f(z) dz = \int_{z_1}^{z_2} F'(z) dz = F(z) \Big|_{z_1}^{z_2} = F(z_2) - F(z_1) \quad (2.24)$$

Example 8: Evaluate $\int_C^{z_2} f(z) dz$ for the given $f(z)$ and path C .

- a) $f(z) = z^2 + 2$ Path C consists of C_1 and C_2 as shown below

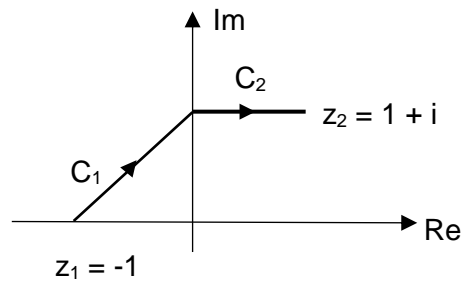


Figure 2.22

- b) $f(z) = \sin 2z$ Path C as shown below

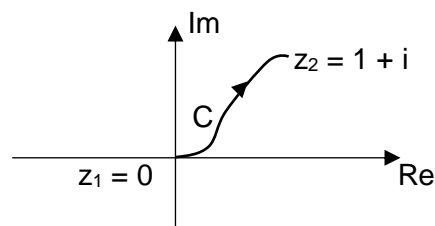


Figure 2.23

- a) Since $f(z) = z^2 + 2$ is analytic everywhere in the region, the integral of $f(z)$ is independent of the path and

$$\int_{-1}^{1+i} (z^2 + 2) dz = \left(\frac{1}{3}z^3 + 2z \right) \Big|_{-1}^{1+i} = \left[\frac{1}{3}(1+i)^3 + 2(1+i) \right] - \left[\frac{1}{3}(-1)^3 + 2(-1) \right] \\ = \frac{1}{3}(11 + i8)$$

- b) Since $f(z) = \sin 2z$ is analytic everywhere in the region, the integral of $f(z)$ is independent of the path and

$$\int_0^{1+i} \sin 2z dz = \left(-\frac{1}{2} \cos 2z \right) \Big|_0^{1+i} = \left(-\frac{1}{2} \right) [\cos(2+i2) - 1] \\ = \left(-\frac{1}{2} \right) [\cos 2 \cosh 2 - i \sin 2 \sinh 2 - 1] = 1.283 + 1.649i$$

Multiply Connected Domain

We now apply Cauchy – Goursat Integral Theorem to multiply connected domains. Let us start with a doubly connected domain as shown in Figure 2.24. We begin with

$$\oint_C f(z) dz = 0$$

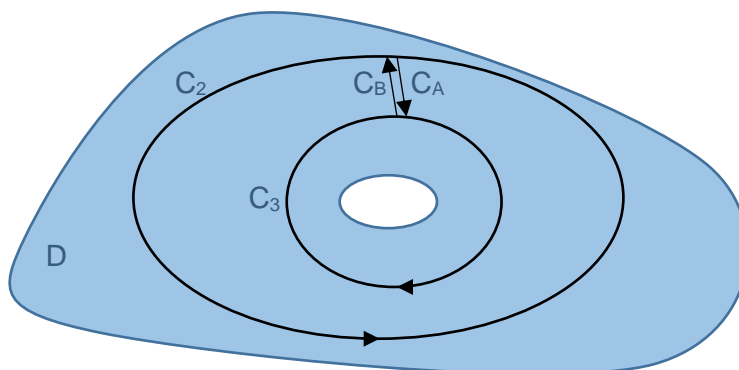


Figure 2.24 Doubly Connected Domain

Curve C consists of curves C_2 , C_A , C_3 , and C_B . In other words let us assume we start at any point on the lower portion of curve C_2 and move the point in counterclockwise direction. Once the point gets to the cuts (C_A and C_B), the point goes on the curve C_A and moves to get to curve C_3 . The point then moves on C_3 in clockwise direction until it reaches the cuts and gets on C_B . The point moves on C_B until it gets back to C_2 . The point moves on C_2 from where the cuts are and completes the contour. Even though the point leaves C_2 at the cuts, it gets back on it after going on C_A , C_3 , and C_B with the directions shown and completes the contour. Function $f(z)$ is analytic everywhere in the region enclosed by curve C described here.

$$\oint_C f(z) dz = \oint_{C_2} f(z) dz + \int_{C_A} f(z) dz + \oint_{C_3} f(z) dz + \int_{C_B} f(z) dz = 0 \quad (2.25)$$

The second and fourth integrals on the right hand side of equation (2.25) cancel. Hence we have

$$\oint_{C_2} f(z) dz = - \oint_{C_3} f(z) dz$$

$$\oint_{C_2} f(z) dz = \oint_{C_1} f(z) dz \quad (2.26)$$

As can be noted, C_1 is the same as C_3 but in counterclockwise direction as shown below.

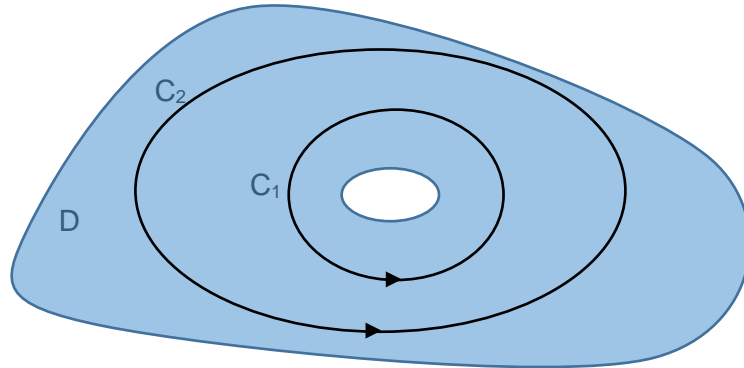


Figure 2.25 Doubly Connected Domain

Referring to example 4 in this Chapter, we saw

$$\oint_C \frac{1}{(z-z_0)} dz = 2\pi i \quad \text{Where } C \text{ is a circle of radius } \rho \text{ and center at } z_0 \text{ in CCW direction}$$

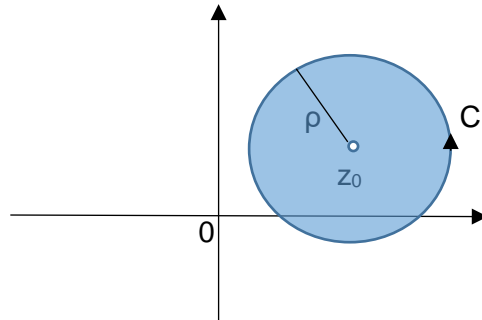


Figure 2.26

Based on equation (2.26) and the above discussion

$$\oint_C \frac{1}{(z-z_0)} dz = 2\pi i$$

for any closed curve C in counterclockwise direction which encloses z_0 .

Based on Cauchy – Goursat Integral Theorem this integral is zero, if curve C does not enclose z_0 , which implies $f(z)$ is analytic in the region.

Obviously the same conclusion can be made, if $z_0 = 0$

$$\oint_C \frac{1}{z} dz = 2\pi i$$

for any closed curve C in counterclockwise direction which encloses $z = 0$

Similarly this integral is zero, if curve C does not enclose $z = 0$.

Let us now consider a triply connected domain as shown in Figure 2.27. Once again we begin with

$$\oint f(z) dz = 0$$

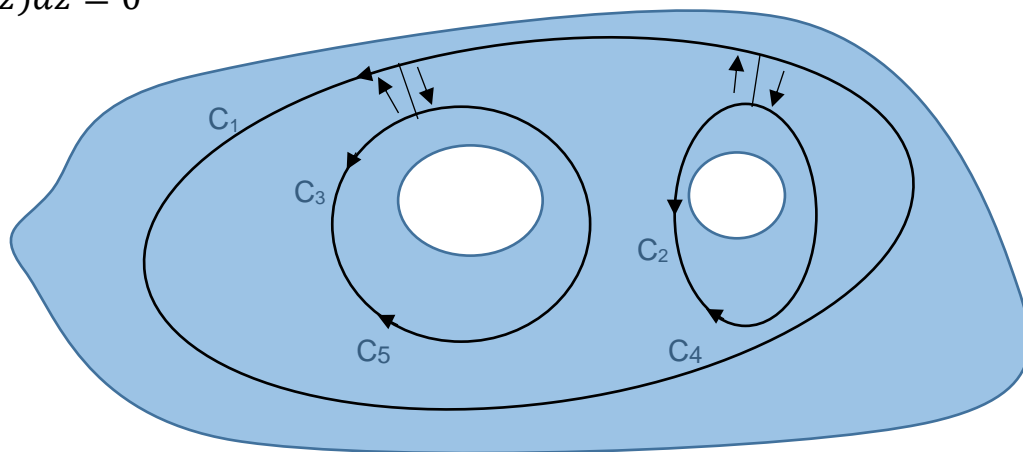


Figure 2.27 Triply Connected Domain

Similar to the Figure 2.24 for the case of doubly connected domain, we make cuts in both direction from curve C_1 to curves C_4 and C_5 . We next apply Cauchy – Goursat Integral Theorem with closed curve C consisting of curve C_1 , C_4 (opposite direction of C_2), C_5 (opposite direction of C_3), and the four cuts (two from C_1 to C_4 and two from C_1 to C_5). We also note the integrals over the cuts cancel. It can be shown that

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz + \oint_{C_3} f(z) dz \quad (2.27)$$

Similar procedure can be used to extend for the cases higher than triply connected. As will be observed similar process will be used in proof of Residue Theorem.

Example 9: Evaluate $\oint_C f(z) dz$ for the given $f(z)$ and path C .

$$f(z) = \frac{z+10}{z^3+4z^2-5z} \quad C: \quad |z| = 4 \text{ in counterclockwise direction}$$

$$z^3 + 4z^2 - 5z = z(z - 1)(z + 5)$$

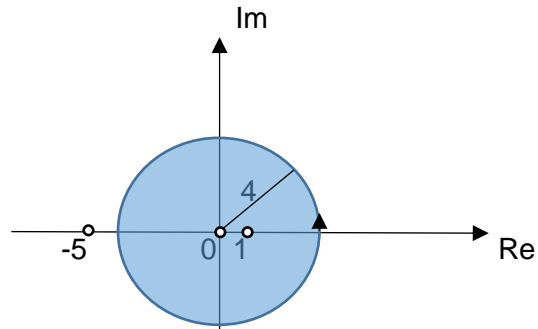


Figure 2.28

Using partial fractions, we write

$$f(z) = \frac{z+10}{z^3+4z^2-5z} = \frac{z+10}{z(z-1)(z+5)} = \frac{-2}{z} + \frac{11/6}{z-1} + \frac{1/6}{z+5}$$

$$\oint_C \frac{z+10}{z(z-1)(z+5)} dz = \oint_C \frac{-2}{z} dz + \oint_C \frac{11/6}{(z-1)} dz + \oint_C \frac{1/6}{(z+5)} dz = -2(2\pi i) + (11/6)(2\pi i) + 0$$

$$= -i\pi/3$$

2.4 – Cauchy's Integral Formulas

In Section 2.3 we saw some applications of Cauchy – Goursat Integral Theorem. Perhaps some of the most significant applications of Cauchy – Goursat Integral Theorem is Cauchy's Integral Formulas. These additional applications are presented in this section.

Theorem 2.6 Cauchy's Integral Formula

If $f(z)$ is an analytic function everywhere in a simply connected domain D and C is a simple closed path in D , then

$$\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0) \quad (2.28)$$

where z_0 is any point inside path C and integration is in counterclockwise direction.

Proof:

We write $f(z)$ as

$$f(z) = f(z_0) + f(z) - f(z_0) \quad (2.29)$$

Substituting equation (2.29) in equation (2.28) results in

$$\begin{aligned} \oint_C \frac{f(z)}{z-z_0} dz &= \oint_C \frac{f(z_0)}{z-z_0} dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0} dz \\ &= f(z_0) \oint_C \frac{1}{z-z_0} dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0} dz \end{aligned} \quad (2.30)$$

From our discussion in Section 2.3, we have

$$f(z_0) \oint_C \frac{1}{z-z_0} dz = 2\pi i f(z_0) \quad (2.31)$$

To show the second integral in equation (2.30) is zero, we can replace C by a small circle C_1 of radius ρ with center at z_0 as shown in Figure 2.29.

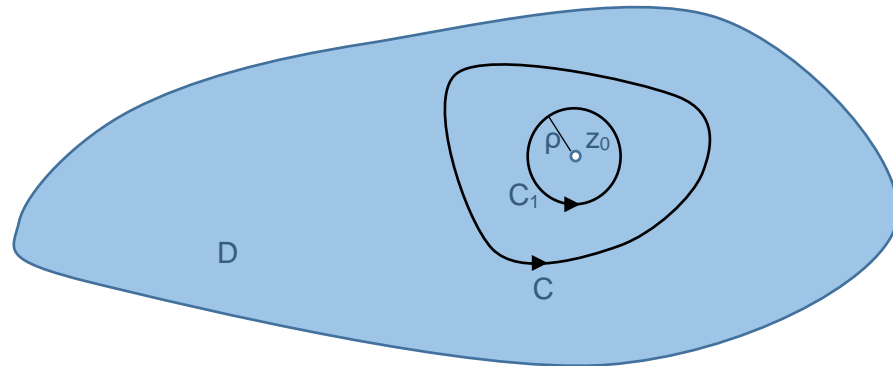


Figure 2.29 Replacing C by Small Circle C_1 Containing z_0

The function $f(z)$ is analytic and hence continuous at z_0 . Based on definition of continuity, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all z satisfying $|z - z_0| < \delta$. By selecting ρ to be smaller than δ and using ML – Inequality Theorem of Section 2.1, we can write

$$\left| \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz \right| \leq ML$$

Here $M = \frac{\epsilon}{\rho}$ and L is the length of curve C_1 ($2\pi\rho$). Hence with $ML = \frac{\epsilon}{\rho} (2\pi\rho) = 2\pi\epsilon$, we have

$$\left| \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz \right| \leq 2\pi\epsilon$$

We can select $\epsilon > 0$ as small as possible ($\epsilon \rightarrow 0+$). Hence the second integral on the right hand side of equation (2.30) is zero.

Example 10: Evaluate $\oint_C \frac{z^2+2}{(z^2+1)(z^2-1)} dz$.

C: $|z - i| = 1$ in counterclockwise direction

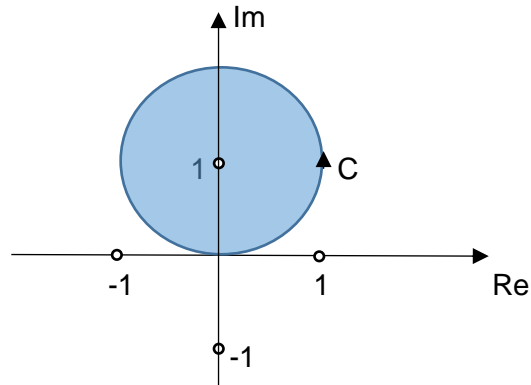


Figure 2.30

$$(z^2 + 1)(z^2 - 1) = (z + i)(z - i)(z + 1)(z - 1)$$

Since $z = i$ is the only point enclosed by C \Rightarrow

$$\oint_C \frac{z^2+2}{(z^2+1)(z^2-1)} dz = \oint_C \frac{(z^2+2)/[(z+i)(z^2-1)]}{(z-i)} dz = 2\pi i \left. \frac{(z^2+2)}{(z+i)(z^2-1)} \right|_{z=i} =$$

$$2\pi i \frac{(-1+2)}{(2i)(-1-1)} = -\pi/2$$

An alternate form of Cauchy's Integral Formula of equation (2.28) can be written as

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz \quad (2.32)$$

Similar to equation (2.32), we now develop equations for $f'(z_0), f''(z_0), \dots, f^{(n)}(z_0)$. This is shown explained in the following theorem.

Theorem 2.7 Cauchy's Integral Formula for Derivatives of Analytic Function f(z)

If f(z) is an analytic function everywhere in a simply connected domain D and C is a simple closed path in D, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad (2.33)$$

where z_0 is any point inside path C and integration is in counterclockwise direction.

Proof:

The proof for the case of $n = 1$ is presented here. We start with $f'(z_0)$ using the definition

$$f'(z_0) = \lim_{\Delta Z \rightarrow 0} \frac{f(z_0 + \Delta Z) - f(z_0)}{\Delta Z}$$

Using equation (2.32), we write

$$\begin{aligned} \frac{f(z_0 + \Delta Z) - f(z_0)}{\Delta Z} &= \frac{1}{2\pi i} \oint_C \frac{1}{\Delta Z} \left[\frac{f(z)}{z - (z_0 + \Delta Z)} \right] dz - \frac{1}{2\pi i} \oint_C \frac{1}{\Delta Z} \left[\frac{f(z)}{z - z_0} \right] dz = \\ &= \frac{1}{2\pi i \Delta Z} \oint_C \frac{(z - z_0)f(z) - [(z - (z_0 + \Delta Z))]f(z)}{[z - (z_0 + \Delta Z)][z - z_0]} dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{[z - z_0 - \Delta Z][z - z_0]} dz \end{aligned} \quad (2.34)$$

If we show the integrals of equation (2.33) for $n = 1$ and (2.34) are the same as $\Delta Z \rightarrow 0$, the theorem is proved. To do so, let us ignore the $(1/2\pi i)$ factor which is the constant coefficient in both and consider the difference between the two.

$$\begin{aligned} \oint_C \frac{f(z)}{[z - z_0 - \Delta Z][z - z_0]} dz - \oint_C \frac{f(z)}{[z - z_0]^2} dz &= \oint_C \frac{f(z)(z - z_0) - f(z)[z - z_0 - \Delta Z]}{[z - z_0 - \Delta Z][z - z_0]^2} dz \\ &= \oint_C \frac{\Delta Z f(z)}{[z - z_0 - \Delta Z][z - z_0]^2} dz \end{aligned} \quad (2.35)$$

The function f(z) is analytic and hence continuous. Therefore f(z) is bounded ($|f(z)| \leq P$). Furthermore, the distance between all z on C and z_0 , is assumed to be greater than some positive real number ρ ($|z - z_0| \geq \rho$). This implies

$$|z - z_0|^2 \geq \rho^2 \quad \Rightarrow \quad \frac{1}{|z - z_0|^2} \leq \frac{1}{\rho^2}$$

We also note that

$$|z - z_0| = |z - z_0 - \Delta z + \Delta z| \leq |z - z_0 - \Delta z| + |\Delta z| \geq \rho \quad \Rightarrow$$

$$|z - z_0 - \Delta z| \geq \rho - |\Delta z|$$

We now assume $\Delta z \leq \rho/2$, we can write

$$|z - z_0 - \Delta z| \geq \rho - \rho/2 \quad \Rightarrow$$

$$|z - z_0 - \Delta z| \geq \rho/2 \quad \Rightarrow \quad \frac{1}{|z - z_0 - \Delta z|} \leq \frac{2}{\rho}$$

Let the length of curve C be L. Using ML – Inequality Theorem of Section 2.1 and equation (2.35), we have

$$\left| \oint_C \frac{\Delta z f(z)}{[z - z_0 - \Delta z][z - z_0]^2} dz \right| \leq (\Delta z)(P) \left(\frac{2}{\rho}\right) \left(\frac{1}{\rho^2}\right) L$$

As we let $\Delta z \rightarrow 0$, the left hand side of equation (2.35) approaches zero and we conclude

$$f^{(1)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz \quad (2.36)$$

$$2\pi i f^{(1)}(z_0) = \oint_C \frac{f(z)}{(z - z_0)^2} dz$$

Example 11: Evaluate $\oint_C \frac{z^2 + 2z + 2}{(z^2 + 4)(z - 2)^2} dz$ for the given f(z) and path C.

C: As shown in Figure 2.31

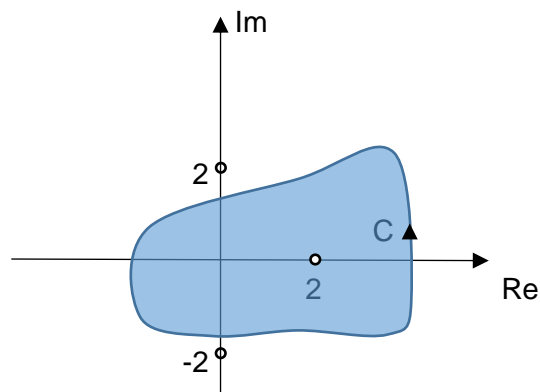


Figure 2.31

$$(z^2 + 4)(z - 2)^2 = (z + i2)(z - i2)(z - 2)(z - 2)$$

Since $z = 2$ and $z = 2$ are the only points enclosed by C , we define $f(z)$ as

$$f(z) = \frac{z^2 + 2z + 2}{(z^2 + 4)}, \text{ and use equation 2.36}$$

$$\oint_C \frac{f(z)}{(z-2)^2} dz = 2\pi i f^{(1)}(2) = 2\pi i \left[\frac{(2z+2)(z^2+4) - 2z(z^2+2z+2)}{(z^2+4)^2} \right]_{z=2} = i\pi/4$$

Let us examine equation (2.33) in the following form,

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \quad (2.37)$$

As discussed in Theorem 2.7, $f(z)$ is bounded ($|f(z)| \leq P$). Without loss of generality, let us assume C is a circle at z_0 and radius ρ . Using equation (2.37), and ML – Inequality Theorem with $L = 2\pi\rho$, we have

$$|f^{(n)}(z_0)| \leq \frac{n!P}{\rho^n} \quad (2.38)$$

Theorem 2.8 Liouville's Theorem

If $f(z)$ is bounded and is an entire function for all values of z , then $f(z)$ is constant.

Proof:

Using equation (2.38) with $n = 1$, we have

$$|f^{(1)}(z_0)| \leq \frac{P}{\rho}$$

Since we can select ρ arbitrary large, this indicates

$$f^{(1)}(z_0) = 0$$

for all z_0 and proves $f(z)$ is constant.

CHAPTER 3

Complex Series and Residue Theorem

Overview

Section 3.1 begins with sequences and series along with ratio and root tests for convergence, followed by geometric and power series in Section 3.2. Taylor and Maclaurin series are discussed in Section 3.3. In Section 3.4 Laurent series and definition of residue are presented. Section 3.5 covers the definition of poles and zeros. Evaluation of residue of a simple pole and repeated pole is discussed in Section 3.6. Residue Theorem is covered in Section 3.7. Section 3.8 presents some applications of Residue Theorem to real integrals.

3.1 – Sequences and Series of Complex Values

Here we begin with the definition of complex sequences and extend to complex series. Many of the definitions, tests, and theorems are similar to what we observed in real calculus.

Complex Sequences

A complex sequence denoted by $\{Z_n\}$ is a collection of terms. The value of n can be $n = 1, 2, 3, \dots$ or $n = 0, 1, 2, \dots$ as

$$\{Z_1, Z_2, Z_3, \dots\} \text{ or } \{Z_0, Z_1, Z_2, \dots\} \quad (3.1)$$

The terms of the sequence in general are complex. As an example

$$\{Z_n\} = \{Z_0, Z_1, Z_2, \dots\} = \{(1 - i), (1 - i)^2, (1 - i)^3, \dots\}$$

Depending on the particular application, n can begin with any integer such as $n = \dots, -1$, or 0, or $1, \dots$. In most application the initial value of n is either 0 or 1. The terms of the sequence can be complex or real consisting of a random values or be characterized by an equation or pattern.

Convergence

The sequence $\{Z_n\} = \{Z_0, Z_1, Z_2, \dots\}$ converges to a value L and is called convergent sequence, if

$$\lim_{n \rightarrow \infty} Z_n = L = L_1 + i L_2 \quad (3.2)$$

Using the definition of limit, equation (3.2) implies that as n increases, for some $n > N$ we can write

$$|Z_n - L| < \varepsilon \quad (3.3)$$

The limit L must be unique. Equation (3.3) simply indicates for $n > N$, all values of Z_n are inside the disk whose center is at L and has a radius of ε as shown in Figure 3.1.

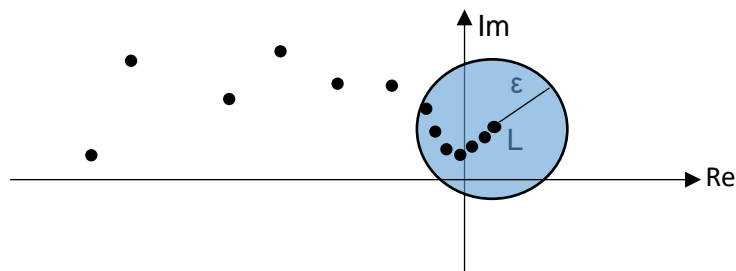


Figure 3.1 Sequence $\{Z_n\}$ Converging to L

Example 1: Indicate if the given sequences are convergent. If convergent evaluate the limit.

- a) $\{Z_n\} = \{(2 + i)^n\}$
 b) $\{Z_n\} = \{(0.5i)^n\}$

- a) $\{(2 + i)^n\} = \{(2 + i), (2 + i)^2, (2 + i)^3, (2 + i)^4, \dots\}$ is divergent
 b) $\{(0.5i)^n\} = \{0.5i, -0.25, -0.125i, 0.0625, \dots\}$ is convergent with the limit $L = 0$

Since the sequence $\{Z_n\} = \{x_n\} + i\{y_n\}$, we can state the convergence of a sequence to $L = L_1 + iL_2$, by considering the convergence of the real and imaginary parts of the sequence to L_1 and L_2 respectively. This is presented in the following theorem.

Theorem 3.1 Convergence of Sequence $\{Z_n\} = \{x_n\} + i\{y_n\}$ to $L = L_1 + iL_2$

A sequence $\{Z_n\} = \{x_n\} + i\{y_n\}$ converges to a complex number $L = L_1 + iL_2$, if and only if the real part of the sequence $\{x_n\}$ converges to L_1 and the imaginary part of the sequence $\{y_n\}$ converges to L_2 .

Proof:

We assume the sequence converges and has a limit $L = L_1 + iL_2$

$$\lim_{n \rightarrow \infty} Z_n = L = L_1 + iL_2 \quad \Rightarrow \quad |Z_n - L| < \varepsilon$$

$$\lim_{n \rightarrow \infty} x_n = L_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = L_2 \quad \Rightarrow \quad \text{as } n \rightarrow \infty$$

$$|x_n - L_1| < \varepsilon \quad \text{and} \quad |y_n - L_2| < \varepsilon$$

Hence as $n \rightarrow \infty$, the x_n and y_n will fall in the intervals $(L_1 - \varepsilon, L_1 + \varepsilon)$ and $(L_2 - \varepsilon, L_2 + \varepsilon)$ respectively. This is shown in Figure 3.2.

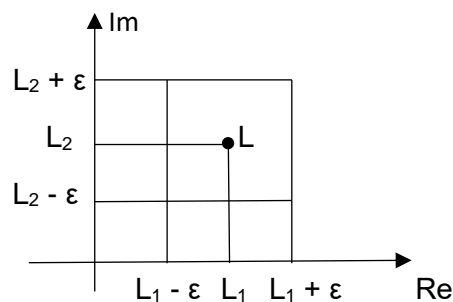


Figure 3.2 Sequence $\{Z_n = x_n + iy_n\}$ Converging to $L = L_1 + iL_2$

Now we consider the inverse of this theorem. If

$$\lim_{n \rightarrow \infty} x_n = L_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = L_2 \quad \Rightarrow$$

for some n large enough $|Z_n - L| < \varepsilon$, where $\{Z_n\} = \{x_n\} + i\{y_n\}$.

Then we can say the series converges to S if for any $\epsilon > 0$ we can find N such that for all $n > N$

$$|R_n| = |S - S_n| < \epsilon$$

Theorem 3.3

A necessary and sufficient condition for the series $S = \sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} (u_n + iv_n)$ to converge to $S = u + iv$ is that $\sum_{n=1}^{\infty} u_n$ converges to u and $\sum_{n=1}^{\infty} v_n$ converges to v .

This can be shown once the complex series S is partitioned to real part and imaginary part.

Theorem 3.4

The series $\sum_{n=1}^{\infty} w_n$ converges absolutely if

$$\lim_{n \rightarrow \infty} \left| \frac{w_{n+1}}{w_n} \right| = r < 1, \quad (3.8)$$

and diverges if

$$\lim_{n \rightarrow \infty} \left| \frac{w_{n+1}}{w_n} \right| = r > 1, \quad (3.9)$$

The proof of this theorem is similar to real series and is based on Comparison Test and Geometric series. The complex form of geometric series is discussed in Section 3.2.

The Ratio Test and the Root Test were utilized in real series. Here we consider the complex forms of these two important tests. The proof of these two tests are similar to real series and are left as an exercise.

Ratio Test

Given the infinite complex series $\sum_{n=1}^{\infty} w_n = w_1 + w_2 + w_3 + \dots$, with

$$\lim_{n \rightarrow \infty} \left| \frac{w_{n+1}}{w_n} \right| = L, \quad \text{then} \quad (3.10)$$

- $\sum_{n=1}^{\infty} w_n$ converges absolutely if $L < 1$
- $\sum_{n=1}^{\infty} w_n$ diverges if $L > 1$
- No conclusion can be made if $L = 1$

Root Test

Given the infinite complex series $\sum_{n=1}^{\infty} w_n = w_1 + w_2 + w_3 + \dots$, with

$$\lim_{n \rightarrow \infty} \sqrt[n]{|w_n|} = L, \quad \text{then} \quad (3.11)$$

- a) $\sum_{n=1}^{\infty} w_n$ converges absolutely if $L < 1$
- b) $\sum_{n=1}^{\infty} w_n$ diverges if $L > 1$
- c) No conclusion can be made if $L = 1$

3.2 – Geometric and Power Series

Geometric Series

A complex geometric series is defined as

$$\sum_{n=0}^{\infty} aw^n = a + aw + aw^2 + aw^3 + \dots = a(1 + w + w^2 + w^3 + \dots) \quad (3.12)$$

Using equation (3.5), the nth partial sum of this is given by

$$S_n = a + aw + aw^2 + aw^3 + \dots + aw^n \quad (3.13)$$

Multiplying equation (3.13) by w results in

$$wS_n = aw + aw^2 + aw^3 + \dots + aw^{n+1} \quad (3.14)$$

Let us now consider $S_n - wS_n$

$$S_n - wS_n = a - aw^{n+1}$$

$$S_n(1 - w) = a(1 - w^{n+1})$$

$$S_n = a \frac{1 - w^{n+1}}{1 - w} \quad (3.15)$$

As $n \rightarrow \infty$, we can conclude

$$\sum_{n=0}^{\infty} aw^n = a + aw + aw^2 + aw^3 + \dots = \frac{a}{1-w} \quad \text{if} \quad |w| < 1$$

and

$$\sum_{n=0}^{\infty} aw^n = a + aw + aw^2 + aw^3 + \dots \quad \text{diverges} \quad \text{if} \quad |w| \geq 1 \quad (3.16)$$

Example 3: Evaluate the sum $\sum_{n=0}^{\infty} (az^{-1})^n$ and indicate the condition for convergence

$$\sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-(az^{-1})} = \frac{z}{z-a} \quad |az^{-1}| < 1 \quad \text{or} \quad |z| > |a|$$

Based on our discussion of sets and domains in Chapter 2, the condition for convergence in this example for $w = az^{-1}$ is $|az^{-1}| < 1$, which includes all the exterior points to a circle with center at $z = 0$ and radius $|a|$.

Power Series

A complex power series is defined as

$$\sum_{n=0}^{\infty} a_n (w - w_0)^n = a_0 + a_1(w - w_0) + a_2(w - w_0)^2 + \dots \quad (3.17)$$

Coefficients a_n , are complex constant and w_0 is also a complex constant in general and is called the center of the power series. In general power series has what is known as a region of convergence defined as interior point of the circle $|w - w_0| = R$. The circle is known as circle of convergence with center at w_0 and the radius is R . The radius R is referred to as radius of convergence.

It is obvious that for $w = w_0$, equation (3.17) converges to a_0 . The power series of equation (3.17) converges everywhere inside the circle of convergence defined by

$$|w - w_0| < R \quad (3.18)$$

and diverges outside the circle of convergence defined by

$$|w - w_0| > R \quad (3.19)$$

Radius of convergence R can be as small or as large as possible. The regions of convergence and divergence are show in Figure 3.3.

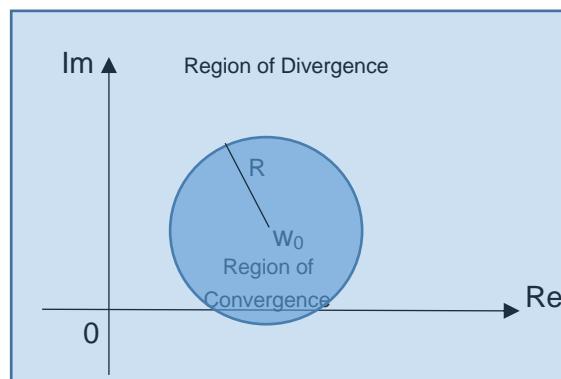


Figure 3.3 Regions of Convergence and Divergence

Evaluating Radius of Convergence

Using the ratio test of equation (3.10), we have

$$\left| \frac{w_{n+1}}{w_n} \right| = L \quad \text{as} \quad n \rightarrow \infty.$$

$$\left| \frac{a_{n+1}(w-w_0)^{n+1}}{a_n(w-w_0)^n} \right| = \left| \frac{a_{n+1}(w-w_0)}{a_n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |w - w_0| = L \quad \text{as} \quad n \rightarrow \infty.$$

Convergence requires $L < 1$. This implies $|w - w_0| < \frac{1}{\left| \frac{a_{n+1}}{a_n} \right|}$ or $|w - w_0| < \left| \frac{a_n}{a_{n+1}} \right|$.

Hence we write the radius of convergence as

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{or} \quad \frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad (3.20)$$

We can now consider two extreme cases.

$R = 0$ if $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow \infty$ as $n \rightarrow \infty \quad \Rightarrow \quad$ Power series converges only at w_0

$R = \infty$ if $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow 0$ as $n \rightarrow \infty \quad \Rightarrow \quad$ Power series converges for all values of w

As an alternate approach the root test of equation (3.11) can be used to evaluate radius of convergence.

$$\sqrt[n]{|a_n(w - w_0)^n|} = L \quad \text{as} \quad n \rightarrow \infty$$

Convergence requires $L < 1$. This implies $|w - w_0| < \frac{1}{\sqrt[n]{|a_n|}}$.

Hence we write the radius of convergence as

$$R = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{|a_n|}} \quad (3.21)$$

Example 4: Indicate the center and the radius of convergence for the given power series.

a) $\sum_{n=0}^{\infty} \frac{(z-2+i3)^n}{(3+i4)^n}$ b) $\sum_{n=0}^{\infty} \frac{(-2)^n(z-i5)^n}{(n+1)!}$

$$a) \sum_{n=0}^{\infty} \frac{(z-2+i3)^n}{(3+i4)^n}$$

The center is at $z = 2 - i3$. Using equation (3.20) \Rightarrow

$$R = \left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{1}{(3+i4)^n} / \frac{1}{(3+i4)^{n+1}} \right| = |3 + i4| = 5 \quad \text{as } n \rightarrow \infty$$

Hence radius of convergence $R = 5$.

We can Use equation (3.21) \Rightarrow

$$R = \frac{1}{\sqrt[n]{|a_n|}} = \frac{1}{\sqrt[n]{\frac{1}{|(3+i4)^n|}}} = |3 + i4| = 5 \quad \text{as } n \rightarrow \infty$$

$$b) \sum_{n=0}^{\infty} \frac{(-2)^n (z-i5)^n}{(n+1)!}$$

The center is at $z = i5$. Using equation (3.20) \Rightarrow

$$R = \left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{(-2)^n}{(n+1)!} / \frac{(-2)^{n+1}}{(n+2)!} \right| = \left| \frac{(n+2)!}{-2} \right| = \infty \quad \text{as } n \rightarrow \infty$$

Hence the series converges for all values of z .

3.3 – Taylor and Maclaurin Series

Taylor Series

Let us begin with power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots$$

then,

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \dots$$

$$f''(z) = 2a_2 + 6a_3(z - z_0) + 12a_4(z - z_0)^2 + \dots$$

$$f'''(z) = 6a_3 + 24a_4(z - z_0) + \dots$$

.....

We note, $f(z_0) = a_0$ $f'(z_0) = a_1$ $f''(z_0) = 2a_2$ $f'''(z_0) = 6a_3$ $f^{(n)}(z_0) = n!a_n \Rightarrow$

$$f(z) = f(z_0) + [f'(z_0)](z - z_0) + [f''(z_0)/2!](z - z_0)^2 + \dots + [f^{(n)}(z_0)/n!](z - z_0)^n + \dots \quad (3.22)$$

Equation (3.22) is known as Taylor series of a complex function $f(z)$. Using Cauchy's Integral Formula of equation (2.28) and Cauchy's Integral Formula for derivatives of equation (2.33), we have

$$a_0 = f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$$

$$a_1 = f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

.....

$$a_n = f^{(n)}(z_0)/n! = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad (3.23)$$

Theorem 3.5 Taylor's Theorem

Let $f(z)$ be an analytic function everywhere in a domain. Let path C be a circle in this domain with center at z_0 and radius r . Then $f(z)$ can be represented as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad (3.24)$$

Proof:

Using Cauchy Integral Formula, we can write

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$$

We now change some of the variables in Cauchy Integral Formula to have $f(z)$ on the left hand side of the equation without altering the actual formula.

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} dw \quad (3.25)$$

Denominator of the integrand in equation (3.25) can be modified as,

$$w - z = (w - z_0) - (z - z_0) = (w - z_0) \left[1 - \frac{z-z_0}{w-z_0} \right] \Rightarrow$$

$$\frac{1}{w-z} = \frac{1}{(w-z_0) \left[1 - \frac{z-z_0}{w-z_0} \right]} = \frac{1}{(w-z_0)} \frac{1}{\left[1 - \frac{z-z_0}{w-z_0} \right]} \quad (3.26)$$

Using equation (3.15) of geometric series we have

$$1 + \zeta + \zeta^2 + \zeta^3 + \dots + \zeta^n = \frac{1-\zeta^{n+1}}{1-\zeta} \Rightarrow$$

$$\frac{1}{1-\zeta} = 1 + \zeta + \zeta^2 + \zeta^3 + \dots + \zeta^n + \frac{\zeta^{n+1}}{1-\zeta} \quad (3.27)$$

Let us apply equation (3.27) to equation (3.26).

$$\begin{aligned} \frac{1}{(w-z)} &= \frac{1}{[1-\frac{z-z_0}{w-z_0}]} = \frac{1}{(w-z_0)} \left[1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{w-z_0}\right)^n \right] + \\ &\quad \frac{1}{(w-z)} \frac{\left(\frac{z-z_0}{w-z_0}\right)^{n+1}}{[1-\frac{z-z_0}{w-z_0}]} \\ \frac{1}{(w-z)} &= \frac{1}{(w-z_0)} \frac{1}{[1-\frac{z-z_0}{w-z_0}]} = \frac{1}{(w-z_0)} \left[1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{w-z_0}\right)^n \right] + \\ &\quad \left(\frac{1}{w-z}\right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} \end{aligned} \quad (3.28)$$

Substituting equation (3.28) in equation (3.25), we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z_0} \left\{ \left[1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{w-z_0}\right)^n \right] + \right. \\ &\quad \left. f(w) \left(\frac{1}{w-z}\right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} \right\} dw \end{aligned} \quad (3.29)$$

In equation (3.29), we define $R_n(z)$ as

$$R_n(z) = \frac{1}{2\pi i} \oint_C f(w) \left(\frac{1}{w-z}\right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} dw \quad (3.30)$$

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C f(w) \frac{1}{(w-z)} \frac{1}{(w-z_0)^{n+1}} dw \quad (3.31)$$

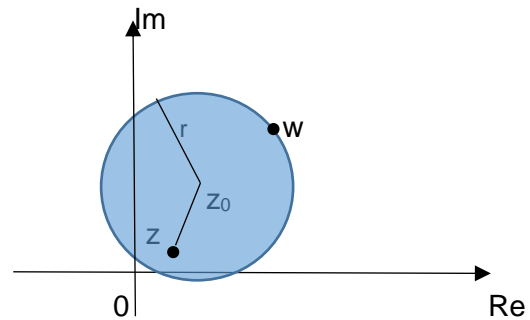


Figure 3.4

Referring to equation (3.30) and Figure 3.4, we have

$$|z - z_0| < |w - z_0| \quad \Rightarrow$$

$$\left| \frac{z-z_0}{w-z_0} \right|^{n+1} < 1.$$

With $|f(w)| < 1$ and length of curve C as $2\pi r$, we can use ML inequality.

$$|R_n(z)| = \left| \frac{1}{2\pi i} \oint_C f(w) \left(\frac{1}{w-z} \right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} dw \right| \leq \frac{1}{2\pi} M \frac{1}{|w-z|} \left| \frac{z-z_0}{w-z_0} \right|^{n+1} \cdot 2\pi r$$

As $n \rightarrow \infty$, the $\left| \frac{z-z_0}{w-z_0} \right|^{n+1} \rightarrow 0$ and as a result $R_n(z)$ approaches zero. Setting $R_n(z) = 0$ and using Cauchy's Integral formula for derivatives (Theorem 2.7), we obtain the Taylor series expansion of equation (3.24).

Maclaurin Series

Maclaurin series is a Taylor series with center at $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0) + \frac{f^{(1)}(0)}{1!} z + \frac{f^{(2)}(0)}{2!} z^2 + \frac{f^{(3)}(0)}{3!} z^3 + \dots \quad (3.32)$$

Example 5: Evaluate Maclaurin series for the given complex functions.

a) $f(z) = \frac{1}{1-z}$ b) $f(z) = e^z$ c) $f(z) = \cos z$ d) $f(z) = \sin z$
 e) $f(z) = \cosh z$ f) $f(z) = \sinh z$

a) $f(z) = \frac{1}{1-z} = (1-z)^{-1}$
 $f'(z) = (1-z)^{-2}$ $f''(z) = 2(1-z)^{-3}$ $f'''(z) = (3)(2)(1-z)^{-4}$
 $f(0) = 1, f'(0) = 1!, f''(0) = 2!, f'''(0) = 3!, \dots$ \Rightarrow
 $f(z) = 1 + z + z^2 + z^3 + \dots = \sum_{n=0}^{\infty} z^n$ Region of Convergence: $|z| < 1$

b) $f(z) = e^z$
 $f'(z) = e^z$ $f''(z) = e^z$ $f'''(z) = e^z$
 $f(0) = 1, f'(0) = 1, f''(0) = 1, f'''(0) = 1, \dots$ \Rightarrow
 $f(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

c) $f(z) = \cos z$
 $f'(z) = -\sin z$ $f''(z) = -\cos z$ $f'''(z) = \sin z$ $f^{(4)}(z) = \cos z$
 $f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0, f^{(4)}(0) = 1, \dots$ \Rightarrow
 $f(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

d) $f(z) = \sin z$
 $f'(z) = \cos z \quad f''(z) = -\sin z \quad f'''(z) = -\cos z \quad f''''(z) = \sin z \quad \dots$
 $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f''''(0) = 0, \dots \Rightarrow$
 $f(z) = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

e) $f(z) = \cosh z$
 $f'(z) = \sinh z \quad f''(z) = \cosh z \quad f'''(z) = \sinh z \quad f''''(z) = \cosh z \quad \dots$
 $f(0) = 1, f'(0) = 0, f''(0) = 1, f'''(0) = 0, f''''(0) = 1, \dots \Rightarrow$
 $f(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$

f) $f(z) = \sinh z$
 $f'(z) = \cosh z \quad f''(z) = \sinh z \quad f'''(z) = \cosh z \quad f''''(z) = \sinh z \dots$
 $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 1, f''''(0) = 0, \dots \Rightarrow$
 $f(z) = \frac{z}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

3.4 – Laurent Series and Residue

Laurent Series

There are many applications where power series expansion of $f(z)$ may have a center at which $f(z)$ be singular. If functions $f(z)$ ceases to be analytic at a point z_0 , then z_0 is referred to as a point of singularity. In such cases Taylor Series expansion does not apply. Instead a different series known as Laurent series is utilized.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n =$$

$$\dots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

It is noted that Laurent series is an extension of Taylor series which includes negative powers of $(z - z_0)$.

Theorem 3.6 Laurent's Theorem

Let $f(z)$ be an analytic function everywhere in a closed domain D as shown in Figure 3.5. The function $f(z)$ can be represented by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n =$$

$$\dots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \quad (3.33)$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad (3.34)$$

C is any closed curve which is completely in D and in counterclockwise direction.

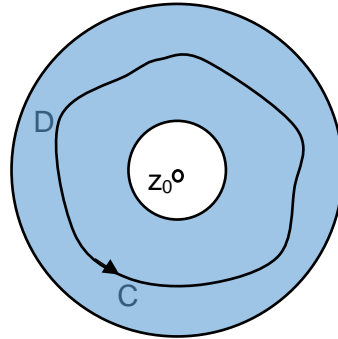


Figure 3.5 Domain D and Point of Singularity z_0

Proof:

Let us consider two concentric circles C_1 and C_3 with center at z_0 and radii r_1 and r_2 as shown in Figure 3.6. We also consider cuts between C_1 and C_3 as we did for the case of doubly connected domain (Figure 2.24). Based on Cauchy's Integral formula, we have

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i} \oint_{C_3} \frac{f(w)}{w-z} dw$$

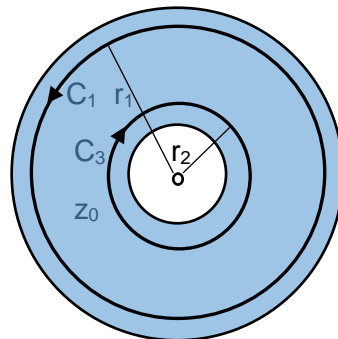


Figure 3.6 Laurent Series-Appling Cauchy's Integral Formula

With curve C_2 the same as C_3 but in opposite direction (counterclockwise), we have

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{z-w} dw \quad (3.35)$$

Following similar procedure as for the case of Taylor's Theorem and using geometric series of equation (3.27), for the first integral we write

$$\frac{1}{(w-z)} = \frac{1}{(w-z_0)} \frac{1}{[1 - \frac{z-z_0}{w-z_0}]} = \frac{1}{(w-z_0)} \left[1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{w-z_0}\right)^n \right] + \left(\frac{1}{w-z}\right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} \quad (3.36)$$

and for the second integral

$$\frac{1}{(z-w)} = \frac{1}{(z-z_0)} \frac{1}{[1 - \frac{w-z_0}{z-z_0}]} = \frac{1}{(z-z_0)} \left[1 + \frac{w-z_0}{z-z_0} + \left(\frac{w-z_0}{z-z_0}\right)^2 + \dots + \left(\frac{w-z_0}{z-z_0}\right)^n \right] + \left(\frac{1}{z-w}\right) \frac{(w-z_0)^{n+1}}{(z-z_0)^{n+1}} \quad (3.37)$$

Substituting equations (3.36) and (3.37) in equation (3.35) results in

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z_0} \left\{ \left[1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{w-z_0}\right)^n \right] + f(w) \left(\frac{1}{w-z}\right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} \right\} dw + \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{z-z_0} \left\{ \left[1 + \frac{w-z_0}{z-z_0} + \left(\frac{w-z_0}{z-z_0}\right)^2 + \dots + \left(\frac{w-z_0}{z-z_0}\right)^n \right] + f(w) \left(\frac{1}{z-w}\right) \frac{(w-z_0)^{n+1}}{(z-z_0)^{n+1}} \right\} dw \quad (3.38)$$

In $f(z)$ above, we define $R_{n1}(z)$ and $R_{n2}(z)$ as

$$R_{n1}(z) = \frac{1}{2\pi i} \oint_{C_1} f(w) \left(\frac{1}{w-z}\right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} dw \quad (3.39)$$

$$R_{n1}(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_{C_1} f(w) \frac{1}{(w-z)} \frac{1}{(w-z_0)^{n+1}} dw \quad (3.40)$$

$$R_{n2}(z) = \frac{1}{2\pi i} \oint_{C_2} f(w) \left(\frac{1}{z-w}\right) \frac{(w-z_0)^{n+1}}{(z-z_0)^{n+1}} dw \quad (3.41)$$

$$R_{n2}(z) = \frac{1}{2\pi i (z-z_0)^{n+1}} \oint_{C_2} f(w) \frac{(w-z_0)^{n+1}}{(z-w)} dw \quad (3.42)$$

Referring to equation (3.39) and Figure 3.5 with z located between concentric circles and w on curve C_1 , we have

$$|z - z_0| < |w - z_0| \quad \Rightarrow \quad \left| \frac{z-z_0}{w-z_0} \right|^{n+1} < 1.$$

With $|f(w)| < M$ and length of curve C_1 as $2\pi r_1$, we can use ML inequality.

$$|R_{n1}(z)| = \left| \frac{1}{2\pi i} \oint_{C_1} f(w) \left(\frac{1}{w-z} \right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} dw \right| \leq \frac{1}{2\pi} M \frac{1}{|w-z|} \left| \frac{z-z_0}{w-z_0} \right|^{n+1} \cdot 2\pi r_1$$

As $n \rightarrow \infty$, the $\left| \frac{z-z_0}{w-z_0} \right|^{n+1} \rightarrow 0$ and as a result $R_{n1}(z)$ approaches zero.

Referring to equation (3.41) and Figure 3.5 with z located between concentric circles and w on curve C_2 , we have

$$|w - z_0| < |z - z_0| \quad \Rightarrow$$

$$\left| \frac{w-z_0}{z-z_0} \right|^{n+1} < 1.$$

With $|f(w)| < M$ and length of curve C_2 as $2\pi r_2$, we can use ML inequality.

$$|R_{n1}(z)| = \left| \frac{1}{2\pi i} \oint_{C_2} f(w) \left(\frac{1}{w-z} \right) \frac{(z-z_0)^{n+1}}{(w-z_0)^{n+1}} dw \right| \leq \frac{1}{2\pi} M \frac{1}{|z-w|} \left| \frac{w-z_0}{z-z_0} \right|^{n+1} \cdot 2\pi r_2$$

As $n \rightarrow \infty$, the $\left| \frac{w-z_0}{z-z_0} \right|^{n+1} \rightarrow 0$ and as a result $R_{n2}(z)$ approaches zero.

Hence from equation (3.38) we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z_0} \left[1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0} \right)^2 + \dots + \left(\frac{z-z_0}{w-z_0} \right)^n \right] dw \\ &+ \frac{1}{2\pi i} \oint_C \frac{f(w)}{z-z_0} \left[1 + \frac{w-z_0}{z-z_0} + \left(\frac{w-z_0}{z-z_0} \right)^2 + \dots + \left(\frac{w-z_0}{z-z_0} \right)^n \right] dw \end{aligned} \quad (3.43)$$

This proves the Laurent's theorem. Furthermore since $f(z)$ is analytic in domain D everywhere between the concentric circles C_1 and C_2 , these curves can be replaced by a closed counterclockwise curve C as shown in Figure 3.5.

It is important to note that equation (3.34) is hardly used to evaluate the coefficient of Laurent series. We usually acquire Laurent series by employing other series for which we are familiar with their power series expansion. Furthermore a given function $f(z)$ can be represented by different Laurent series depending on how the region of convergence is defined for that particular $f(z)$. This is illustrated in the following examples.

Example 6: Evaluate Laurent series for the given complex functions.

$$\text{a) } f(z) = \frac{z + \cos z}{z^2} \quad \text{b) } f(z) = z e^{2/z} \quad \text{c) } f(z) = \frac{-z+26}{z^2+2z-8}$$

$$\begin{aligned} \text{a) } f(z) &= \frac{z + \cos z}{z^2} \\ f(z) &= \frac{1}{z^2} \left(z + 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \right) = \frac{1}{z} + \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} + \dots \quad |z| > 0 \end{aligned}$$

$$b) f(z) = Ze^{2/Z} = Z \left[1 + \frac{\left(\frac{2}{Z}\right)}{1!} + \frac{\left(\frac{4}{Z^2}\right)}{2!} + \frac{\left(\frac{8}{Z^3}\right)}{3!} + \dots \right] = Z + 2 + \frac{4}{2!Z} + \frac{8}{3!Z^2} + \dots \quad |z| > 0$$

$$c) f(z) = \frac{-Z+26}{Z^2+2Z-8} \quad \text{applying partial fractions}$$

$$f(z) = \frac{4}{Z-2} - \frac{5}{Z+4}$$

We can consider variety of regions of convergence for which Laurent series can be represented.

$$i) \quad 2 < |z| < 4$$

$$\frac{4}{Z-2} = \frac{4}{Z(1-\frac{2}{Z})} = \frac{4}{Z} \left[1 + \left(\frac{2}{Z}\right) + \left(\frac{2}{Z}\right)^2 + \left(\frac{2}{Z}\right)^3 + \dots \right] = \frac{4}{Z} + \frac{8}{Z^2} + \frac{16}{Z^3} + \dots$$

$$\frac{-5}{Z+4} = \frac{-5}{4(1+\frac{Z}{4})} = \frac{-5}{4} \left[1 - \left(\frac{Z}{4}\right) + \left(\frac{Z}{4}\right)^2 - \left(\frac{Z}{4}\right)^3 + \dots \right] = -\frac{5}{4} + \frac{5Z}{16} - \frac{5Z^2}{64} + \dots$$

$$f(z) = \dots + \frac{16}{Z^3} + \frac{8}{Z^2} + \frac{4}{Z} - \frac{5}{4} + \frac{5Z}{16} - \frac{5Z^2}{64} + \dots$$

$$ii) \quad |z| < 2$$

$$\frac{4}{Z-2} = \frac{4}{-2(1-\frac{Z}{2})} = -2 \left[1 + \left(\frac{Z}{2}\right) + \left(\frac{Z}{2}\right)^2 + \left(\frac{Z}{2}\right)^3 + \dots \right] = -2 - Z - \frac{Z^2}{2} + \dots$$

$$\frac{-5}{Z+4} = \text{same as (i)}$$

$$f(z) = -2 - Z - \frac{Z^2}{2} + \dots - \frac{5}{4} + \frac{5Z}{16} - \frac{5Z^2}{64} + \dots = -\frac{13}{4} - \frac{11Z}{16} - \frac{37Z^2}{64} + \dots$$

$$iii) \quad |z| > 4$$

$$\frac{4}{Z-2} = \text{same as (i)}$$

$$\frac{-5}{Z+4} = \frac{-5}{Z(1+\frac{4}{Z})} = -\frac{5}{Z} \left[1 - \left(\frac{4}{Z}\right) + \left(\frac{4}{Z}\right)^2 - \left(\frac{4}{Z}\right)^3 + \dots \right] = -\frac{5}{Z} + \frac{20}{Z^2} - \frac{80}{Z^3} + \dots$$

$$f(z) = \frac{4}{Z} + \frac{8}{Z^2} + \frac{16}{Z^3} + \dots - \frac{5}{Z} + \frac{20}{Z^2} - \frac{80}{Z^3} + \dots = -\frac{1}{Z} + \frac{28}{Z^2} - \frac{64}{Z^3} + \dots$$

Example 7: Evaluate Laurent series for $f(z) = \frac{1}{1-z}$ in the annulus $0 < |z+1| < 2$.

$$f(z) = \frac{1}{1-z} = \frac{1}{2-(z+1)} = \frac{1}{2[1-\frac{(z+1)}{2}]} = \frac{1}{2} \left[1 + \left(\frac{z+1}{2}\right) + \left(\frac{z+1}{2}\right)^2 + \dots \right]$$

Residue

We discussed Laurent series expansion in the previous section.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \sum_{n=-1}^{-\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} a_n (z-z_0)^n = \dots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

The coefficient of Laurent series is given as

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz. \text{ We also note}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz \quad (3.44)$$

The coefficient a_{-1} is defined as the residue of $f(z)$ at the point of singularity z_0 . We also note residue of $f(z)$ is simply the coefficient of $\frac{1}{z-z_0}$ in Laurent series. The importance of residue should be clear from equation (3.44) and will be fully utilized.

3.5 – Poles and Zeros

Points of Singularity and Poles of $f(z)$

We saw in Section 3.4 that when a function $f(z)$ ceases to be analytic at a point z_0 , then z_0 is referred to as a point of singularity and we say $f(z)$ is singular at z_0 . If $f(z)$ has a singular point at z_0 , but there are no other singular points of $f(z)$ in some neighborhood of z_0 , then z_0 is known as an isolated singular point. Essential singularity is when Laurent series contains infinite negative powers of $z - z_0$. When Laurent series contains a finite number of negative powers of $z - z_0$ as shown below, then $z = z_0$ is defined as the pole of $f(z)$.

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \quad (3.45)$$

If $a_{-m} \neq 0$ in equation (3.45), the pole at z_0 is said to be of order m or repeated m times. A simple pole or a pole of order 1 is when all the coefficients of negative powers of $(z - z_0)$ in a Laurent series are zero, except a_{-1} as shown in equation (3.46).

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \quad (3.46)$$

Zeros of $f(z)$

The function $f(z)$ is said to have a zero at z_0 , if $f(z_0) = 0$. Similar to poles, zeros can be simple or of order 1 or repeated m times or of order m . A function $f(z)$ has a zero of order m at $z = z_0$ if

$$f(z_0) = 0, \quad f^{(1)}(z_0) = 0, \quad f^{(2)}(z_0) = 0, \dots, f^{(m-1)}(z_0) = 0, \text{ but } f^{(m)}(z_0) \neq 0$$

The poles and zeros have many different applications in science and engineering. Some such examples include Control Systems, Robotics, Filter Design, etc.

Example 8: Evaluate all the poles and the zeros of $H(z)$ and indicate the order of each.

$$\text{a) } H(z) = \frac{(Z^2+4)(Z-1)^2}{Z^2(Z^2+6Z+25)^3(Z+5)} \quad \text{b) } H(z) = \frac{Z-1}{e^Z - 1}$$

$$\text{a) } H(z) = \frac{(Z^2+4)(Z-1)^2}{Z^2(Z^2+6Z+25)^3(Z+5)}$$

Zeros: $\pm i2$ (order 1 or simple), 1 (order 2)

Poles: 0 (order 2), $-3 \pm i4$ (order 3), -5 (order 1 or simple)

$$\text{b) } H(z) = \frac{Z-1}{e^Z - 1}$$

Zeros: 1 (order 1 or simple)

Poles: $\pm i2n\pi$ (order 1 or simple)

3.6 – Evaluation of Residue

In Section 3.4 we saw Laurent series as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=-\infty}^{-1} a_n(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^n = \dots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

and defined residue of $f(z)$ at z_0 as

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz$$

We also noted residue of $f(z)$ to be the coefficient of $\frac{1}{z-z_0}$. To evaluate residue of $f(z)$, we must consider two cases. In Case 1, we consider $f(z)$ to have a simple pole at z_0 . In Case 2, we consider $f(z)$ to have a repeated pole of order m at z_0 . In either case the objective is to evaluate a_{-1} .

Case 1 – Residue of $f(z)$ with a Simple Pole

Using Laurent series, we have

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \quad (3.47)$$

Multiplying both sides of equation (3.47) by $(z-z_0)$ results in

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + \dots$$

After substituting $z = z_0$ on both sides, we have

$$\text{Res } f(z)|_{z=z_0} = \lim_{Z \rightarrow Z_0} (Z - Z_0)f(z) \quad (3.48)$$

This process is not so much different from partial fractions. We now consider an alternate method of evaluation residue of $f(z)$ at z_0 .

An Alternate Approach:

Consider $f(z)$ represented as $f(Z) = \frac{N(Z)}{D(Z)}$. Consider expanding $D(z)$ in Taylor series at z_0 .

$$f(Z) = \frac{N(Z)}{D(Z)} = \frac{N(Z)}{D(Z_0) + (Z - Z_0)D'(Z_0) + \left(\frac{1}{2!}\right)(Z - Z_0)^2 D''(Z_0) + \dots}$$

Since $f(z)$ has a simple pole at $z = z_0$, this implies $D(z_0) = 0$. Applying equation (3.48), we write

$$\begin{aligned} \text{Res } f(z)|_{z=z_0} &= \lim_{Z \rightarrow Z_0} (Z - Z_0) \frac{N(Z)}{(Z - Z_0)D'(Z_0) + \left(\frac{1}{2!}\right)(Z - Z_0)^2 D''(Z_0) + \dots} \\ &= \lim_{Z \rightarrow Z_0} \frac{N(Z)}{D'(Z_0) + \left(\frac{1}{2!}\right)(Z - Z_0)D''(Z_0) + \dots} \end{aligned}$$

$$\text{Res } f(z)|_{z=z_0} = \frac{N(Z_0)}{D'(Z_0)} \quad (3.49)$$

Equations (3.48) or (3.49) can be used to evaluate the residue of $f(z)$ at z_0 . The preference of using one or the other has to do with how the denominator is represented. If the denominator is already in factored form, perhaps equation (3.48) is easier to use. However if denominator is not written in factored form, but all the simple poles are known, perhaps equation (3.49) should be utilized. For example if $D(z) = z^4 - 1$ with roots as $-1, 1, -i,$ and i either of the equations can be used in finding the residues. However if $D(z) = e^z - 1$ (with roots as $\pm i2n\pi$), using equation (3.49) should be considered.

Case 2 – Residue of $f(z)$ with a Repeated Pole of Order m

Using Laurent series, we have

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-(m-1)}}{(z-z_0)^{(m-1)}} + \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \quad (3.50)$$

Multiplying both sides of equation (3.50) by $(z - z_0)^m$ results in

$$(z - z_0)^m f(z) = a_{-m} + a_{-(m-1)}(z - z_0) + \dots + a_{-2}(z - z_0)^{m-2} + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + a_1(z - z_0)^{m+1} + a_2(z - z_0)^{m+2} + \dots$$

If we substitute $z = z_0$ at this stage, we obtain a_{-m} . To evaluate a_{-1} we take the derivative of both sides with respect to z , $(m-1)$ times and then let $z = z_0$ on both sides. All the terms prior to a_{-1} vanish and all the terms after a_{-1} contain $(z - z_0)$ to a power 1 or higher and as a result are eliminated when $z = z_0$ is substituted. Hence we have

$$\frac{d^{(m-1)}}{dz^{m-1}}(z - z_0)^m f(z)|_{z=z_0} = (m - 1)! a_{-1} = (m - 1)! \text{Res } f(z)|_{z=z_0} \quad \text{or}$$

$$\text{Res } f(z)|_{z=z_0} = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{(m-1)}}{dz^{m-1}} [(z - z_0)^m f(z)] \quad (3.51)$$

Example 9: Evaluate the residues at all the poles of $f(z)$.

$$\text{a) } f(z) = \frac{(z^2+4)}{z(z^2+1)} \quad \text{b) } f(z) = \frac{(z+1)^2}{z(z^2-4z+4)}$$

$$\text{a) } f(z) = \frac{(z^2+4)}{z(z^2+1)} = \frac{(z^2+4)}{z(z-i)(z+i)} = \frac{(z^2+4)}{(z^3+z)}$$

Poles: 0 (order 1), $-i$ (order 1), i (order 1)

Using equation (3.48)

$$\text{Res } f(z)|_{z=0} = \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} \frac{(z^2+4)}{(z^2+1)} = 4$$

$$\text{Res } f(z)|_{z=i} = \lim_{z \rightarrow i} (z - i) f(z) = \lim_{z \rightarrow i} \frac{(z^2+4)}{z(z+i)} = -\frac{3}{2}$$

$$\text{Res } f(z)|_{z=-i} = \lim_{z \rightarrow -i} (z + i) f(z) = \lim_{z \rightarrow -i} \frac{(z^2+4)}{z(z-i)} = -\frac{3}{2}$$

Using equation (3.49)

$$\frac{N(z)}{D'(z)} = \frac{(z^2+4)}{(3z^2+1)}$$

$$\text{Res } f(z)|_{z=0} = \frac{N(0)}{D'(0)} = 4$$

$$\text{Res } f(z)|_{z=i} = \frac{N(i)}{D'(i)} = -\frac{3}{2}$$

$$\text{Res } f(z)|_{z=-i} = \frac{N(-i)}{D'(-i)} = -\frac{3}{2}$$

$$\text{b) } f(z) = \frac{(z+1)^2}{z(z^2-4z+4)} = \frac{(z+1)^2}{z(z-2)^2}$$

Poles: 0 (order 1), 2 (order 2)

Using equation (3.48)

$$\text{Res } f(z)|_{z=0} = \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} \frac{(z+1)^2}{(z-2)^2} = \frac{1}{4}$$

Using equation (3.51)

$$\begin{aligned} \text{Res } f(z)|_{z=2} &= \frac{1}{(2-1)!} \lim_{z \rightarrow 2} \frac{d}{dz} [(z-2)^2 f(z)] = \lim_{z \rightarrow 2} \frac{d}{dz} \left[\frac{(z+1)^2}{z} \right] \\ &= \lim_{z \rightarrow 2} \left[\frac{2z(z+1) - (z+1)^2}{z^2} \right] = \frac{3}{4} \end{aligned}$$

Example 10: Evaluate residue of $f(z) = \frac{z^2+2z+10}{e^z-1}$ at $z = 0$.

Using equation (3.49)

$$\frac{N(z)}{D'(z)} = \frac{z^2+2z+10}{e^z}$$

$$\text{Res } f(z)|_{z=0} = \frac{N(0)}{D'(0)} = 10$$

3.7 – Residue Theorem

Theorem 3.7 Residue Theorem

Let C be a simple closed path in counterclockwise direction entirely in a simply closed domain D . If $f(z)$ is analytic everywhere on and inside C except at a finite number of points of singularities at z_1, z_2, \dots, z_m all located inside C , then

$$\oint_C f(z) dz = 2\pi i \sum_{l=1}^m \text{Res } f(z)|_{z_l} \quad (3.52)$$

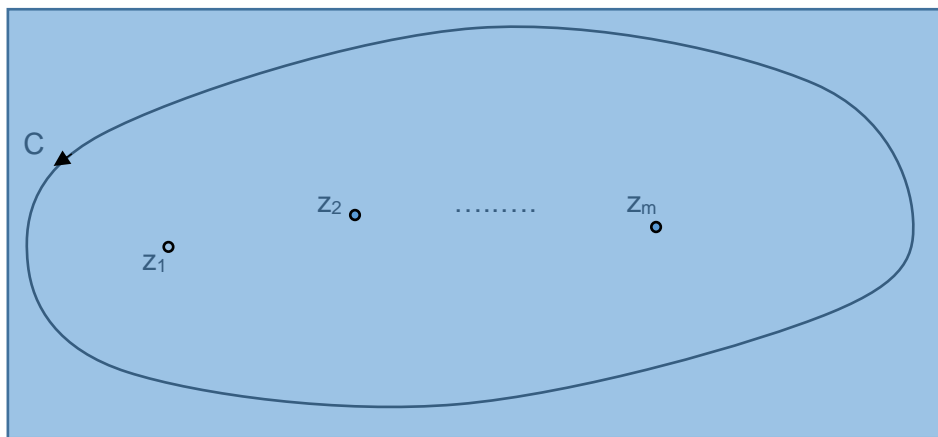


Figure 3.7 Closed Path Containing Singularity Points z_1, z_2, \dots, z_m

Proof:

Let us enclose each point of singularity Z_l with a circle C_l^* as shown in Figure 3.8. The radius of each circle is selected small enough such that all the m circles are inside curve C and are completely separated. These m circles (C_1^* , C_2^* , ..., C_m^*) are in clockwise direction as shown. Let us now consider a cut from curve C to each of these m circles in both directions.

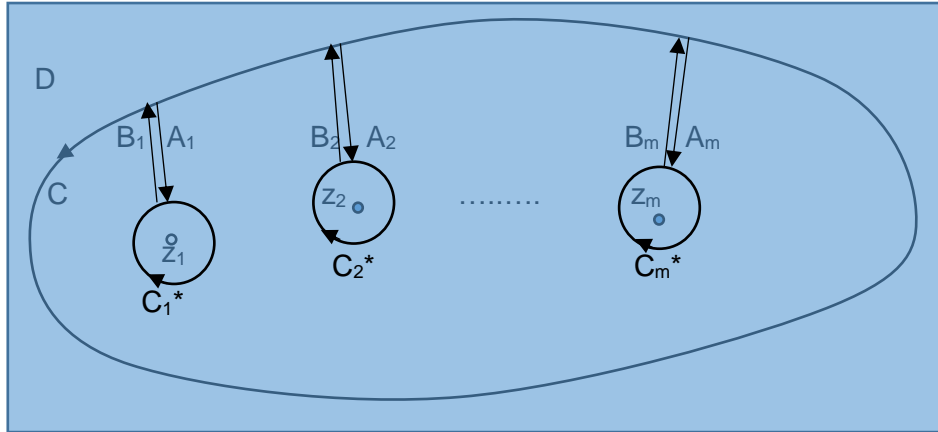


Figure 3.8 Closed Path C Containing Singularity Points z_1, z_2, \dots, z_m

The function $f(z)$ is analytic everywhere in the multiply connected region which bounded by C , $A_m, C_m^*, B_m, \dots, A_2, C_2^*, B_2, A_1, C_1^*, B_1$. Applying Cauchy's Integral theorem and considering the line integrals due to the cuts cancel each out, we have

$$\oint_C f(z)dz + \oint_{C_m^*} f(z)dz + \dots + \oint_{C_2^*} f(z)dz + \oint_{C_1^*} f(z)dz = 0 \quad \Rightarrow$$

$$\oint_C f(z)dz = -\oint_{C_m^*} f(z)dz - \dots - \oint_{C_2^*} f(z)dz - \oint_{C_1^*} f(z)dz \quad \Rightarrow$$

$$\oint_C f(z)dz = \oint_{C_m} f(z)dz + \dots + \oint_{C_2} f(z)dz + \oint_{C_1} f(z)dz \quad (3.53)$$

In equation (3.53), C_m, \dots, C_2 , and C_1 are the same as C_m^*, \dots, C_2^* , and C_1^* respectively but in counterclockwise direction. Using Laurent series and results obtained in Section 3.5, we write

$$\oint_C f(z)dz = 2\pi i \operatorname{Res} f(z)|_{z=z_m} + \dots + 2\pi i \operatorname{Res} f(z)|_{z=z_2} + 2\pi i \operatorname{Res} f(z)|_{z=z_1}$$

$$\oint_C f(z)dz = 2\pi i \sum_{l=1}^m \operatorname{Res} f(z)|_{z=z_l}$$

Example 11: Evaluate the following integrals.

a) $\oint_C \frac{z^2+4z+5}{(z-3)^2(z^2+16)} dz$ C: $|z - 1| = 1.5$ Counterclockwise

b) $\oint_C \frac{z+2}{z^2(z-5)} dz$ C: $|z - i| = 1.5$ Counterclockwise

c) $\oint_C \frac{10}{z(z^3-8)} dz$ C: $|z - 1| = 1.5$ Counterclockwise

d) $\oint_C \frac{\cos z}{z^3} dz$ C: $|z| = 5$ Clockwise

a) $\oint_C \frac{z^2+4z+5}{(z-3)^2(z^2+16)} dz = 0$

poles at $z = 3, 3, -i4, i4$ are all outside C

b) $\oint_C \frac{z+2}{z^2(z-5)} dz$

pole of order 2 at $z=0$ is inside C and simple pole at $z = 5$ is outside C

$$\begin{aligned} \oint_C \frac{z+2}{z^2(z-5)} dz &= 2\pi i \operatorname{Res}f(z)|_{z=0} \\ &= 2\pi i \frac{1}{(2-1)!} \lim_{z \rightarrow 0} \frac{d}{dz} \left[z^2 \frac{z+2}{z^2(z-5)} \right] = 2\pi i \left(\frac{-7}{25} \right) = \frac{-14\pi i}{25} \end{aligned}$$

c) $\oint_C \frac{10}{z(z^3-8)} dz$ $z(z^3 - 8) = 0 \Rightarrow$ simple poles at: $z = 0, 2, 2e^{i2\pi/3}, 2e^{i4\pi/3}$

simple poles at $z = 0, 2$ are inside C and simple poles at $z = 2e^{i2\pi/3}$ and $2e^{i4\pi/3}$ are outside C

$$\begin{aligned} \oint_C \frac{10}{z(z^3-8)} dz &= 2\pi i \{ [\operatorname{Res}f(z)|_{z=0} + [\operatorname{Res}f(z)|_{z=2}] \} \\ &= 2\pi i \left[\lim_{z \rightarrow 0} \frac{10}{4z^3-8} + \lim_{z \rightarrow 2} \frac{10}{4z^3-8} \right] = 2\pi i \left(-\frac{5}{4} + \frac{5}{12} \right) = -\frac{i5\pi}{3} \end{aligned}$$

d) $\oint_C \frac{\cos z}{z^3} dz$

pole of order 3 at $z=0$ is inside C.

$$\oint_C \frac{\cos z}{z^3} dz = -2\pi i [\operatorname{Res}f(z)|_{z=0}] = -2\pi i \frac{1}{(3-1)!} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \left[z^3 \frac{\cos z}{z^3} \right] = -2\pi i \left(-\frac{1}{2} \right) = \pi i$$

one should note, $\frac{\cos z}{z^3} = \frac{1}{z^3} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \right) \Rightarrow (\operatorname{Res}f(z)|_{z=0}) = -\frac{1}{2!} = -\frac{1}{2}$

3.8 – Applications of Residue Theorem to Real Integrals

In this section, we consider a few applications of Residue Theorem involving real integrals.

I - Integral of a rational function of $\cos \theta$ and $\sin \theta$ from 0 to 2π

Consider the following integral

$$\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta \quad (3.54)$$

The integrand is a rational function of $\cos \theta$ and $\sin \theta$ and the lower and upper limits of integration are from 0 to 2π respectively. To apply Residue Theorem, we define z as

$$z = e^{i\theta} \quad (3.55)$$

We note that as θ is varied from 0 to 2π , the variable z as defined by equation (3.55) represents one complete revolution of a closed circle of radius 1 with center at the origin starting at $z = 1$. Furthermore $\cos \theta$, $\sin \theta$, and $d\theta$ in equation (3.54) can be written as

$$\begin{aligned} \cos \theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(Z + Z^{-1}) \\ \sin \theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(Z - Z^{-1}) \\ d\theta &= dZ/ie^{i\theta} = dZ/iZ \end{aligned} \quad (3.56)$$

Substituting equation (3.56) in equation (3.54) results in

$$\oint_C f(z) dz \quad \text{C: } |Z| = 1 \text{ in counterclockwise direction}$$

Example 12: Evaluate the following integrals.

$$\begin{aligned} &\int_0^{2\pi} \frac{d\theta}{10-6\cos\theta} \\ &\int_0^{2\pi} \frac{d\theta}{10-6\cos\theta} = \oint_C \frac{1}{10-3(Z+Z^{-1})iZ} \frac{1}{iZ} dz = \frac{1}{-i} \oint_C \frac{1}{3Z^2-10Z+3} dz = \frac{1}{-i3} \oint_C \frac{1}{Z^2-(\frac{10}{3})Z+1} dz \\ &Z^2 - \left(\frac{10}{3}\right)Z + 1 = 0 \quad \Rightarrow \quad \text{simple poles at: } z = 1/3, 3 \\ &\int_0^{2\pi} \frac{d\theta}{10-6\cos\theta} = \frac{1}{-i3} 2\pi i [\text{Res}f(z)|_{z=\frac{1}{3}}] = \frac{-2\pi}{3} \left[\lim_{Z \rightarrow \frac{1}{3}} \frac{1}{2Z - \frac{10}{3}} \right] = \frac{\pi}{4} \end{aligned}$$

II – Improper integral of a rational function $f(x)$ with degree of denominator at least 2 more than the degree of numerator. Furthermore denominator of $f(x)$ contains no real roots.

Consider the following integral

$$\int_{-\infty}^{\infty} f(x) dx \quad (3.57)$$

The integrand is a rational function x and the lower and upper limits of integration are from $-\infty$ to ∞ respectively. To apply Residue Theorem, consider the following integral with closed path C consisting of real axis from $-r$ to $+r$ and upper semicircle with radius r (curve C_1) as shown in Figure 3.8. We consider $r \rightarrow \infty$ and as a result path C encloses the entire upper half plane.

$$\begin{aligned} \oint_C f(z) dz &= \int_{C_1} f(z) dz + \lim_{r \rightarrow \infty} \int_{-r}^r f(z) dz \\ &= 2\pi i \sum \text{Res } f(z) | @ \text{poles in upper half plane} \end{aligned} \quad (3.58)$$

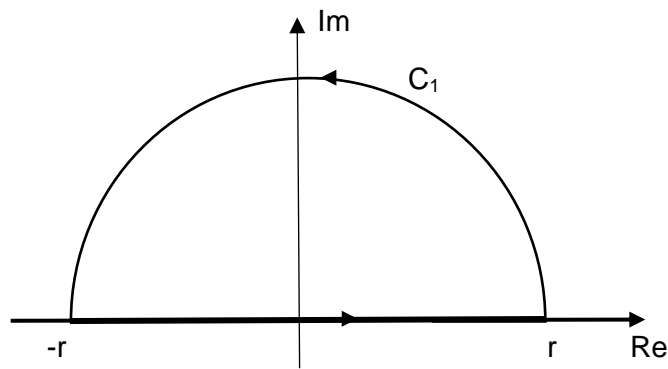


Figure 3.8 Closed Path C

Referring to the integral on semicircle portion of equation (3.58), we have

$$|f(z)| < \frac{M_1}{r^2}$$

Since degree of denominator is at least 2 more than the degree of numerator and M_1 is a constant. We also have

$$L = \text{length of semicircle} = \pi r$$

Using ML inequality, we have

$$\left| \int_{C_1} f(z) dz \right| < \frac{M_1}{r^2} \cdot \pi r$$

As $r \rightarrow \infty$, this integral approaches zero. This implies

$$\lim_{r \rightarrow \infty} \int_{-r}^r f(z) dz = \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum \text{Res } f(z) | @ \text{poles in upper half plane} \quad (3.59)$$

Example 13: Evaluate the following integrals.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+2x+2)(x^2+4)^2} \quad \Rightarrow \quad f(z) = \frac{1}{(z^2+2z+2)(z^2+4)^2}$$

Poles in the upper half plane: $-1 + i$ (simple), $i2$ (order 2)

$$\text{Res } f(z) |_{z=-1+i} = \lim_{z \rightarrow -1+i} (z+1-i)f(z) = \lim_{z \rightarrow -1+i} \frac{1}{(z+1+i)(z^2+4)^2} = \frac{1}{(32+i24)}$$

$$\text{Res } f(z) |_{z=i2} = \frac{1}{(2-1)!} \lim_{z \rightarrow i2} \frac{d}{dz} [(z-i2)^2 f(z)] = \lim_{z \rightarrow i2} \frac{d}{dz} \left[\frac{1}{(z^2+2z+2)(z+i2)^2} \right]$$

$$= - \lim_{z \rightarrow i2} \left[\frac{(2z+2)(z+i2)+2(z^2+2z+2)}{(z^2+2z+2)^2 (z+i2)^3} \right] = \frac{20-i16}{(64)(-16+i12)}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+2x+2)(x^2+4)^2} = 2\pi i \left[\frac{1}{(32+i24)} + \frac{20-i16}{(64)(-16+i12)} \right] = \frac{23\pi}{800}$$

III – Improper integral of functions $f(x) \cos \omega x$ and $f(x) \sin \omega x$, where $f(x)$ is a rational function with degree of denominator at least 2 more than the degree of numerator. Furthermore denominator of $f(x)$ contains no real roots.

Consider the following integrals

$$\int_{-\infty}^{\infty} f(x) \cos \omega x \, dx \quad (3.60)$$

$$\int_{-\infty}^{\infty} f(x) \sin \omega x \, dx$$

To simplify the problem, let us consider the following integral that contains both integrals of equation (3.60) as noted below.

$$\int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx = \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx + i \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx \quad (3.61)$$

One may note that equation (3.61) represents Fourier Transform. We follow the same exact procedure as the previous case except replacing $f(z)$ by $f(z) e^{i\omega z}$. Furthermore $|e^{i\omega z}| \leq 1$ and using ML inequality, we have

$$| \int_{C_1} f(z) e^{i\omega z} \, dz | < \frac{M_1}{r^2} (1) \cdot \pi r$$

As $r \rightarrow \infty$, this integral approaches zero and we have

$$\lim_{r \rightarrow \infty} \int_{-r}^r f(z) e^{i\omega z} \, dz = \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx$$

$$= 2\pi i \sum \text{Res } f(z)e^{i\omega z} | @ \text{poles in upper half plane} \quad (3.62)$$

Using Euler's equation and equating both sides of equation (3.62) results in

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx &= -2\pi \sum \text{Im}[\text{Res } f(z)e^{i\omega z} | @ \text{poles in upper half plane}] \\ \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx &= 2\pi \sum \text{Re}[\text{Res } f(z)e^{i\omega z} | @ \text{poles in upper half plane}] \end{aligned} \quad (3.63)$$

Example 14: Evaluate the following integrals.

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+1)(x^2+4)} \quad \Rightarrow \quad f(z)e^{iz} = \frac{Ze^{iz}}{(Z^2+1)(Z^2+4)}$$

Poles in the upper half plane: i (simple), $i2$ (simple)

$$\text{Res } f(z)e^{iz} |_{Z=i} = \lim_{Z \rightarrow i} (Z-i)f(z)e^{iz} = \lim_{Z \rightarrow i} \frac{Ze^{iz}}{(Z+i)(Z^2+4)} = \frac{i e^{-1}}{i2(3)} = \frac{e^{-1}}{6}$$

$$\text{Res } f(z)e^{iz} |_{Z=i2} = \lim_{Z \rightarrow i} (Z-i2)f(z)e^{iz} = \lim_{Z \rightarrow i} \frac{Ze^{iz}}{(Z^2+1)(Z+i2)} = \frac{i2 e^{-2}}{(-3)i4} = -\frac{e^{-2}}{6}$$

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+1)(x^2+4)} = 2\pi \sum \text{Re}[\text{Res } f(z)e^{i\omega z} | @ \text{poles in upper half plane}] = \frac{\pi}{3} (e^{-1} - e^{-2})$$

From example 14, we note $\int_{-\infty}^{\infty} \frac{x \cos x \, dx}{(x^2+1)(x^2+4)} = 0$.

CHAPTER 4

Linear Algebra

Matrix Theory

Overview

Section 4.1 begins with definition of matrices and basic operations. Determinant of a square matrix is discussed in Section 4.2. System of linear equations is covered in section 4.3 and solutions are obtained using Gaussian elimination method. Linear independence and dependence and rank of a matrix is presented in Section 4.4. In section 4.5, inverse of a matrix is discussed. Section 4.6 presents solving linear system of equations using inverse of a matrix and Cramer's rule. Section 4.7 defines orthogonal matrices.

4.1 – Matrices and Basic Operations

We begin with the definition of matrices and extend to other definitions related to matrices. The fundamental operations of matrices are also discussed in this section.

Matrices

A matrix is a rectangular array of elements, numbers, or functions defined as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})_{mn} \quad (4.1)$$

Matrix \mathbf{A} is an $m \times n$ matrix (m rows and n columns). This indicates the size of the matrix. The elements of the matrix are designated by a_{ij} , where i is the row number and j is the column number. In other words a_{ij} is the element of the matrix which is in the i -th row and j -th column of the matrix.

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 5 & 7 \end{bmatrix} \quad \text{is a } 2 \times 2 \text{ matrix}$$

$$\mathbf{B} = \begin{bmatrix} 6 & 9 \\ -2 & 3 \\ -7 & 4 \end{bmatrix} \quad \text{is a } 3 \times 2 \text{ matrix}$$

$$\mathbf{C} = \begin{bmatrix} 2 & x & -1 \\ \sin x & 4 & 12 \\ 0 & x^3 & \cos x \end{bmatrix} \quad \text{is a } 3 \times 3 \text{ matrix}$$

Square Matrix

If $m = n$, the matrix is referred to as an $m \times m$ square matrix or a matrix of order m .

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} = (a_{ij})_{mm} \quad (4.2)$$

The main diagonal of a square matrix is the one that contains elements $a_{11}, a_{22}, \dots, a_{mm}$. For example

$$\mathbf{A}_1 = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 8 \\ 4 & 2 & 10 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 1.5 & 3 & 0 & 8 \\ 9 & 4.7 & 1 & 1 \\ 11 & -1 & -3 & 3 \end{bmatrix}$$

\mathbf{A}_1 and \mathbf{A}_2 are square matrices of order 3 and \mathbf{A}_3 is a square matrix of order 4.

Row Vector and Column Vector

If $m = 1$, the matrix is referred to as a $1 \times n$ or simply a row vector. If $n = 1$, the matrix is referred to as a $m \times 1$ or simply a column vector.

$$\text{Row Vector:} \quad [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \quad (4.3)$$

$$\text{Column Vector:} \quad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \quad (4.4)$$

$$\text{Example of a row vector:} \quad [1 \quad 0 \quad -2]$$

$$\text{Example of a column vector:} \quad \begin{bmatrix} 6 \\ 9 \\ -1 \\ 3 \\ 10 \end{bmatrix}$$

Basic Matrix OperationsEquality

Two $m \times n$ matrices $\mathbf{A} = (a_{ij})_{mn}$ and $\mathbf{B} = (b_{ij})_{mn}$ are equal if and only if

$$(a_{ij})_{mn} = (b_{ij})_{mn} \quad (4.5)$$

Example 1: Given $\mathbf{A} = \mathbf{B}$, evaluate a , b , c

$$\mathbf{A} = \begin{bmatrix} a+3 & 4 & 10+c \\ 0 & b-2 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 15 & 4 & -5 \\ 0 & 12 & 9 \end{bmatrix}$$

$$\begin{array}{l} \mathbf{A} = \mathbf{B} \quad \Rightarrow \\ a+3 = 15 \quad a = 12 \\ b-2 = 12 \quad b = 14 \\ 10+c = -5 \quad c = -15 \end{array}$$

Addition

Given $\mathbf{A} = (a_{ij})_{mn}$ and $\mathbf{B} = (b_{ij})_{mn}$, then

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad (c_{ij})_{mn} = (a_{ij} + b_{ij})_{mn} \quad (4.6)$$

Example 2:

- Evaluate $\mathbf{A} + \mathbf{B}$
- Evaluate $\mathbf{A} + \mathbf{A}$
- Evaluate $\mathbf{A} + \mathbf{C}$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 3 \\ 0 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -4 & 2.3 \\ 8 & 1 \\ 5 & 3.4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 2 \\ -3 & 8 \end{bmatrix}$$

$$\text{a) } \mathbf{A} + \mathbf{B} = \begin{bmatrix} -3 & 5.3 \\ 6 & 4 \\ 5 & 12.4 \end{bmatrix}$$

$$\text{b) } \mathbf{A} + \mathbf{A} = \begin{bmatrix} 2 & 6 \\ -4 & 6 \\ 0 & 18 \end{bmatrix}$$

$$\text{c) } \mathbf{A} + \mathbf{C} \quad \text{Cannot be added. } \mathbf{A} \text{ is } 3 \times 2 \text{ and } \mathbf{C} \text{ is } 2 \times 2$$

Subtraction

Given $\mathbf{A} = (a_{ij})_{mn}$ and $\mathbf{B} = (b_{ij})_{mn}$, then

$$\mathbf{C} = \mathbf{A} - \mathbf{B} \quad (c_{ij})_{mn} = (a_{ij} - b_{ij})_{mn} \quad (4.7)$$

Example 3: Evaluate $\mathbf{A} - \mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} -3 & 3 & 7 & -1 \\ 4 & 2 & 3 & 8 \\ -5 & 8 & 1 & -6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & -10 & -5 \\ 2 & 6 & 20 & -2 \\ 8 & 10 & 3 & -1 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} -3 & 2 & 17 & 4 \\ 2 & -4 & -17 & 10 \\ -13 & -2 & -2 & -5 \end{bmatrix}$$

Multiplication of a Matrix by a Scalar

Given $\mathbf{A} = (a_{ij})_{mn}$ and a scalar k , then

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix} = (ka_{ij})_{mn} \quad (4.8)$$

Example 4: Evaluate $-2\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -4 \\ 2 & -5 & 6 \end{bmatrix}$$

$$-2\mathbf{A} = -2 \begin{bmatrix} 1 & 3 & -4 \\ 2 & -5 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 8 \\ -4 & 10 & -12 \end{bmatrix}$$

Multiplication of Matrices

Let \mathbf{A} be an $m \times n$ matrix and \mathbf{B} be a $n \times p$ matrix, then $\mathbf{C} = \mathbf{AB}$ can be evaluated and \mathbf{C} is a $m \times p$ matrix and is defined as

$$\mathbf{C} = \mathbf{AB} \quad \text{with } i = 1, 2, \dots, m \quad \text{and} \quad j = 1, 2, \dots, p$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj} \quad (4.9)$$

The number of columns of matrix $\mathbf{A}(p)$ must be the same as the number of rows of matrix $\mathbf{B}(p)$ to multiply \mathbf{A} by \mathbf{B} . Furthermore it should be clear to evaluate c_{ij} , the i -th row of matrix \mathbf{A} is multiplied by the corresponding elements in the j -th column of matrix \mathbf{B} as shown below

$$c_{ij} = [a_{i1} \quad a_{i2} \quad \dots \quad a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj} \quad (4.10)$$

We can conclude that matrix multiplication is not commutative. Also two square matrices are multiplied, they have to be of the same order.

Let \mathbf{A} be an $m \times n$ matrix and \mathbf{B} be a $n \times p$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = (a_{ij})_{mn} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} = (b_{ij})_{np}$$

then \mathbf{AB} can be written as

$$\mathbf{AB} = [\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \dots \quad \mathbf{Ab}_p] \quad (4.11)$$

where $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ are column vectors of \mathbf{B} matrix as shown below.

$$\mathbf{b}_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} \quad \dots \quad \mathbf{b}_p = \begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{np} \end{bmatrix} \quad (4.12)$$

Example 5: Evaluate \mathbf{AB}

$$\text{a) } \mathbf{A} = \begin{bmatrix} 1 & 3 & -4 \\ 2 & -5 & 6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -2 & 3 \\ 0 & 9 \end{bmatrix}$$

$$\begin{aligned}\mathbf{AB} &= \begin{bmatrix} 1 & 3 & -4 \\ 2 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 3 \\ 0 & 9 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1) + (3)(-2) + (-4)(0) & (1)(3) + (3)(3) + (-4)(9) \\ (2)(1) + (-5)(-2) + (6)(0) & (2)(3) + (-5)(3) + (6)(9) \end{bmatrix} = \begin{bmatrix} -5 & -24 \\ 12 & 45 \end{bmatrix}\end{aligned}$$

$$\text{b) } \mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \quad \mathbf{B} = [2 \quad -4 \quad 3]$$

$$\mathbf{AB} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} [2 \quad -4 \quad 3] = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -8 & 6 \\ 10 & -20 & 15 \end{bmatrix}$$

$$\text{c) } \mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 7 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -2 & 3 \\ 0 & 9 \end{bmatrix}$$

\mathbf{AB} cannot be performed. \mathbf{A} is a 2×2 and \mathbf{B} is a 3×2 . Number of columns of \mathbf{A} is not the same as the number of rows of \mathbf{B} .

Transpose of a Matrix

Let \mathbf{A} be an $m \times n$ matrix as shown in equation (4.1). Transpose of \mathbf{A} is denoted by \mathbf{A}^T and is defined as

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \quad (4.13)$$

As can be observed the first column of \mathbf{A} is the first row of \mathbf{A}^T , second column of \mathbf{A} is the second row of \mathbf{A}^T and so on. Hence the rows and columns are exchanged between \mathbf{A} and \mathbf{A}^T . From this definition it should be clear that

$$(\mathbf{A}^T)^T = \mathbf{A} \quad (4.14)$$

Example 6: Evaluate \mathbf{A}^T

$$\text{a) } \mathbf{A} = \begin{bmatrix} 10 & -5 & -7 \\ 12 & -9 & 16 \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} 10 & 12 \\ -5 & -9 \\ -7 & 16 \end{bmatrix}$$

$$\text{b) } \mathbf{A} = \begin{bmatrix} 10 \\ -2 \\ 5 \end{bmatrix}$$

$$\mathbf{A}^T = [10 \quad -2 \quad 5]$$

$$\text{c) } \mathbf{A} = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -8 & 6 \\ 10 & -20 & 15 \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} 2 & 4 & 10 \\ -4 & -8 & -20 \\ 3 & 6 & 15 \end{bmatrix}$$

Conjugate of a Matrix

Let \mathbf{A} be an $m \times n$ matrix. The conjugate of \mathbf{A} is denoted by $\bar{\mathbf{A}}$ and is defined as

$$\begin{aligned} \mathbf{A} &= (a_{ij})_{mn} \\ \bar{\mathbf{A}} &= (\bar{a}_{ij})_{mn} \end{aligned} \tag{4.15}$$

This indicates the elements of \mathbf{A} and $\bar{\mathbf{A}}$ are complex conjugate of each other. Furthermore if $\mathbf{A} = \bar{\mathbf{A}}$, then \mathbf{A} is a real matrix.

Example 7: Evaluate $\bar{\mathbf{A}}$

$$\mathbf{A} = \begin{bmatrix} 5 - i & i3 & -7 - i2 \\ 12 & 2 + i3 & -1 + i \end{bmatrix}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} 5 + i & -i3 & -7 + i2 \\ 12 & 2 - i3 & -1 - i \end{bmatrix}$$

Tranjugate of a Matrix

Let \mathbf{A} be an $m \times n$ matrix. The Tranjugate of \mathbf{A} is defined as $(\bar{\mathbf{A}})^T$ and is defined as

$$\begin{aligned} \mathbf{A} &= (a_{ij})_{mn} \\ (\bar{\mathbf{A}})^T &= (\bar{a}_{ji})_{nm} \end{aligned} \tag{4.16}$$

We note that $(\bar{\mathbf{A}})^T = \overline{\mathbf{A}^T}$.

Example 8: Evaluate $(\bar{\mathbf{A}})^T$

$$\mathbf{A} = \begin{bmatrix} 5 - i & i3 & -7 - i2 \\ 12 & 2 + i3 & -1 + i \end{bmatrix}$$

$$(\bar{\mathbf{A}})^T = \overline{\mathbf{A}^T} = \begin{bmatrix} 5 + i & 12 \\ -i3 & 2 - i3 \\ -7 + i2 & -1 - i \end{bmatrix}$$

Properties of Matrix Addition

Let \mathbf{A} , \mathbf{B} , and \mathbf{C} be $m \times n$ matrices and $\mathbf{0}$ designate a zero $m \times n$ matrix.

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Commutative
$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$	Associative
$\mathbf{A} + \mathbf{0} = \mathbf{A}$	Additive Identity
$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$	Additive Inverse
$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$	Transpose of Sum

Properties of Matrix and Scalar Multiplication

Let k_1 and k_2 be a scalars. Let \mathbf{A} , \mathbf{B} , and \mathbf{C} be of the proper size so each operation can be performed. For example for addition matrices must be of the same exact size and for multiplication the columns of the first matrix must be equal to the rows of the second matrix.

$(k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}$	Distributive
$k_1(\mathbf{A} + \mathbf{B}) = k_1\mathbf{A} + k_1\mathbf{B}$	Distributive
$(k_1k_2)\mathbf{A} = k_1(k_2\mathbf{A}) = k_2(k_1\mathbf{A})$	
$1\mathbf{A} = \mathbf{A}$	
$k_1(\mathbf{AB}) = (k_1\mathbf{A})\mathbf{B} = \mathbf{A}(k_1\mathbf{B})$	
$\mathbf{AB} \neq \mathbf{BA}$	Commutative
$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$	Associative
$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$	Distributive
$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$	Distributive
$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$	Transpose of Product
$(k_1\mathbf{A})^T = k_1\mathbf{A}^T$	

It is already indicated that $\mathbf{AB} \neq \mathbf{BA}$. One can make up an example that commutative law of multiplication applies. However as an example if \mathbf{A} is 3×4 and \mathbf{B} is 4×2 , \mathbf{AB} can be evaluated, but \mathbf{BA} cannot even be calculated. Another rule of algebra that does not apply to linear algebra is $\mathbf{AB} = \mathbf{0}$ does not suggest that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. In fact neither \mathbf{A} nor \mathbf{B} need to be a zero for the product to be a zero matrix.

Definition of Some Important Matrices

Zero or Null Matrix

A zero or a null matrix is a matrix whose elements are all zero. This matrix is denoted by $\mathbf{0}$. Here are some examples

$$\mathbf{0} = [0 \quad 0 \quad 0]$$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Diagonal Matrix

A square matrix whose elements above or below the main diagonal are all zero. Here are some examples

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Scalar Matrix

A diagonal matrix whose diagonal elements are the same scalar a . This matrix is denoted by \mathbf{S} . Here are some examples

$$\mathbf{S} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

The pre-multiplication or post-multiplication of a matrix \mathbf{A} by a scalar matrix when the matrix sizes allow it is the same as multiplying matrix \mathbf{A} by a constant. For example

$$\mathbf{SA} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 3 & 8 & 9 \end{bmatrix} = \begin{bmatrix} a & 5a & 2a \\ 3a & 8a & 9a \end{bmatrix} = a\mathbf{A}$$

$$\mathbf{AS} = \begin{bmatrix} 1 & 3 & 5 \\ 6 & 7 & 2 \\ -1 & -2 & 9 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} a & 3a & 5a \\ 6a & 7a & 2a \\ -a & -2a & 9a \end{bmatrix} = a\mathbf{A}$$

If \mathbf{A} and \mathbf{S} are square matrices of the same order, then

$$\mathbf{SA} = \mathbf{AS} = a\mathbf{A} \quad (4.17)$$

Identity or Unit Matrix

A diagonal matrix whose diagonal elements are all 1. This matrix is denoted by \mathbf{I} . Here are some examples

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similar conclusions can be made as for the case of scalar matrix with $a = 1$. Furthermore if \mathbf{A} and \mathbf{I} are square matrices of the same order, then

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A} \quad (4.18)$$

Symmetric Matrix

Matrix \mathbf{A} is symmetric if transpose of the matrix is the same as the matrix ($\mathbf{A}^T = \mathbf{A}$ or $a_{ij} = a_{ji}$). Clearly this requires the matrix to be a square matrix. Here are some examples

$$\mathbf{A} = \begin{bmatrix} 2 & -6 \\ -6 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & -3 & 4 \\ -3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

Skew-Symmetric Matrix

Matrix \mathbf{A} is skew-symmetric if the transpose of the matrix is the same as the negative of matrix ($\mathbf{A}^T = -\mathbf{A}$ or $a_{ij} = -a_{ji}$). Clearly this requires the matrix to be a square matrix and the elements on the main diagonal to be zero. Here are some examples

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & -8 & 5 \\ -2 & 0 & 4 & -4 \\ 8 & -4 & 0 & 0 \\ -5 & 4 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

Hermitian Matrix

Matrix \mathbf{A} is hermitian if transpose of the matrix is the same as the conjugate of matrix ($\mathbf{A}^T = \overline{\mathbf{A}}$ or $a_{ji} = \overline{a_{ij}}$). Clearly this requires the matrix to be a square matrix and the elements on the main diagonal to be real. Here are some examples

$$\mathbf{A} = \begin{bmatrix} 2 & -3 + i2 \\ -3 - i2 & 10 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & i3 & 2 + i4 \\ -i3 & 11 & 0 \\ 2 - i4 & 0 & 17 \end{bmatrix}$$

Skew-Hermitian Matrix

Matrix \mathbf{A} is skew-hermitian if transpose of the matrix is the same as the negative of the conjugate of matrix ($\mathbf{A}^T = -\overline{\mathbf{A}}$ or $a_{ji} = -\overline{a_{ij}}$). Clearly this requires the matrix to be a square matrix and the elements on the main diagonal to be pure imaginary or zero. Here are some examples

$$\mathbf{A} = \begin{bmatrix} i2 & 3 + i2 \\ -3 + i2 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} i2 & 5 + i & i5 \\ -5 + i & i11 & -2 \\ i5 & 2 & -i17 \end{bmatrix}$$

Upper Triangular Matrix

Square matrix \mathbf{A} is an upper triangular matrix if all its elements below the main diagonal are zero ($a_{ij} = 0$ for $i > j$). Here are some examples

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & -4 \\ 0 & 11 & 0 \\ 0 & 0 & 17 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 10 & 2 & -8 & 5 \\ 0 & 0 & 0 & -14 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Lower Triangular Matrix

Square matrix \mathbf{A} is a lower triangular matrix if all its elements above the main diagonal are zero ($a_{ij} = 0$ for $i < j$). Here are some examples

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 9 & 0 \\ 6 & -3 & 8 \end{bmatrix}$$

Submatrices of a Matrix

Submatrix of a matrix is any matrix obtained by omitting some rows or columns of the matrix. An example of all submatrices of a 3x2 matrix \mathbf{A} is shown below.

$$\mathbf{A} = \begin{bmatrix} 10 & 8 \\ -2 & 5 \\ 4 & 9 \end{bmatrix}$$

Submatrices:

$$3 \times 2 \quad \begin{bmatrix} 10 & 8 \\ -2 & 5 \\ 4 & 9 \end{bmatrix}$$

$$3 \times 1 \quad \begin{bmatrix} 10 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix}$$

$$2 \times 2 \quad \begin{bmatrix} 10 & 8 \\ -2 & 5 \end{bmatrix}, \begin{bmatrix} 10 & 8 \\ 4 & 9 \end{bmatrix}, \begin{bmatrix} -2 & 5 \\ 4 & 9 \end{bmatrix}$$

$$\begin{array}{ll}
 2 \times 1 & [10], [-2], [10], [8], [5], [8] \\
 & [-2], [4], [4], [5], [9], [9] \\
 1 \times 2 & [10 \ 8], [-2 \ 5], [4 \ 9] \\
 1 \times 1 & [10], [8], [-2], [5], [4], [9]
 \end{array}$$

4.2 – Determinants

Given an $n \times n$ square matrix \mathbf{A}

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad (4.19)$$

the determinant of \mathbf{A} is represented by $\det \mathbf{A}$ or $|\mathbf{A}|$ and is a single number.

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \quad (4.20)$$

Although the vertical lines in equation (4.2) are used to denote absolute value as well, this does not imply that $\det \mathbf{A}$ is positive. Determinant of an $n \times n$ square matrix \mathbf{A} is written as

$$\det \mathbf{A} = |\mathbf{A}| = \sum_{j=1}^n a_{ij} C_{ij} \quad i = 1, \text{ or } 2, \dots, \text{ or } n \quad (4.21)$$

or

$$\det \mathbf{A} = |\mathbf{A}| = \sum_{i=1}^n a_{ij} C_{ij} \quad j = 1, \text{ or } 2, \dots, \text{ or } n \quad (4.22)$$

The term C_{ij} is known as the cofactor of a_{ij} and is equal to

$$C_{ij} = (-1)^{i+j} M_{ij} \quad (4.23)$$

M_{ij} is called the minor of a_{ij} and is the determinant of remaining square submatrix of \mathbf{A} after deleting the i -th row and the j -th column of \mathbf{A} . Hence equations (4.21) and (4.22) can be written as

$$\det \mathbf{A} = |\mathbf{A}| = \sum_{j=1}^n (-1)^{i+j} a_{ij} M_{ij} \quad i = 1, \text{ or } 2, \dots, \text{ or } n \quad (4.24)$$

or

$$\det \mathbf{A} = |\mathbf{A}| = \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{ij} \quad j = 1, \text{ or } 2, \dots, \text{ or } n \quad (4.25)$$

Equations (4.21) and (4.24) are referred to as evaluating $\det \mathbf{A}$ by row expansion and equations (4.22) and (4.25) are referred to as evaluating $\det \mathbf{A}$ by column expansion. To better understand these equations, let $i = 1$ in equations (4.21) and (4.24).

$$\begin{aligned} \det \mathbf{A} &= a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n} \\ &= a_{11}M_{11} - a_{12}M_{12} + \dots + a_{1n}(-1)^{1+n}M_{1n} \end{aligned} \quad (4.26)$$

As can be observed $a_{11}, a_{12}, \dots, a_{1n}$ are elements of the first row of \mathbf{A} and $M_{11}, M_{12}, \dots, M_{1n}$ are the corresponding minors of these elements respectively. Similarly, let $j = 2$ in equations (4.22) and (4.25).

$$\begin{aligned} \det \mathbf{A} &= a_{12}C_{12} + a_{22}C_{22} + \dots + a_{n2}C_{1n} \\ &= -a_{12}M_{12} + a_{22}M_{22} + \dots + a_{n2}(-1)^{2+n}M_{n2} \end{aligned} \quad (4.27)$$

As can be observed $a_{12}, a_{22}, \dots, a_{n2}$ are elements of the second column of \mathbf{A} and $M_{12}, M_{22}, \dots, M_{n2}$ are the corresponding minors of these elements respectively. Clearly in computation of determinants one should expand about a row or column with most number of zeros.

We now consider determinants of some square matrices.

Determinant of 1x1 Matrix

$$\begin{aligned} \mathbf{A} &= [a_{11}] \\ \det \mathbf{A} = |\mathbf{A}| &= a_{11} \end{aligned}$$

Determinant of 2x2 Matrix

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ \text{using equation (4.24) and } i &= 1 \\ \det \mathbf{A} = |\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{1+1}a_{11}M_{11} + (-1)^{1+2}a_{12}M_{12} = a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

$$\begin{aligned} \text{using equation (4.24) and } i &= 2 \\ \det \mathbf{A} = |\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{2+1}a_{21}M_{21} + (-1)^{2+2}a_{22}M_{22} = -a_{21}a_{12} + a_{22}a_{11} \end{aligned}$$

$$\begin{aligned} \text{using equation (4.25) and } j &= 1 \\ \det \mathbf{A} = |\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{1+1}a_{11}M_{11} + (-1)^{2+1}a_{21}M_{21} = a_{11}a_{22} - a_{21}a_{12} \end{aligned}$$

$$\begin{aligned} \text{using equation (4.25) and } j &= 2 \\ \det \mathbf{A} = |\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{1+2}a_{12}M_{12} + (-1)^{2+2}a_{22}M_{22} = -a_{12}a_{21} + a_{22}a_{11} \end{aligned}$$

As can be observed equations (4.24) and (4.25) provide the same results regardless of what values of i (1 or 2) or j (1 or 2) is selected.

Determinant of 3x3 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

using equation (4.24) and $i = 1$

$$\begin{aligned} \det A = |\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1}a_{11}M_{11} + (-1)^{1+2}a_{12}M_{12} + (-1)^{1+3}a_{13}M_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

One can use $i = 2$ or 3 in equation (4.24) or $j = 1, 2, \text{ or } 3$ in equation (4.25) and the results would be the same. As observed a 3x3 determinant breaks down into three 2x2 determinants. Similarly a 4x4 determinant breaks down into four 3x3 determinants and so on.

Example 9: Evaluate det A

a) $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 5 & -2 \end{bmatrix}$

$$\det \mathbf{A} = \begin{vmatrix} 5 & 4 \\ 5 & -2 \end{vmatrix} = (5)(-2) - (4)(5) = -30$$

b) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 1 \\ -2 & 0 & 7 \end{bmatrix}$

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} 2 & -1 & 3 \\ 4 & 5 & 1 \\ -2 & 0 & 7 \end{vmatrix} = (2) \begin{vmatrix} 5 & 1 \\ 0 & 7 \end{vmatrix} - (-1) \begin{vmatrix} 4 & 1 \\ -2 & 7 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ -2 & 0 \end{vmatrix} = (2)(35) + (28+2) + 3(10) \\ &= 130 \end{aligned}$$

From equations (4.24) and (4.25) we can make some simple observations.

- The determinant of a diagonal matrix is the product of the main diagonal elements.
- The determinant of an identity matrix is 1.
- The determinant of a triangular matrix is the product of the main diagonal elements.

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 30$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 10 & -2 & 50 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{vmatrix} = -150$$

For (a) and (b), we can simply consider evaluating the determinant by expansion about any row or column. For (c), we can consider evaluating the determinant by expansion about the first column for an upper triangle and expansion about the first for a lower triangle.

Properties of Determinants

- 1) Determinant of square matrix \mathbf{A} is equal to determinant of \mathbf{A}^T ($|\mathbf{A}|=|\mathbf{A}^T|$).
- 2) If all elements of a row or a column of a square matrix \mathbf{A} is zero, then $\det \mathbf{A} = 0$
- 3) If all elements of one row or column of a square matrix \mathbf{A} are multiplied by k , then the value of the determinant is multiplied by k .
- 4) If all elements of a square matrix \mathbf{A} are multiplied by k , then the value of the determinant is multiplied by k^n .
- 5) If elements of two rows or columns of a square matrix \mathbf{A} are proportional or the same, the value of $\det \mathbf{A} = 0$.
- 6) If any two rows or columns of a square matrix \mathbf{A} are interchanged, the value of determinant of the new matrix is negative of determinant of original matrix \mathbf{A} .
- 7) If any row of a determinant is multiplied by a constant and added to the corresponding elements of any other row, the value of a determinant is not altered. Similarly if any column of a determinant is multiplied by a constant and added to the corresponding elements of any other column, the value of a determinant is not altered.
- 8) If \mathbf{A} and \mathbf{B} are $n \times n$ square matrices, then $\det (\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$.

We now consider some examples applied to each of these properties.

Example 10:

$$\text{a) } \mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix} \quad \det \mathbf{A} = 2(-2) - (-1)(-3 + 2) + 4(4) = 11$$

$$\mathbf{A}^T = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 2 & 0 \\ 4 & 1 & -1 \end{bmatrix} \quad \det \mathbf{A}^T = 2(-2) - (3)(1) - 2(-1 - 8) = 11$$

$$\text{b) } \mathbf{A} = \begin{bmatrix} 2 & 10 & -2 \\ 0 & 0 & 0 \\ 4 & 1 & -1 \end{bmatrix} \quad \text{Using equation (4.24) with } i = 2 \text{ (expansion about the 2}^{\text{nd}} \text{ row)}$$

$$\det \mathbf{A} = 0$$

$$\text{c) } \mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix} \quad \det \mathbf{A} = 11 \text{ from part (a)}$$

$$\mathbf{B} = \begin{bmatrix} 10 & -5 & 20 \\ 3 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix} \quad \text{First row of A is multiplied by 5} \quad \Rightarrow$$

$$\det \mathbf{B} = 10(-2) - (-5)(-3 + 2) + 20(4) = 55$$

$$\text{d) } \mathbf{A} = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} \quad \det \mathbf{A} = 2 \quad \mathbf{B} = 5\mathbf{A} = \begin{bmatrix} 25 & -20 \\ 15 & -10 \end{bmatrix}$$

$$\det \mathbf{B} = -250 + 300 = 50 = 2(5)^2$$

e) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -2 & 4 & -1 \end{bmatrix}$ rows 1 and 2 are proportional

$$\det \mathbf{A} = 2(2 - 24) + 1(-4 + 12) + 3(16 - 4) = -44 + 8 + 36 = 0$$

f) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix}$ $\det \mathbf{A} = 11$ from part (a). Interchanging 1st and 3rd column.

$\mathbf{B} = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ expanding about the 3rd row

$$\det \mathbf{B} = -1(-3 - 4) + (-2)(8 + 1) = 7 - 18 = -11$$

g) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix}$ $\det \mathbf{A} = 11$ from part (a).

Replacing the 3rd row by multiplying the 1st row by 2 and adding it to the 3rd row.

$\mathbf{B} = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 2 & -2 & 7 \end{bmatrix}$ $\det \mathbf{B} = 2(14 + 2) - (-1)(21 - 2) + 4(-6 - 4) = 11$

h) $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ -3 & 9 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & -2 \\ 3 & -2 \end{bmatrix}$

$$\det \mathbf{A} = 18 - 15 = 3 \quad \det \mathbf{B} = -8 + 6 = -2 \quad (\det \mathbf{A})(\det \mathbf{B}) = -6$$

$\mathbf{AB} = \begin{bmatrix} 2 & -5 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -7 & 6 \\ 15 & -12 \end{bmatrix}$ $\det (\mathbf{AB}) = 84 - 90 = -6$

4.3 – Systems of Linear Equations

A linear set of equations with unknown x_1, x_2, \dots, x_n and known constant coefficients a_{ij} ($i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$) and known constants b_i ($i = 1, 2, \dots, m$) is described by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \tag{4.28}$$

If all b_i ($i = 1, 2, \dots, m$) are zero, the linear system of equations is called homogeneous. If there is at least one $b_i \neq 0$, the linear system of equations is referred to as nonhomogeneous. Clearly a homogeneous linear system of equations with n variables x_1, x_2, \dots, x_n is satisfied if $x_i = 0$ for all $i = 1, 2, \dots, n$. This solution is referred to as the trivial solution of linear system of homogeneous equations.

Solution of equation (4.28) consists of x_1, x_2, \dots, x_n that satisfy every single equation of equation (4.28). If there is no solution, then the linear system is called inconsistent. If there is at least one set of x_1, x_2, \dots, x_n that satisfies equation (4.28), then the linear system is said to be consistent. It is possible that a consistent linear system can have infinite solutions.

Let us consider some simple example of a linear system of equations with a unique solution, infinite solution, and no solution

$$\begin{aligned} \text{a) } & 3x_1 + 2x_2 = 2 \\ & -4x_1 + 2x_2 = -12 \quad \Rightarrow \end{aligned}$$

$$x_1 = 2 \text{ and } x_2 = -2 \quad \underline{\text{unique solution satisfying both equations}}$$

$$\begin{aligned} \text{b) } & x_1 - 4x_2 = 5 \\ & -2x_1 + 8x_2 = -10 \quad \Rightarrow \end{aligned}$$

$$x_1 = 4x_2 + 5 \quad \text{select any value for evaluate } x_1, \text{ then evaluate } x_2$$

infinite solutions satisfying both equations

$$\begin{aligned} \text{c) } & x_1 + 2x_2 = 4 \\ & 5x_1 + 10x_2 = -12 \quad \Rightarrow \end{aligned}$$

no solutions exist

Defining matrices **A**, **x**, and **B** as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

then equation (4.28) can be written in matrix form as

$$\mathbf{Ax} = \mathbf{B} \quad (4.29)$$

We also define the augmented matrix of the system which consists of matrix **A** and matrix **B** placed in the last column to the right of **A** as

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right] \quad (4.30)$$

This augmented matrix simply represents equation (4.28) with a_{ij} as the coefficients of x_j and b_i as the known constants. Since the augmented matrix of equation (4.30) represents the linear system of equations and based on properties of equality, the following row operations can be performed

- 1) Any row can be multiplied or divided by a nonzero constant.
- 2) Location of any row can be interchanged with any other row.
- 3) Any row can be replaced by multiplying any other row by a constant and adding it to that row.

Gaussian Elimination Method

Gaussian elimination method can be used to solve equation (4.28). The procedure is to use the augmented matrix of equation (4.30) and through row operations indicated above, write the augmented matrix in what is referred to as row echelon form. The main objective is by using row operations, transform the augmented matrix into a matrix with zeros below or above its diagonal elements. This is to transform the augmented matrix to a matrix resembling an upper or a lower triangular matrices. For the sake of understanding, let us assume that using row operations we have managed to transform an augmented matrix for a linear system of equations with three unknown variables x_1, x_2 , and x_3 into the following form

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad [\text{modified } \mathbf{A} \mid \text{modified } \mathbf{B}] \quad (4.31)$$

This modified augmented matrix is known as row echelon form. Starting with the last row, we write

$$\begin{array}{llll} x_3 = 2 & & & x_3 = 2 \\ x_2 + 3x_3 = 5 & \Rightarrow & x_2 + 3(2) = 5 & x_2 = -1 \\ x_1 + 3x_2 - 2x_3 = -9 & & x_1 + 3(-1) - 2(2) = -9 & x_1 = -2 \end{array}$$

Once the row echelon form is obtained, this last step is referred to a back substitution. Depending on if there is unique solution, infinite solutions, or no solution the row echelon form may appear different. Let us assume we are given m equations with m unknown.

- a) All zero rows are below any nonzero rows
- b) The first nonzero entry in any row is 1. This entry of 1 should appear with $m - 1$ zeros in the columns to the left in the same row.
- c) In the next row, the leading entry of 1 appears to the right of the 1 in the row above it.

In the modified augmented matrix of equation (4.31), it is for convenient in computations to have the diagonal elements of modified \mathbf{A} as 1. Certainly any nonzero constants in these positions serve the same purpose and the final solutions don't change.

Example 11: Using Gaussian elimination method solve for x_1 , x_2 , and x_3 .

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\2x_1 - x_2 + x_3 &= -1 \\5x_1 + x_2 + 6x_3 &= 8\end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & -1 \\ 5 & 1 & 6 & 8 \end{array} \right]$$

replace row 3 by multiplying row 1 by -5 and adding to row 3: $(3) \rightarrow -5(1) + (3) \Rightarrow$

replace row 2 by multiplying row 1 by -2 and adding to row 2: $(2) \rightarrow -2(1) + (2) \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & -4 & 1 & -2 \end{array} \right] \quad (3) \rightarrow -\frac{4}{3}(2) + (3) \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & 0 & 7/3 & 14/3 \end{array} \right] \quad (2) \rightarrow -\frac{1}{3}(2) \text{ and } (3) \rightarrow \frac{3}{7}(3) \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 1 & 2 \end{array} \right] \text{ This is in row echelon form. Performing back substitution } \Rightarrow$$

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\x_2 + (1/3)x_3 &= 5/3 \\x_3 &= 2\end{aligned}$$

$$\begin{aligned}x_3 &= 2 \\x_2 &= -(2/3) + 5/3 = 1 \\x_1 &= -2 - 1 + 2 = -1\end{aligned}$$

The problem in example 11 has a unique solution. We now consider two examples, one with infinite solutions and one with no solution.

Example 12: Using Gaussian elimination method solve for x_1 , x_2 , and x_3 .

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\2x_1 - x_2 + x_3 &= -1 \\4x_1 + x_2 + 3x_3 &= 3\end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & -1 \\ 4 & 1 & 3 & 3 \end{array} \right]$$

replace row 3 by multiplying row 1 by -4 and adding to row 3: $(3) \rightarrow -4(1) + (3) \Rightarrow$

replace row 2 by multiplying row 1 by -2 and adding to row 2: $(2) \rightarrow -2(1) + (2) \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & -3 & -1 & -5 \end{array} \right]$$

$(3) \rightarrow -1(2) + (3) \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$(2) \rightarrow -\frac{1}{3}(2) \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is in row echelon form. Performing back substitution \Rightarrow

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 + (1/3)x_3 &= 5/3 \end{aligned}$$

The problem in example 12 has infinite solutions. In the above two equations, any arbitrary value for x_1 can be selected and x_2 and x_3 can then be evaluated.

Example 13: Using Gaussian elimination method solve for x_1 , x_2 , and x_3 .

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= -1 \\ 4x_1 + x_2 + 3x_3 &= 5 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & -1 \\ 4 & 1 & 3 & 5 \end{array} \right]$$

replace row 3 by multiplying row 1 by -4 and adding to row 3: $(3) \rightarrow -4(1) + (3) \Rightarrow$

replace row 2 by multiplying row 1 by -2 and adding to row 2: $(2) \rightarrow -2(1) + (2) \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & -3 & -1 & -3 \end{array} \right]$$

$(3) \rightarrow -1(2) + (3) \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & 0 & 0 & 2 \end{array} \right] \quad \text{The last two rows of the modified augmented matrix} \quad \Rightarrow$$

$$\begin{aligned} -3x_2 - x_3 &= -5 \\ -3x_2 - x_3 &= -3 \end{aligned}$$

The problem in example 13 has no solution. This is clear from the above two equations or can be observed from the last row of the modified augmented matrix indicating

$$0x_1 + 0x_2 + 0x_3 = 2 \quad \text{or} \quad 0 = 2$$

Example 14: Using Gaussian elimination method solve the following homogenous equations for x_1 , x_2 , and x_3 .

$$\begin{aligned} 3x_1 - 2x_2 + 5x_3 &= 0 \\ x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -2 & 5 & 0 \\ 1 & 2 & -1 & 0 \end{array} \right] \quad (1) \leftrightarrow (2) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & -2 & 5 & 0 \end{array} \right] \quad (2) \rightarrow -3(1) + (2) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -8 & 8 & 0 \end{array} \right] \quad (2) \rightarrow -\frac{1}{8}(2) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right] \quad \Rightarrow$$

$$x_1 + 2x_2 - x_3 = 0$$

$$x_2 - x_3 = 0 \quad \Rightarrow$$

$$x_2 = x_3 \quad \text{and} \quad x_1 = -x_2$$

It can be observed that even though problem of example 14 is homogeneous, it has other solutions besides the trivial solution discussed earlier ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$). In this example x_2 and x_3 are equal and $x_1 = -x_2$. This provides infinite solutions.

4.4 – Linear Independence and Dependence and Rank of a Matrix

Linear Independence and Dependence

Let us assume we are given m row vectors ($1 \times n$) or m column vectors ($n \times 1$) as shown below

$$\begin{aligned} \mathbf{v}_1 &= [v_{11} \quad v_{12} \quad \dots \quad v_{1n}] \\ \mathbf{v}_2 &= [v_{21} \quad v_{22} \quad \dots \quad v_{2n}] \\ &\dots\dots\dots \\ \mathbf{v}_m &= [v_{m1} \quad v_{m2} \quad \dots \quad v_{mn}] \end{aligned} \quad (4.32)$$

or

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{bmatrix} \quad \dots\dots\dots \quad \mathbf{v}_m = \begin{bmatrix} v_{1m} \\ v_{2m} \\ \vdots \\ v_{nm} \end{bmatrix} \quad (4.33)$$

Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are linearly dependent if and only if for given scalars k_1, k_2, \dots, k_m not all zero, the following equation is satisfied.

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_m\mathbf{v}_m = \mathbf{0} \quad (4.34)$$

Similarly vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are said to be linearly independent if and only if the only way for equation (4.34) to be satisfied is to have $k_1 = k_2 = \dots = k_m = 0$.

As an example the following row vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

$$\begin{aligned} \mathbf{v}_1 &= [-2 \quad 1 \quad 4 \quad -8] \\ \mathbf{v}_2 &= [1 \quad 0 \quad 2 \quad -3] \\ \mathbf{v}_3 &= [-1 \quad 2 \quad 14 \quad -25] \end{aligned}$$

The linear combination of these three vectors $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ is a 1×4 zero vector for $k_1 = 2, k_2 = 3, k_3 = -1$.

$$\begin{aligned} 2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 &= 2[-2 \quad 1 \quad 4 \quad -8] + 3[1 \quad 0 \quad 2 \quad -3] - [-1 \quad 2 \quad 14 \quad -25] \\ &= [0 \quad 0 \quad 0 \quad 0] \end{aligned}$$

Below is an example of three column vectors that are linearly independent.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \quad k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In other words, the only way the linear combination of these three vectors results in a column vector of $\mathbf{0}$ is if $k_1 = 0, k_2 = 0, \text{ and } k_3 = 0$.

As can be observed from the examples, it is not obvious to detect if a given set of row vectors or column vectors are linearly independent or dependent. However after rank of a matrix is defined, more methodical approaches are used for this purpose.

Rank of a Matrix

Rank of an $m \times n$ matrix \mathbf{A} is denoted by $\text{rank } \mathbf{A}$. The rank of \mathbf{A} is the maximum number of linearly independent row vectors of \mathbf{A} . It can also be stated that the rank of \mathbf{A} is the maximum number of linearly independent column vectors of \mathbf{A} . This indicates that rank of \mathbf{A} is the same as rank of \mathbf{A}^T . Proof of $\text{rank } \mathbf{A} = \text{rank } \mathbf{A}^T$ is left as an exercise. We proceed to evaluate rank of a matrix using two different method: 1) Row Reduction and 2) Determinant.

Evaluating Rank \mathbf{A} by Row Reduction

Using row reduction and after transforming a given matrix to row echelon form, the rank of matrix can be determined by examining the number of nonzero rows. As an example consider matrix \mathbf{A} whose rows consists of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 examined in an earlier example where we found out the three row vectors were linearly dependent.

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} -2 & 1 & 4 & -8 \\ 1 & 0 & 2 & -3 \\ -1 & 2 & 14 & -25 \end{bmatrix} && (1) \leftrightarrow (2) \Rightarrow \\ & \begin{bmatrix} 1 & 0 & 2 & -3 \\ -2 & 1 & 4 & -8 \\ -1 & 2 & 14 & -25 \end{bmatrix} && (2) \rightarrow 2(1) + (2) \ \& \ (3) \rightarrow (1) + (3) \Rightarrow \\ & \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 8 & -14 \\ 0 & 2 & 16 & -28 \end{bmatrix} && (3) \rightarrow -2(2) + (3) \Rightarrow \\ & \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 8 & -14 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The rank of matrix \mathbf{A} is 2 which is equal to the number of nonzero rows after \mathbf{A} is transformed to row echelon form.

We now consider \mathbf{A}^T .

$$\begin{aligned} \mathbf{A}^T &= \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 2 \\ 4 & 2 & 14 \\ -8 & -3 & -25 \end{bmatrix} && (1) \leftrightarrow (2) \Rightarrow \\ & \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -1 \\ 4 & 2 & 14 \\ -8 & -3 & -25 \end{bmatrix} && (2) \rightarrow 2(1) + (2) \ \& \ (3) \rightarrow -4(1) + (3) \ \& \ (4) \rightarrow 8(1) + (4) \Rightarrow \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & -3 & -9 \end{bmatrix} \quad (3) \rightarrow -2(2) + (3) \ \& \ (4) \rightarrow 3(2) + (4) \quad \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of matrix \mathbf{A}^T is 2 which is equal to the number of nonzero rows after \mathbf{A}^T is transformed to row echelon form. As expected rank of \mathbf{A} is the same as rank of \mathbf{A}^T .

Evaluating Rank of \mathbf{A} Using Determinant of \mathbf{A}

Let \mathbf{A} be an $m \times n$ matrix. The rank of \mathbf{A} is k , if and only if there exists at least one $k \times k$ submatrix of \mathbf{A} whose determinant is not zero, while every square submatrix of \mathbf{A} with the order greater than k has a determinant equal to zero. This implies that rank of an $m \times m$ square matrix \mathbf{A} is m , if $\det \mathbf{A}$ is not zero. In the previous two examples, it was noted that

$$\text{rank } \mathbf{A} = \text{rank} \begin{bmatrix} -2 & 1 & 4 & -8 \\ 1 & 0 & 2 & -3 \\ -1 & 2 & 14 & -25 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 8 & -14 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2$$

Rank of a matrix is the same as the rank of row echelon form of that matrix. It is true that row operation changes the determinant of submatrices of a matrix to a zero value. Hence when an entire row in a row echelon form becomes zero such as in the example shown, the rank of the original Matrix is one less than the highest order. If two rows row in a row echelon form become zero, the rank of the original Matrix is two less than the highest order and so on.

Example 15: Evaluate rank \mathbf{A} .

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 3 & -3 \end{bmatrix}$$

$$\det \mathbf{A} = 1(-3-3) - (-1)(-9-1) + 2(9-1) = -6 - 10 + 16 = 0 \quad \Rightarrow \quad \text{rank } \mathbf{A} \neq 3$$

Consider the 2×2 submatrix $\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$ of \mathbf{A} and note $\det \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} \neq 0$. Hence $\text{rank } \mathbf{A} = 2$

4.5 – Inverse of a Matrix

Given an $n \times n$ matrix \mathbf{A} , another $n \times n$ matrix \mathbf{B} , and an $n \times n$ identity matrix \mathbf{I} such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \quad (4.35)$$

then matrix \mathbf{B} is defined as the inverse of matrix \mathbf{A} and is denoted by \mathbf{A}^{-1} . Hence we write

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad (4.36)$$

Matrix \mathbf{A} is said to be nonsingular if its inverse exist and singular if matrix \mathbf{A} does not have an inverse. The term -1 in \mathbf{A}^{-1} should not be mistaken for power or exponent. It is simply a notation. The inverse of an $m \times m$ matrix exists, if and only if the rank of the matrix is m or $\det \mathbf{A} \neq 0$. This is based on our discussion in the previous section of rank of \mathbf{A} and using determinant of a matrix to evaluate it. We now present two methods for evaluating \mathbf{A}^{-1} .

Evaluating \mathbf{A}^{-1} Using Row Operations-Reductions

This method is based on Section 4.3 and is very much similar to Gaussian elimination method. The objective is to evaluate \mathbf{B} such that $\mathbf{AB} = \mathbf{I}$. Since the unknown is \mathbf{B} , we set up an $m \times 2m$ augmented matrix consisting of the known matrix \mathbf{A} and the known identity matrix \mathbf{I} as shown below.

$$[\mathbf{A} | \mathbf{I}]$$

Using row operations and reductions as described in Section 4.3, we transform the augmented matrix to

$$[\mathbf{I} | \mathbf{C}]$$

The Matrix \mathbf{C} is the same as the unknown matrix \mathbf{B} which is \mathbf{A}^{-1} . Let us consider an example.

Example 16: Evaluate \mathbf{A}^{-1} .

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 3 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & 0 & 1 \end{array} \right] \quad (1) \leftrightarrow (3) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} -2 & 0 & -1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & 1 & 0 & 0 \end{array} \right] \quad (1) \rightarrow -\frac{1}{2}(1) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 0 & 0 & -1/2 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & 1 & 0 & 0 \end{array} \right] \quad (2) \rightarrow -3(1) + (2) \ \& \ (3) \rightarrow -2(1) + (3) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 0 & 0 & -1/2 \\ 0 & 2 & -1/2 & 0 & 1 & 3/2 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array} \right] \quad (3) \rightarrow \frac{1}{2}(2) + (3) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 0 & 0 & -1/2 \\ 0 & 2 & -1/2 & 0 & 1 & 3/2 \\ 0 & 0 & 3/4 & 1 & 1/2 & 7/4 \end{array} \right] \quad (2) \rightarrow \frac{1}{2}(2) \ \& \ (3) \rightarrow \frac{4}{3}(3) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 0 & 0 & -1/2 \\ 0 & 1 & -1/4 & 0 & 1/2 & 3/4 \\ 0 & 0 & 1 & 4/3 & 2/3 & 7/3 \end{array} \right] \quad (2) \rightarrow \frac{1}{4}(3) + (2) \ \& \ (1) \rightarrow -\frac{1}{2}(3) + (1) \quad \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -1/3 & -5/3 \\ 0 & 1 & 0 & 1/3 & 2/3 & 4/3 \\ 0 & 0 & 1 & 4/3 & 2/3 & 7/3 \end{array} \right] \quad \Rightarrow$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -2/3 & -1/3 & -5/3 \\ 1/3 & 2/3 & 4/3 \\ 4/3 & 2/3 & 7/3 \end{bmatrix}$$

This answer can be easily verified by confirming $\mathbf{AA}^{-1} = \mathbf{I}$ or $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Evaluating \mathbf{A}^{-1} Using Determinant and Adjoint of \mathbf{A}

Before evaluating the inverse of a matrix, we define adjoint of a matrix as

$$\text{Adj } \mathbf{A} = [\text{Cof } \mathbf{A}]^T = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} \quad (4.37)$$

Cofactor C_{ij} of the element a_{ij} was defined in Section 4.2. The method evaluating \mathbf{A}^{-1} is presented in the following theorem.

Theorem 4.1

Let \mathbf{A} be an $n \times n$ nonsingular matrix as shown

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Then \mathbf{A}^{-1} is

$$\mathbf{A}^{-1} = \frac{[\text{Cof } \mathbf{A}]^T}{|\mathbf{A}|} = \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} \quad (4.38)$$

Proof:

The proof of this theorem is based on using the procedure for row expansion of equation (4.21) in Section 4.2 in evaluating determinant as well as recognizing that sum of product of elements of any row of \mathbf{A} and the corresponding cofactors of any other row of \mathbf{A} is zero. In other words we have

$$\begin{aligned} a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n} &= |\mathbf{A}| \\ a_{21}C_{21} + a_{22}C_{22} + \dots + a_{2n}C_{2n} &= |\mathbf{A}| \\ \dots & \\ a_{n1}C_{n1} + a_{n2}C_{n2} + \dots + a_{nn}C_{nn} &= |\mathbf{A}| \end{aligned} \quad (4.39)$$

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = 0 \quad i \neq j \quad i = 1, 2, \dots, n \ \& \ j = 1, 2, \dots, n \quad (4.40)$$

As indicated, equation (4.39) is simply determinant of \mathbf{A} using row expansion. Clearly any of the n -rows can be used. Equation (4.40) is true because in evaluating $C_{j1}, C_{j2}, \dots, C_{jn}$ the elements of the i -th row of \mathbf{A} ($a_{i1}, a_{i2}, \dots, a_{in}$) are utilized again. Since $a_{i1}, a_{i2}, \dots, a_{in}$ are also the coefficients in equation (4.40), this is simply the same as finding the determinant of a matrix with two identical rows. The determinant of a matrix with two identical row is zero. Using equations (4.39) and (4.40), we can write

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} = \begin{bmatrix} |\mathbf{A}| & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & |\mathbf{A}| \end{bmatrix} \quad (4.41)$$

It is observed that \mathbf{A} is post multiplied by adjoint of \mathbf{A} or $[\text{Cof } \mathbf{A}]^T$ in equation (4.41). Hence we have

$$\mathbf{A} \text{Adj } \mathbf{A} = |\mathbf{A}| \mathbf{I} \quad \Rightarrow$$

$$\mathbf{A} \left[\frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} \right] = \mathbf{I} \quad (4.42)$$

Equation (4.42) implies

$$\mathbf{A}^{-1} = \frac{[\text{Cof } \mathbf{A}]^T}{|\mathbf{A}|} = \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|}$$

Example 17: Evaluate \mathbf{A}^{-1} .

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 3 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

$$|\mathbf{A}| = -2(-1 - 4) - 1(4 + 3) = 3$$

$$\text{Adj } \mathbf{A} = [\text{Cof } \mathbf{A}]^T = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -2 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 4 \\ -1 & 2 & 2 \\ -5 & 4 & 7 \end{bmatrix}^T$$

$$\text{Adj } \mathbf{A} = \begin{bmatrix} -2 & -1 & -5 \\ 1 & 2 & 4 \\ 4 & 2 & 7 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{[\text{Cof } \mathbf{A}]^T}{|\mathbf{A}|} = \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = (1/3) \begin{bmatrix} -2 & -1 & -5 \\ 1 & 2 & 4 \\ 4 & 2 & 7 \end{bmatrix} = \begin{bmatrix} -2/3 & -1/3 & -5/3 \\ 1/3 & 2/3 & 4/3 \\ 4/3 & 2/3 & 7/3 \end{bmatrix}$$

Once again the answer can be easily verified by confirming $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ or $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

In many applications one is required to evaluate inverse of a 2x2 matrix. The method of determinant and adjoint is now applied for this particular case.

$$\mathbf{A}^{-1} = \frac{[\text{Cof } \mathbf{A}]^T}{|\mathbf{A}|} = \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}^T$$

$$\mathbf{A}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \quad (4.43)$$

In summary, the inverse of a 2x2 matrix is obtained by changing the position of diagonal elements, changing the sign of non-diagonal elements, and dividing each element by the determinant of the matrix.

Example 18: Evaluate \mathbf{A}^{-1} .

$$\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix}$$

$$|\mathbf{A}| = -6 + 4 = -2$$

$$\mathbf{A}^{-1} = \left(\frac{1}{-2}\right) \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1/2 & -3/2 \end{bmatrix}$$

Some Properties of Inverse of a Matrix

a) Inverse of a nonsingular diagonal matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/a_{11} & 0 & \dots & 0 \\ 0 & 1/a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/a_{nn} \end{bmatrix} \quad (4.44)$$

b) Inverse of an inverse

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A} \quad (4.45)$$

c) Inverse of a product

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad (4.46)$$

This property can be extended to n matrices.

$$(\mathbf{A}_1\mathbf{A}_2 \dots \mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1} \dots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1} \quad (4.47)$$

4.6 – Solving Linear System of Equations Using Inverse of a Matrix and Cramer's Rule

A linear set of n equations with n unknown x_1, x_2, \dots, x_n and known constant coefficients a_{ij} ($i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$) and known constants b_i ($i = 1, 2, \dots, n$) is described by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots & \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad (4.48)$$

Equation (4.48) can be written as

$$\mathbf{AX} = \mathbf{B} \quad (4.49)$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (4.50)$$

Pre-multiplying both sides of equation (4.49), we have

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \quad (4.51)$$

Theorem 4.2

Given $\mathbf{AX} = \mathbf{B}$ as in equation (4.49) and if $\det \mathbf{A} = |\mathbf{A}| \neq 0$, the unknown variables x_1, x_2, \dots, x_n is given by

$$x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|}, x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|}, \dots, x_n = \frac{|\mathbf{A}_n|}{|\mathbf{A}|} \quad (4.52)$$

where

$$\mathbf{A}_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}, \quad \dots \dots \dots$$

$$\mathbf{A}_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{bmatrix} \quad (4.53)$$

Proof:

The proof of this theorem is based on using equation (4.51) and utilizing the procedure outlined in the previous section in evaluating inverse of a matrix.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (4.54)$$

Setting the two sides of equation (4.54) equal and remembering column expansion for evaluating determinants of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ we have

$$x_1 = \frac{1}{|\mathbf{A}|} [b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}] = \frac{|\mathbf{A}_1|}{|\mathbf{A}|}$$

$$x_2 = \frac{1}{|\mathbf{A}|} [b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2}] = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} \quad (4.55)$$

.....

$$x_n = \frac{1}{|\mathbf{A}|} [b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn}] = \frac{|\mathbf{A}_n|}{|\mathbf{A}|}$$

Example 19: Solve the system of equations shown using

a) Inverse – equation (4.51)

b) Cramer's rule – equation (4.52)

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 - x_3 = 5$$

$$x_1 - 2x_2 + 2x_3 = -9$$

$$\text{a) } \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -9 \end{bmatrix} \quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} \mathbf{B} \quad \Rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 5 \\ -9 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -2 & -5 & -4 \\ -4 & 1 & 3 \\ -1 & 3 & -2 \end{bmatrix}^T \begin{bmatrix} 0 \\ 5 \\ -9 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -2 & -4 & -1 \\ -5 & 1 & 3 \\ -4 & 3 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

b) $|\mathbf{A}| = -11$

$$x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \left(\frac{1}{-11} \right) \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & -1 \\ -9 & -2 & 2 \end{vmatrix} = \left(\frac{1}{-11} \right) [1(10 - 9) - 1(-10)] = 1$$

$$x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \left(\frac{1}{-11} \right) \begin{vmatrix} 1 & 0 & 1 \\ 2 & 5 & -1 \\ 1 & -9 & 2 \end{vmatrix} = \left(\frac{1}{-11} \right) [1(10 - 9) + 1(-18 - 5)] = 2$$

$$x_3 = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \left(\frac{1}{-11} \right) \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & -2 & -9 \end{vmatrix} = \left(\frac{1}{-11} \right) [1(10) - 1(-18 - 5)] = -3$$

Using Theorem 4.2, it can be observed if equation (4.48) is homogeneous and $\det \mathbf{A} \neq 0$, the solution is simply $x_1 = 0, x_2 = 0, \dots, x_n = 0$. Furthermore, if equation (4.48) is homogeneous and $\det \mathbf{A} = 0$, we have nontrivial solutions.

4.7 – Orthogonal Matrices

Having defined inverse of a matrix, we can now define orthogonal matrices. Matrix \mathbf{A} is orthogonal if inverse of \mathbf{A} is equal to transpose of \mathbf{A}

$$\mathbf{A}^T = \mathbf{A}^{-1} \quad (4.56)$$

An example of an orthogonal matrix is

$$\mathbf{A} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$$

where

$$\mathbf{A}^T = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \frac{1}{0.36+0.64} \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$$

The determinant of \mathbf{A} is 1. This is not coincidental. The determinant of an orthogonal matrix is 1 or -1 . To show this, we write

$$|\mathbf{A}\mathbf{A}^{-1}| = |\mathbf{I}| = 1$$

Since $\mathbf{A}^T = \mathbf{A}^{-1}$

$$|\mathbf{A}\mathbf{A}^T| = 1$$

Referring to the properties of determinant; $|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$ and $|\mathbf{A}| = |\mathbf{A}^T|$ we write

$$|\mathbf{A}\mathbf{A}^T| = |\mathbf{A}| \cdot |\mathbf{A}^T| = (|\mathbf{A}|)^2 = 1 \quad \Rightarrow \quad |\mathbf{A}| = \pm 1$$

CHAPTER 5

Linear Algebra

Eigenvalue-Eigenvector

State Variable Equations

Overview

Section 5.1 starts with the discussion of eigenvalues and eigenvectors. Similarity transformation and diagonalization is addressed in Section 5.2. Section 5.3 explains bilinear and quadratic form. As application to linear algebra, Section 5.3 introduces state variable equations and different realization block diagrams. Solution of state variable equations in time-domain and in s-domain are covered in Sections 5.5 and 5.6 respectively. The idea and details of linear transformation and diagonalization applied to state variable equations is presented in Section 5.7.

$\mathbf{A}\mathbf{X} = \lambda\mathbf{X}$ which is the same as equation (5.3) and remembering the complex conjugate of a product is the product of complex conjugate.

For repeated eigenvalue of matrix \mathbf{A} , we define algebraic multiplicity and geometric multiplicity as follows. Algebraic multiplicity of eigenvalue λ is multiplicity of λ . Geometric multiplicity of repeated eigenvalue λ is the number of linearly independent eigenvectors corresponding to λ .

Using equation (5.5), it can be shown the eigenvalues of a diagonal or triangular matrix are the diagonal elements (see example 4 on page 129).

From the properties of determinant discussed in Chapter 4 ($\det \mathbf{A} = \det \mathbf{A}^T$), we can write the eigenvalues of a matrix \mathbf{A} is the same as the eigenvalues of transpose of \mathbf{A} .

Using equation (5.3), we have

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{X}$$

Pre-multiplying both side by \mathbf{A}^{-1}

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{X} = \lambda\mathbf{A}^{-1}\mathbf{X} \quad \Rightarrow$$

$$\mathbf{A}^{-1}\mathbf{X} = (1/\lambda)\mathbf{X} \quad (5.6)$$

Equation (5.6) implies that the eigenvalues of the inverse of a nonsingular matrix \mathbf{A} is the inverse of the eigenvalues of \mathbf{A} . However the corresponding eigenvectors are the same.

Example 1: Evaluate the eigenvalues and eigenvectors of \mathbf{A}

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} 2 - \lambda & 1 \\ 3 & 4 - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - 6\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = 1, 5$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{0}$$

$$\begin{bmatrix} 2 - \lambda & 1 \\ 3 & 4 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow$$

$$\begin{aligned} (2 - \lambda)x_1 + x_2 &= 0 \\ 3x_1 + (4 - \lambda)x_2 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= 1 \\ x_1 + x_2 &= 0 \\ 3x_1 + 3x_2 &= 0 \end{aligned}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \lambda_2 &= 5 \\ -3x_1 + x_2 &= 0 \\ 3x_1 - x_2 &= 0 \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

Example 2: Evaluate the eigenvalues and eigenvectors of **A**

$$\mathbf{A} = \begin{bmatrix} -2 & -1 \\ 5 & -4 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} -2 - \lambda & -1 \\ 5 & -4 - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 6\lambda + 13 = 0 \quad \Rightarrow \quad \lambda = -3 \pm i2$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 - \lambda & -1 \\ 5 & -4 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} (-2 - \lambda)x_1 - x_2 &= 0 \\ 5x_1 + (-4 - \lambda)x_2 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= -3 + i2 \\ (1 - i2)x_1 - x_2 &= 0 \\ 5x_1 + (-1 - i2)x_2 &= 0 \end{aligned}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 - i2 \end{bmatrix}$$

$$\begin{aligned} \lambda_2 &= -3 - i2 \\ (1 + i2)x_1 - x_2 &= 0 \\ 5x_1 + (-1 + i2)x_2 &= 0 \end{aligned}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 + i2 \end{bmatrix}$$

As discussed above, the eigenvalues and eigenvectors are complex conjugate of one another.

Example 3: Evaluate the eigenvalues and eigenvectors of **A**

$$\mathbf{A} = \begin{bmatrix} -5 & -10 & 9 \\ 2 & 3 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} -5 - \lambda & -10 & 9 \\ 2 & 3 - \lambda & -1 \\ 1 & 2 & -1 - \lambda \end{vmatrix} = 0 \quad \Rightarrow$$

$$(-5 - \lambda)[(3 - \lambda)(-1 - \lambda) + 2] + 10[-2 - 2\lambda + 1] + 9[4 - 3 + \lambda] = 0 \quad \Rightarrow$$

$$\lambda^3 + 3\lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda = 1, -2, -2$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{0}$$

$$\begin{bmatrix} -5 - \lambda & -10 & 9 \\ 2 & 3 - \lambda & -1 \\ 1 & 2 & -1 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow$$

$$(-5 - \lambda)x_1 - 10x_2 + 9x_3 = 0$$

$$2x_1 + (3 - \lambda)x_2 - x_3 = 0$$

$$x_1 + 2x_2 + (-1 - \lambda)x_3 = 0$$

$$\lambda_1 = 1$$

$$-6x_1 - 10x_2 + 9x_3 = 0$$

$$2x_1 + 2x_2 - x_3 = 0$$

$$x_1 + 2x_2 - 2x_3 = 0$$

$$\text{Let } x_1 = 1 \quad \Rightarrow$$

$$-10x_2 + 9x_3 = 6$$

$$2x_2 - x_3 = -2$$

$$2x_2 - 2x_3 = -1$$

$$\text{using any two equations} \quad \Rightarrow$$

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ -3/2 \\ -1 \end{bmatrix}$$

$$\lambda_2 = \lambda_3 = -2$$

$$-3x_1 - 10x_2 + 9x_3 = 0$$

$$2x_1 + 5x_2 - x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$\text{Let } x_1 = 1 \quad \Rightarrow$$

$$-10x_2 + 9x_3 = 3$$

$$5x_2 - x_3 = -2$$

$$2x_2 + x_3 = -1$$

$$\text{using any two equations} \quad \Rightarrow$$

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ -3/7 \\ -1/7 \end{bmatrix}$$

In example 3, the algebraic multiplicity is 2 for $\lambda = -2$. However the geometric multiplicity which is the number of linearly independent eigenvector for $\lambda = -2$ is 1.

Example 4: Evaluate the eigenvalues and eigenvectors of **A**

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} -1-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)(-2-\lambda)(-3-\lambda) = 0 \Rightarrow \lambda = -1, -2, -3$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{0}$$

$$\begin{bmatrix} -1-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} (-1-\lambda)x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + (-2-\lambda)x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + (-3-\lambda)x_3 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= -1 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 - x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 - 2x_3 &= 0 \end{aligned}$$

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \lambda_2 &= -2 \\ x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 - x_3 &= 0 \end{aligned}$$

$$\mathbf{X}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \lambda_3 &= -3 \\ 2x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

$$\mathbf{X}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5.2 – Similarity Transformation and Diagonalization

Given an $n \times n$ nonsingular matrix \mathbf{T} and an $n \times n$ matrix \mathbf{A} , we define an $n \times n$ matrix \mathbf{A}_T to be similar to \mathbf{A} such that

$$\mathbf{A}_T = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad (5.7)$$

Transformation from \mathbf{A} to \mathbf{A}_T is referred to as the similarity transformation. The notation \mathbf{T} is used here to designate transformation matrix.

Theorem 5.1

Eigenvalues of the similar matrix \mathbf{A}_T are the same as those of \mathbf{A} and the corresponding eigenvector of \mathbf{A}_T is $\mathbf{X}_T = \mathbf{T}^{-1}\mathbf{X}$, where \mathbf{X} is the eigenvector of matrix \mathbf{A} .

Proof:

We begin with equation (5.3)

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{X}$$

Pre-multiplying both sides by \mathbf{T}^{-1} results in

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{X} = \lambda\mathbf{T}^{-1}\mathbf{X}$$

Placing an $n \times n$ identity matrix between \mathbf{A} and \mathbf{X} on the left and then writing \mathbf{I} as $\mathbf{T}\mathbf{T}^{-1}$

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{I}\mathbf{X} = \lambda\mathbf{T}^{-1}\mathbf{X}$$

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{T}^{-1}\mathbf{X} = \lambda\mathbf{T}^{-1}\mathbf{X} \quad \Rightarrow$$

$$\mathbf{A}_T\mathbf{X}_T = \lambda\mathbf{X}_T$$

Example 5: Evaluate \mathbf{A}_T for the given \mathbf{A} and \mathbf{T} .

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 1 & -5 \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\mathbf{T}^{-1} = (1/5) \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A}_T = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = (1/5) \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -11 \\ 0 & -2 \end{bmatrix}$$

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad \begin{bmatrix} -1 - \lambda & -3 \\ 1 & -5 - \lambda \end{bmatrix} \Rightarrow \lambda^2 + 6\lambda + 8 = 0 \quad \Rightarrow \lambda = -2, -4$$

$$|\mathbf{A}_T - \lambda \mathbf{I}| = 0 \quad \begin{bmatrix} -4 - \lambda & -11 \\ 0 & -2 - \lambda \end{bmatrix} \Rightarrow (\lambda + 4)(\lambda + 2) = 0 \Rightarrow \lambda = -2, -4$$

Using Theorem 5.1, eigenvectors \mathbf{X}_T of \mathbf{A}_T can be evaluated and it can be verified that $\mathbf{X}_T = \mathbf{T}^{-1}\mathbf{X}$ for the corresponding eigenvalues.

Theorem 5.2 Diagonalization

If an $n \times n$ matrix \mathbf{A} has n distinct eigenvalues and n linearly independent eigenvectors and the columns of the transformation matrix in the equation $\mathbf{A}_T = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ are made up of the eigenvectors of matrix \mathbf{A} , then the resulting \mathbf{A}_T is a diagonal matrix whose diagonal elements are the eigenvalues of \mathbf{A} .

Proof:

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalue of \mathbf{A} and $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ be the corresponding eigenvectors of \mathbf{A} . Using equation (5.3), we have

$$\begin{aligned} \mathbf{A}\mathbf{X}_1 &= \lambda_1\mathbf{X}_1 \\ \mathbf{A}\mathbf{X}_2 &= \lambda_2\mathbf{X}_2 \\ &\dots\dots\dots \\ \mathbf{A}\mathbf{X}_n &= \lambda_n\mathbf{X}_n \end{aligned}$$

These equations can be written as

$$\mathbf{A} \cdot [\mathbf{X}_1 | \mathbf{X}_2 | \dots | \mathbf{X}_n] = [\mathbf{X}_1 | \mathbf{X}_2 | \dots | \mathbf{X}_n] \cdot \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad (5.8)$$

We define \mathbf{T} as

$$\mathbf{T} = [\mathbf{X}_1 | \mathbf{X}_2 | \dots | \mathbf{X}_n] \quad (5.9)$$

Hence equation (5.8) can be written as

$$\mathbf{A}\mathbf{T} = \mathbf{T} \cdot \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad (5.10)$$

Pre-multiplying both sides of (5.10) by \mathbf{T}^{-1} gives

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad (5.11)$$

$$Q = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3 & 0 & 0 \\ 6 & 11 & 0 \\ -14 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$Q = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3 & 3 & -7 \\ 3 & 11 & -1 \\ -7 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

As observed for the first two representations matrix **A** is triangular. In the last case matrix **A** is symmetrical. From equation (5.18) it should be clear that coefficients of $x_1^2, x_2^2, \dots, x_n^2$ are the diagonal elements. And coefficient of $x_i x_j$ ($i \neq j$) are sum of the elements in the i th row j th column and j th row i th column of **A**. As an example coefficient of $x_1 x_2$ is $a_{12} + a_{21}$ which provides infinite possibilities. Hence for triangular representation of **A**, one of these coefficients is assumed zero (either the first or the second element). For symmetrical representation of **A**, half of coefficient of $x_i x_j$ ($i \neq j$) is placed in a_{ij} position and half is placed in a_{ji} position.

5.4 – State Variable Equations

As an application of linear algebra, state variable for continuous time domain system is discussed in this section. As systems become more complex and the use of computers becomes vital in solving these complex systems with multiple inputs and multiple outputs, modeling and solving systems using state variables becomes essential.

The most general form of state variable equation is represented by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (5.20)$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (5.21)$$

$\mathbf{x}(t)$ is an $n \times 1$ state vector consisting of elements $x_1(t), x_2(t), \dots, x_n(t)$, and $\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}}{dt}$ is the corresponding derivative of the state vector. $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ is an $n \times 1$ vector function and $\mathbf{u}(t)$ is an $m \times 1$ input vector consisting of elements $u_1(t), u_2(t), \dots, u_m(t)$. $\mathbf{y}(t)$ is a $p \times 1$ vector consisting of elements $y_1(t), y_2(t), \dots, y_p(t)$ and $\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)$ is an $m \times 1$ vector function. In our discussion we consider the general form of linear time-invariant system of state variable equation described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (5.22)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (5.23)$$

Where

$$\text{State Vector: } \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$$

$$\text{Input Vector: } \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$\text{Output Vector: } \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

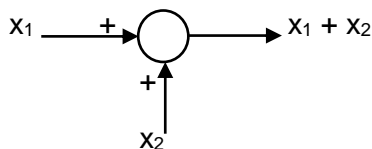
$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix}$$

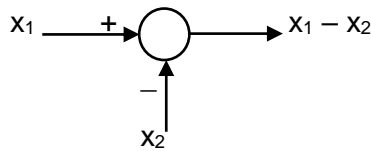
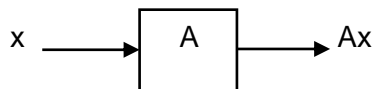
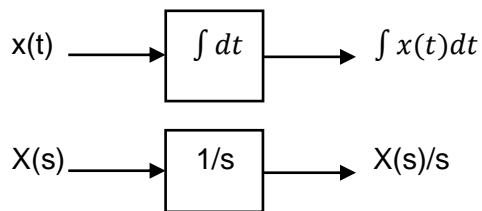
$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pn} \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p1} & d_{p2} & \dots & d_{pm} \end{bmatrix}$$

For the sake of simplicity and without loss of generality we assume single input $u(t)$ in our discussion. There are many methods and possible procedures that can be used to write state variable system of equations for a linear time-invariant system in the form of equations (5.22) and (5.23). The most popular of these are controllable canonical form, observable canonical form, diagonal form, and Jordan canonical form. Diagonal form is for the case when matrix \mathbf{A} has distinct eigenvalues. Diagonalization will also be presented in Section 5.7 when we discuss linear transformation. Jordan canonical form is when matrix \mathbf{A} has repeated eigenvalues. Jordan canonical form of state variable modeling is not discussed in this chapter and is left as an exercise. Hence we begin with state variable modeling using controllable canonical form, observable canonical form, and diagonal form. To do so we first begin by basic components utilized in system realizations and simulations.

Summing Point



ComparatorAmplifier/AttenuatorIntegrator

Integrators are used in place of differentiators for realizations and simulations, since they are less susceptible to noise, especially at higher frequencies.

Controllable Canonical Form

Consider a linear constant coefficient nonhomogeneous n -th order system with input $u(t)$ and output $y(t)$ described by differential equation

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_n \frac{d^n u}{dt^n} + b_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u \quad (5.24)$$

The corresponding system transfer function $H(s) = Y(s)/U(s)$ is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \quad (5.25)$$

We now define $X(s)$ as

$$X(s) = \left[\frac{1}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right] U(s) \quad (5.26)$$

Hence $Y(s)$ can be written as

$$Y(s) = [b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0] X(s) \tag{5.27}$$

Using equation (5.26), we have

$$s^n X(s) = -a_0 X(s) - a_1 s X(s) - \dots - a_{n-1} s^{n-1} X(s) + U(s) \tag{5.28}$$

The block diagram of Equation (5.28) is shown in Figure 5.1.

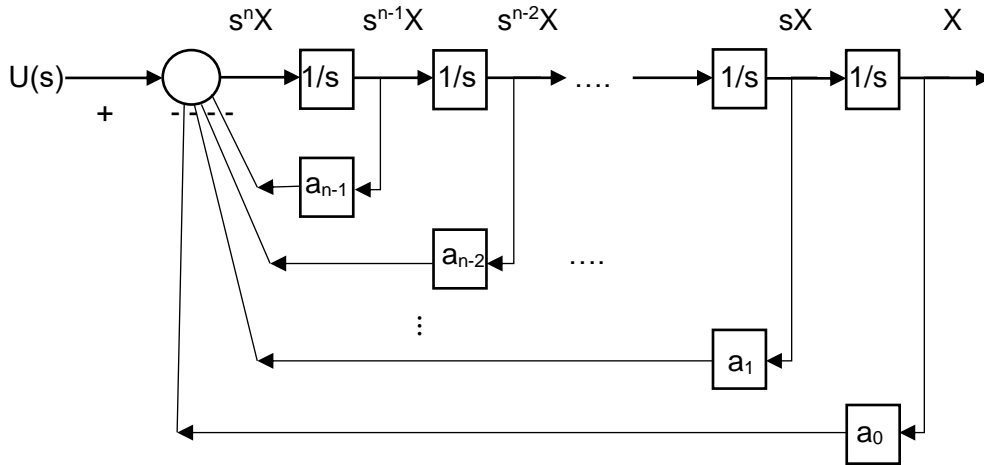


Figure 5.1 Block Diagram Realization of Equations (5.26) and (5.28)

Using equation (5.27), the simulation diagram of the entire system as defined by equations (5.24) and (5.25) can shown as

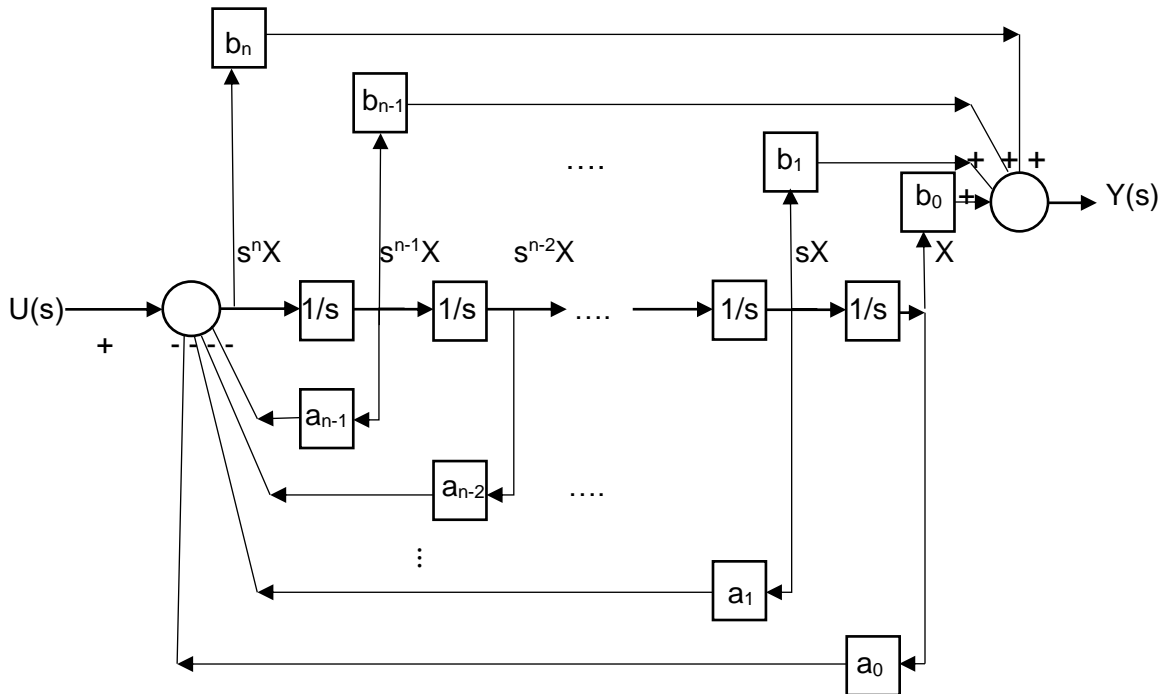


Figure 5.2 Block Diagram Realization of Equations (5.24) and (5.25)

Using the realization diagram of Figure 5.2, we define

$$\begin{aligned}
 X_1(s) &= X(s) \\
 sX_1(s) &= X_2(s) = sX(s) \\
 sX_2(s) &= X_3(s) = s^2X(s) \\
 &\dots\dots\dots \\
 sX_{n-1}(s) &= X_n(s) = s^{n-1}X(s) \\
 sX_n(s) &= -a_0X_1(s) - a_1X_2(s) - \dots - a_{n-1}X_n(s) + U(s)
 \end{aligned} \tag{5.29}$$

and based on the above equation and the realization diagram we write

$$Y(s) = b_0X_1(s) + b_1X_2(s) + \dots + b_{n-1}X_n(s) + b_n sX_n(s) \tag{5.30}$$

We now substitute $sX_n(s)$ from (5.29)

$$\begin{aligned}
 Y(s) &= b_0X_1(s) + b_1X_2(s) + \dots + b_{n-1}X_n(s) + b_n[-a_0X_1(s) - a_1X_2(s) - \dots - a_{n-1}X_n(s) + U(s)] \\
 Y(s) &= (b_0 - a_0b_n)X_1(s) + (b_1 - a_1b_n)X_2(s) + \dots + (b_{n-1} - a_{n-1}b_n)X_n(s) + b_nU(s)
 \end{aligned} \tag{5.31}$$

Outputs of integrators from the right side are defined as X_1, X_2, \dots, X_n with inputs sX_1, sX_2, \dots, sX_n respectively. Converting equations (5.29) and (5.31) from s-domain to time domain gives

$$\begin{aligned}
 \dot{x}_1(t) &= x_2(t) \\
 \dot{x}_2(t) &= x_3(t) \\
 &\dots\dots\dots \\
 \dot{x}_{n-1}(t) &= x_n(t) \\
 \dot{x}_n(t) &= -a_0x_1(t) - a_1x_2(t) - \dots - a_{n-1}x_n(t) + u(t) \\
 y(t) &= (b_0 - a_0 b_n)x_1(t) + (b_1 - a_1 b_n)x_2(t) + \dots + (b_{n-1} - a_{n-1} b_n)x_n(t) + b_n u(t)
 \end{aligned} \tag{5.32}$$

These equations can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t) \tag{5.33}$$

$$y(t) = [b_0 - a_0 b_n \quad b_1 - a_1 b_n \quad \dots \quad b_{n-1} - a_{n-1} b_n] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + b_n u(t) \tag{5.34}$$

It is important to note the output of integrators are the state variables x_1, x_2, \dots, x_n . In most cases the degree of the numerator of the system transfer function is less than the degree of the denominator. This means in such cases $b_n = 0$. This simplifies the state variable equations.

Example 8: Obtain the state variable representation of the given system in controllable canonical form.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 5s^2 + 10}{s^3 + 6s^2 + 11s + 6}$$

Either the system transfer function or the corresponding system differential equation can be used for this purpose. The system differential equation is

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 2\frac{d^3u}{dt^3} + 5\frac{d^2u}{dt^2} + 10u$$

By inspection $a_0 = 6$, $a_1 = 11$, $a_2 = 6$, $b_3 = 2$, $b_2 = 5$, $b_1 = 0$, $b_0 = 10$. Using equation (5.33) and (5.34) we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [10 - (6)(2) \quad 0 - (11)(2) \quad 5 - (6)(2)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u(t)$$

$$y(t) = [-2 \quad -22 \quad -7] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u(t)$$

Example 9: Obtain the state variable representation of the given system in controllable canonical form and draw the realization block diagram.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s+10}{s^2+6s+25}$$

The system differential equation is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = \frac{du}{dt} + 10u$$

By inspection $a_0 = 25$, $a_1 = 6$, $b_2 = 0$, $b_1 = 1$, $b_0 = 10$. Using equation (5.33) and (5.34) we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [10 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u(t)$$

The realization block diagram is shown in Figure 5.3 below.

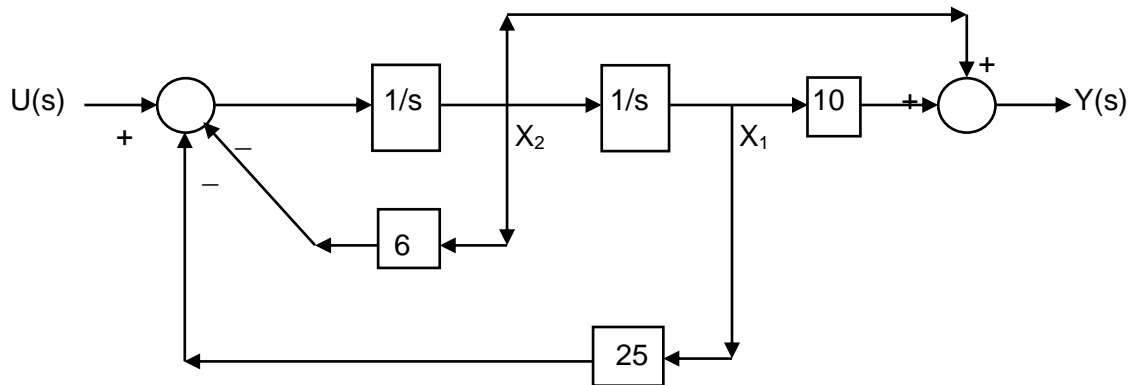


Figure 5.3 Block Diagram Realization of Example 9

It is important to point out that the realization block diagram can be presented in s-domain or time-domain. Figure 5.4 shows the realization diagram of example 9 in time-domain.

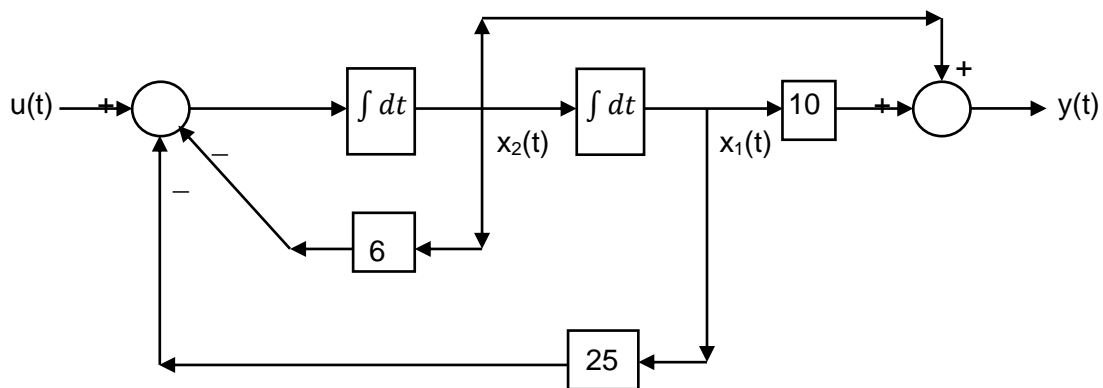


Figure 5.4 Block Diagram Realization of Example 9

Observable Canonical Form

Using the system transfer function of equation (5.25)

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

we can write

$$[s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0] Y(s) = [b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0] U(s)$$

$$s^n Y(s) = [b_n s^n U(s)] + [b_{n-1} U(s) - a_{n-1} Y(s)] s^{n-1} + \dots + [b_1 U(s) - a_1 Y(s)] s + [b_0 U(s) - a_0 Y(s)]$$

Dividing both sides by s^n results in

$$Y(s) = [b_n U(s)] + [b_{n-1} U(s) - a_{n-1} Y(s)]s^{-1} + \dots + [b_1 U(s) - a_1 Y(s)]s^{-n+1} + [b_0 U(s) - a_0 Y(s)]s^{-n} \tag{5.35}$$

Using n integrators, summing points, and amplifiers, the block diagram of equation (5.35) is shown in Figure 5.5.

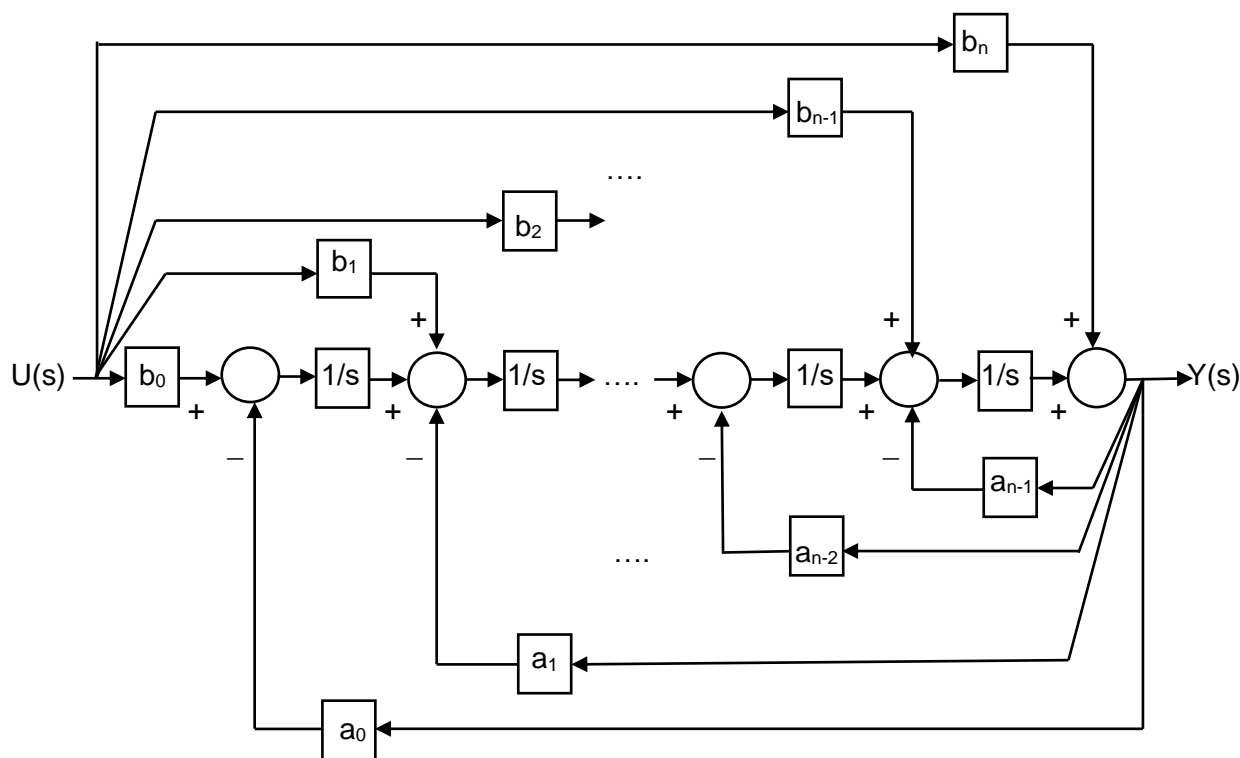


Figure 5.5 Block Diagram Realization of Equation (5.35)

Using the realization diagram of Figure 5.5, we define the outputs of integrators starting from the left as X_1, X_2, \dots, X_n . The input of the integrators are $sX_1(s), sX_2(s), \dots, sX_n(s)$ respectively. Hence we have

$$\begin{aligned} sX_1(s) &= b_0 U(s) - a_0 Y(s) \\ sX_2(s) &= X_1(s) + b_1 U(s) - a_1 Y(s) \\ &\dots \dots \dots \end{aligned} \tag{5.36}$$

$$\begin{aligned} sX_{n-1}(s) &= X_{n-2}(s) + b_{n-2} U(s) - a_{n-2} Y(s) \\ sX_n(s) &= X_{n-1}(s) + b_{n-1} U(s) - a_{n-1} Y(s) \end{aligned}$$

$$Y(s) = X_n(s) + b_n U(s) \tag{5.37}$$

Substituting equation (5.37) in equation (5.36) results

$$\begin{aligned}
sX_1(s) &= b_0U(s) - a_0[X_n(s) + b_nU(s)] = -a_0X_n(s) + [b_0 - a_0b_n]U(s) \\
sX_2(s) &= X_1(s) + b_1U(s) - a_1[X_n(s) + b_nU(s)] = X_1(s) - a_1X_n(s) + [b_1 - a_1b_n]U(s) \\
&\dots\dots\dots \\
sX_{n-1}(s) &= X_{n-2}(s) + b_{n-2}U(s) - a_{n-2}[X_n(s) + b_nU(s)] = X_{n-2}(s) - a_{n-2}X_n(s) + [b_{n-2} - a_{n-2}b_n]U(s) \\
sX_n(s) &= X_{n-1}(s) + b_{n-1}U(s) - a_{n-1}[X_n(s) + b_nU(s)] = X_{n-1}(s) - a_{n-1}X_n(s) + [b_{n-1} - a_{n-1}b_n]U(s)
\end{aligned} \tag{5.38}$$

Converting equations (5.38) and (5.37) from s-domain to time domain gives

$$\begin{aligned}
\dot{x}_1(t) &= -a_0x_n(t) + [b_0 - a_0b_n]u(t) \\
\dot{x}_2(t) &= x_1(t) - a_1x_n(t) + [b_1 - a_1b_n]u(t) \\
&\dots\dots\dots \\
\dot{x}_{n-1}(t) &= x_{n-2}(t) - a_{n-2}x_n(t) + [b_{n-2} - a_{n-2}b_n]u(t) \\
\dot{x}_n(t) &= x_{n-1}(t) - a_{n-1}x_n(t) + [b_{n-1} - a_{n-1}b_n]u(t)
\end{aligned} \tag{5.39}$$

$$y(t) = x_n(t) + b_nu(t)$$

These equations can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_0 - a_0b_n \\ b_1 - a_1b_n \\ \vdots \\ b_{n-2} - a_{n-2}b_n \\ b_{n-1} - a_{n-1}b_n \end{bmatrix} u(t) \tag{5.40}$$

$$y(t) = [0 \quad 0 \quad \dots \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + b_nu(t) \tag{5.41}$$

It is important to note the output of integrators are the state variables x_1, x_2, \dots, x_n . In most cases the degree of the numerator of the system transfer function is less than the degree of the denominator. This means in such cases $b_n = 0$. This simplifies the state variable equations.

Example 10: Obtain the state variable representation of the given system in observable canonical form.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 5s^2 + 10}{s^3 + 6s^2 + 11s + 6}$$

Either the system transfer function or the corresponding system differential equation can be used for this purpose. The system differential equation is

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 2\frac{d^3u}{dt^3} + 5\frac{d^2u}{dt^2} + 10u$$

By inspection $a_0 = 6$, $a_1 = 11$, $a_2 = 6$, $b_3 = 2$, $b_2 = 5$, $b_1 = 0$, $b_0 = 10$. Using equation (5.39) and (5.40) we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 - (6)(2) \\ 0 - (11)(2) \\ 5 - (6)(2) \end{bmatrix} u(t) = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -2 \\ -22 \\ -7 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u(t)$$

Example 11: Obtain the state variable representation of the given system in observable canonical form and draw the realization block diagram.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s+10}{s^2+6s+25}$$

The system differential equation is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = \frac{du}{dt} + 10u$$

By inspection $a_0 = 25$, $a_1 = 6$, $b_2 = 0$, $b_1 = 1$, $b_0 = 10$. Using equation (5.33) and (5.34) we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -25 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u(t)$$

The realization block diagram is shown in Figure 5.6 below

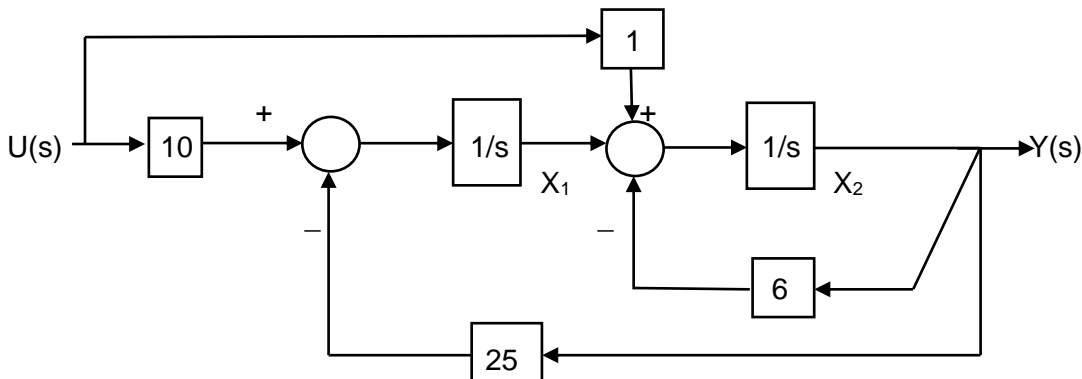


Figure 5.6 Block Diagram Realization of Example 11

Once again the realization block diagram can be presented in s-domain or time-domain. Figure 5.7 shows the realization diagram of example 11 in time-domain.

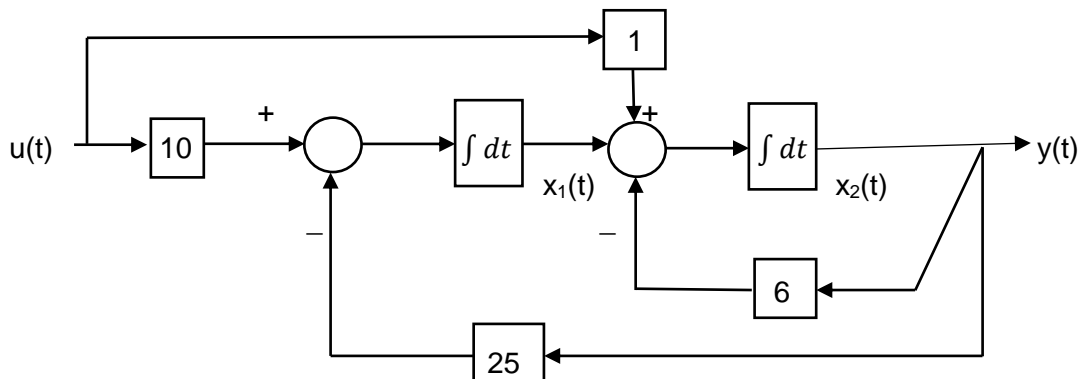


Figure 5.7 Block Diagram Realization of Example 11

Diagonal Form

Using the system transfer function of equation (5.25)

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

In this case we assume the roots of the denominator $-p_1, -p_2, \dots, -p_n$ (poles of the system) are distinct. If $b_n \neq 0$, we need to perform the division first

$$H(s) = \frac{Y(s)}{U(s)} = b_n + \frac{R(s)}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad (5.42)$$

In equation (5.42), $R(s)$ is the remainder polynomial. After performing partial fractions we obtain

$$H(s) = \frac{Y(s)}{U(s)} = b_n + \frac{C_1}{s+p_1} + \frac{C_2}{s+p_2} + \dots + \frac{C_n}{s+p_n} \quad (5.43)$$

Clearly if $b_n = 0$, no division is required and only partial fractions must be performed. Using equation (5.43) we have

$$Y(s) = b_n U(s) + \frac{C_1}{s+p_1} U(s) + \frac{C_2}{s+p_2} U(s) + \dots + \frac{C_n}{s+p_n} U(s) \quad (5.44)$$

We define state variables X_1, X_2, \dots, X_n as

$$X_1(s) = \frac{1}{s+P_1}U(s), X_2(s) = \frac{1}{s+P_2}U(s), \dots, X_n(s) = \frac{1}{s+P_n}U(s) \quad (5.45)$$

Hence from equation (5.44) we write

$$Y(s) = b_nU(s) + C_1X_1(s) + C_2X_2(s) + \dots + C_nX_n(s) \quad (5.46)$$

Before presenting the complete block diagram realization of equation (5.43), let us consider realization of

$$X_1(s) = \frac{1}{s+P_1}U(s) \quad \Rightarrow \quad sX_1(s) = -P_1X_1(s) + U(s) \quad (5.47)$$

The block diagram realization of equation (5.47) is shown in Figure (5.8) below

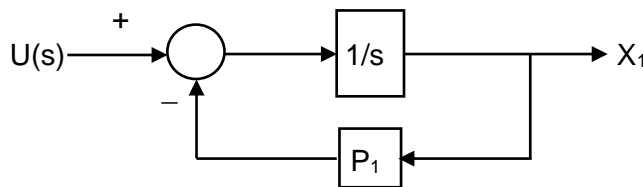


Figure 5.8 Block Diagram Realization of Equation (5.47)

Using n integrators, summing points, and amplifiers, and Figure (5.8), the complete realization block diagram of equation (5.44) is shown in Figure 5.9.

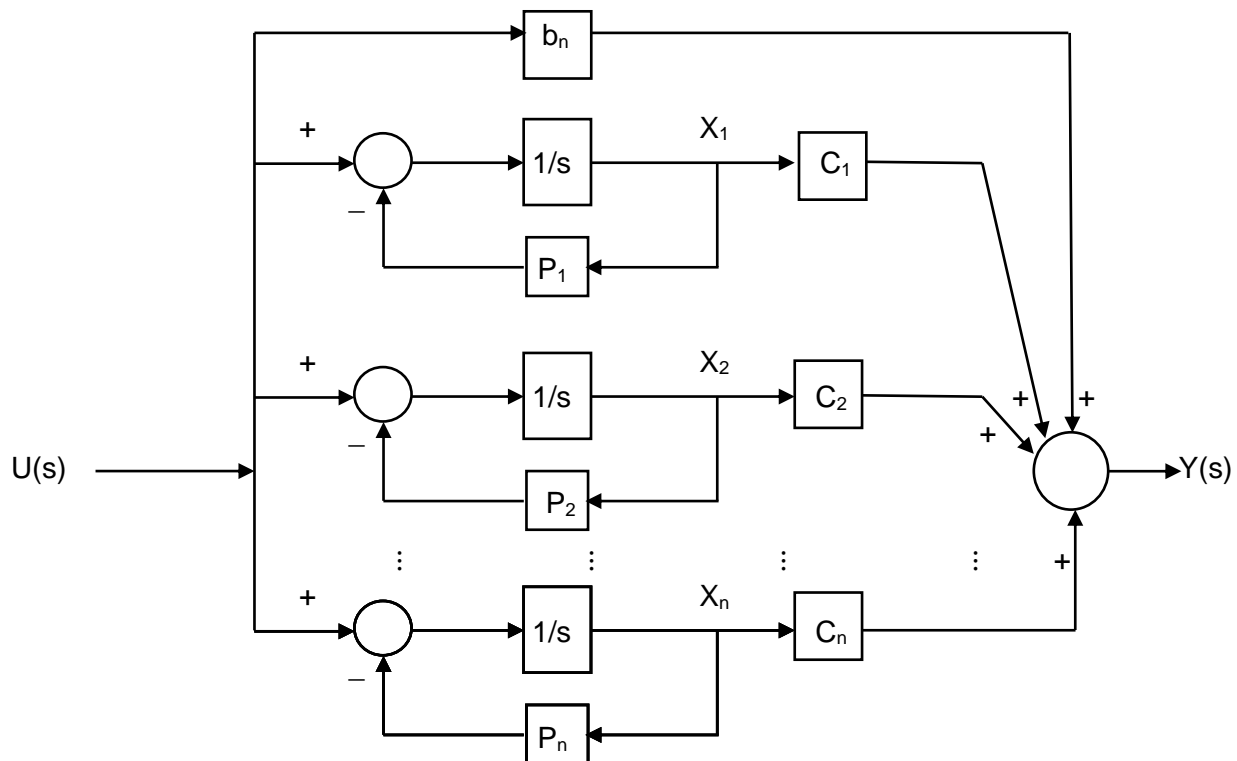


Figure 5.9 Block Diagram Realization of Equation (5.43)

Using the equations obtained and the realization diagram of Figure 5.9 we have

$$\begin{aligned} sX_1(s) &= -P_1X_1(s) + U(s) \\ sX_2(s) &= -P_2X_2(s) + U(s) \\ &\dots\dots\dots \\ sX_n(s) &= -P_nX_n(s) + U(s) \end{aligned} \quad (5.48)$$

$$Y(s) = b_nU(s) + C_1X_1(s) + C_2X_2(s) + \dots + C_nX_n(s) \quad (5.49)$$

Converting equations (5.47) and (5.48) from s-domain to time domain gives

$$\begin{aligned} \dot{x}_1(t) &= -P_1x_1(t) + u(t) \\ \dot{x}_2(t) &= -P_2x_2(t) + u(t) \\ &\dots\dots\dots \\ \dot{x}_n(t) &= -P_nx_n(t) + u(t) \end{aligned} \quad (5.50)$$

$$y(t) = C_1x_1(t) + C_2x_2(t) + \dots + C_nx_n(t) + b_nu(t)$$

These equations can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} -P_1 & 0 & \dots & 0 \\ 0 & -P_2 & & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -P_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t) \quad (5.51)$$

$$y(t) = [C_1 \quad C_2 \quad \dots \quad C_n] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + b_nu(t) \quad (5.52)$$

As before the output of integrators are the state variables x_1, x_2, \dots, x_n and in most cases the degree of the numerator of the system transfer function is less than the degree of the denominator. This means in such cases $b_n = 0$.

Example 12: Obtain the state variable representation of the given system in diagonal form.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{5s^2 + 28s + 35}{s^2 + 5s + 6}$$

After dividing and doing partial fractions we have

$$\frac{Y(s)}{U(s)} = 5 + \frac{-1}{s+2} + \frac{4}{s+3}$$

using equations (5.51) and (5.52) we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [-1 \quad 4] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 5u(t)$$

The realization block diagram is shown in Figure 5.10 below.

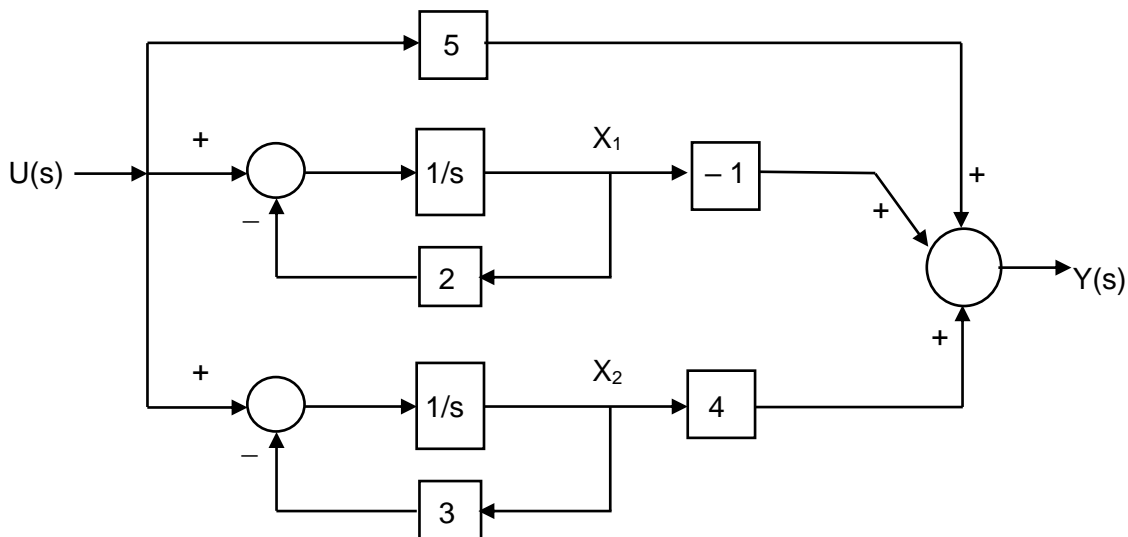


Figure 5.10 Block Diagram Realization of Example 12

Figure 5.11 shows the realization diagram of example 12 in time-domain.

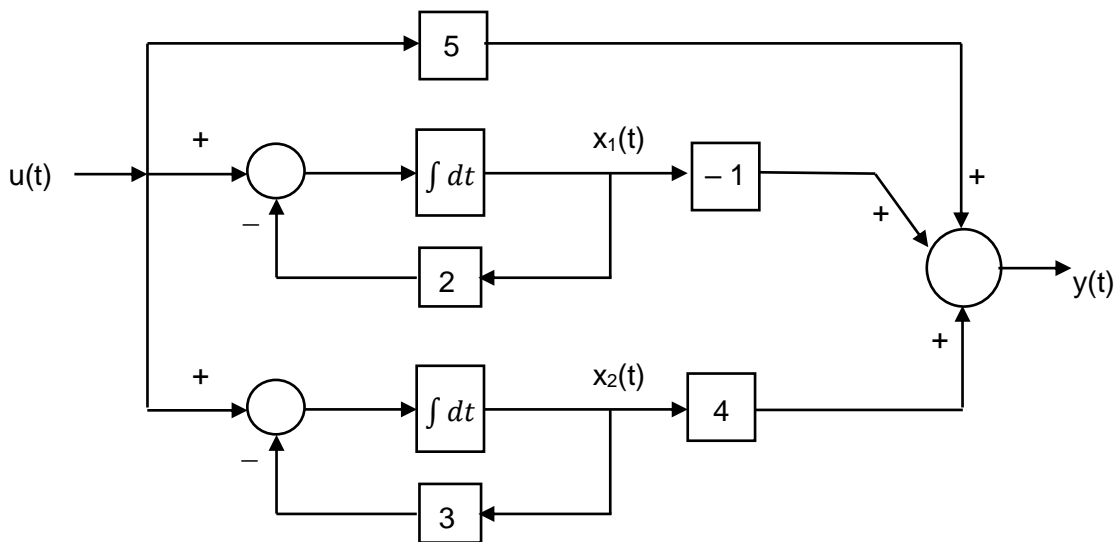


Figure 5.11 Block Diagram Realization of Example 12

5.5 – Solution of State Variable Equations – Time Domain

In the previous section the time invariant state variable equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (5.53)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (5.54)$$

were presented using variety of forms. It was also emphasized the forms introduced and the corresponding realization diagrams are not unique. In this section the solution of state variable equations using time-domain is considered.

Homogeneous Solution

Homogeneous matrix differential equation of equation (5.53) is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad (5.55)$$

Similar to scalar version of equation (5.55), the solution is assumed to be of the form

$$\mathbf{x}(t) = \mathbf{c}_0 + \mathbf{c}_1 t + \mathbf{c}_2 t^2 + \mathbf{c}_3 t^3 + \dots + \mathbf{c}_k t^k + \dots \quad (5.56)$$

Substituting equation (5.56) in equation (5.55) results in

$$\mathbf{c}_1 + 2\mathbf{c}_2 t + 3\mathbf{c}_3 t^2 + \dots + k\mathbf{c}_k t^{k-1} + \dots = \mathbf{A} [\mathbf{c}_0 + \mathbf{c}_1 t + \mathbf{c}_2 t^2 + \dots + \mathbf{c}_k t^k + \dots]$$

Equating coefficients of the like terms and recognizing $\mathbf{x}(0) = \mathbf{c}_0$ from equation (5.56) gives

$$\begin{aligned} \mathbf{c}_1 &= \mathbf{A}\mathbf{c}_0 = \mathbf{A}\mathbf{x}(0) \\ \mathbf{c}_2 &= (1/2)\mathbf{A}\mathbf{c}_1 = (1/2)\mathbf{A}^2\mathbf{x}(0) \\ \mathbf{c}_3 &= (1/3)\mathbf{A}\mathbf{c}_2 = (1/3 \times 2)\mathbf{A}^3\mathbf{x}(0) \\ &\dots\dots\dots \\ \mathbf{c}_k &= (1/k!) \mathbf{A}^k\mathbf{x}(0) \end{aligned} \quad (5.57)$$

Substituting equation (5.57) in equation (5.56)

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}(0) + \mathbf{A}\mathbf{x}(0)t + (1/2)\mathbf{A}^2\mathbf{x}(0)t^2 + (1/3 \times 2)\mathbf{A}^3\mathbf{x}(0)t^3 + \dots + (1/k!) \mathbf{A}^k\mathbf{x}(0)t^k + \dots \\ \mathbf{x}(t) &= [\mathbf{I} + \mathbf{A}t + (1/2!)\mathbf{A}^2t^2 + (1/3!)\mathbf{A}^3t^3 + \dots + (1/k!) \mathbf{A}^k t^k + \dots] \mathbf{x}(0) \end{aligned} \quad (5.58)$$

We define

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + (1/2!)\mathbf{A}^2t^2 + (1/3!)\mathbf{A}^3t^3 + \dots + (1/k!) \mathbf{A}^k t^k + \dots \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \end{aligned} \quad (5.59)$$

The $n \times n$ matrix $e^{\mathbf{A}t}$ is known as state transition matrix and in many applications is denoted by $\phi(t)$. Hence the homogeneous solution of equation (5.55) is

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0) = \boldsymbol{\phi}(t)\mathbf{x}(0) \quad (5.60)$$

From equation (5.59), we note

$$e^{\mathbf{A}0} = e^{\mathbf{0}} = \mathbf{I} \quad (5.61)$$

and

$$\begin{aligned} \frac{de^{\mathbf{A}t}}{dt} &= \mathbf{A} + \mathbf{A}^2t + [1/2!] \mathbf{A}^3t^2 + \dots + [1/(k-1)!] \mathbf{A}^{kt^{k-1}} + \dots \\ &= \mathbf{A} [\mathbf{I} + \mathbf{A}t + (1/2!)\mathbf{A}^2t^2 + (1/3!)\mathbf{A}^3t^3 + \dots + (1/k!) \mathbf{A}^{kt^k} + \dots] \\ &= [\mathbf{I} + \mathbf{A}t + (1/2!)\mathbf{A}^2t^2 + (1/3!)\mathbf{A}^3t^3 + \dots + (1/k!) \mathbf{A}^{kt^k} + \dots] \mathbf{A} \\ &= \mathbf{A} e^{\mathbf{A}t} = e^{\mathbf{A}t} \mathbf{A} \end{aligned} \quad (5.62)$$

Let us consider the product of $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$ using equation (5.59)

$$\begin{aligned} e^{\mathbf{A}t} e^{-\mathbf{A}t} &= e^{-\mathbf{A}t} e^{\mathbf{A}t} \\ &= [\mathbf{I} + \mathbf{A}t + (1/2!)\mathbf{A}^2t^2 + (1/3!)\mathbf{A}^3t^3 + \dots + (1/k!) \mathbf{A}^{kt^k} + \dots] \times \\ &\quad [\mathbf{I} - \mathbf{A}t + (1/2!)\mathbf{A}^2t^2 - (1/3!)\mathbf{A}^3t^3 + \dots + (-1)^k(1/k!) \mathbf{A}^{kt^k} + \dots] \\ &= \mathbf{I} + [\mathbf{A}t - \mathbf{A}t] + [(1/2!)\mathbf{A}^2t^2 + (1/2!)\mathbf{A}^2t^2 - \mathbf{A}^2t^2] + \\ &\quad [(1/3!)\mathbf{A}^3t^3 - (1/3!)\mathbf{A}^3t^3 + (1/2!)\mathbf{A}^3t^3 - (1/2!)\mathbf{A}^3t^3] + \dots \\ &= \mathbf{I} \end{aligned}$$

This indicates $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$ are inverse of each other.

$$\begin{aligned} [e^{\mathbf{A}t}]^{-1} &= e^{-\mathbf{A}t} \\ [e^{-\mathbf{A}t}]^{-1} &= e^{\mathbf{A}t} \end{aligned} \quad (5.63)$$

Now we examine the product of $e^{\mathbf{A}t}$ and $e^{\mathbf{B}t}$ using equation (5.59)

$$\begin{aligned} e^{\mathbf{A}t} e^{\mathbf{B}t} &= [\mathbf{I} + \mathbf{A}t + (1/2!)\mathbf{A}^2t^2 + (1/3!)\mathbf{A}^3t^3 + \dots + (1/k!) \mathbf{A}^{kt^k} + \dots] \times \\ &\quad [\mathbf{I} + \mathbf{B}t + (1/2!)\mathbf{B}^2t^2 + (1/3!)\mathbf{B}^3t^3 + \dots + (1/k!) \mathbf{B}^{kt^k} + \dots] \\ &= \mathbf{I} + [\mathbf{A} + \mathbf{B}]t + [(1/2!)\mathbf{A}^2 + \mathbf{A}\mathbf{B} + (1/2!)\mathbf{B}^2]t^2 + \\ &\quad [(1/3!)\mathbf{A}^3t^3 + (1/2!)\mathbf{A}^2\mathbf{B}t^3 + (1/2!)\mathbf{A}\mathbf{B}^2t^3 + (1/3!)\mathbf{B}^3t^3] + \dots \end{aligned} \quad (5.64)$$

$$\begin{aligned} e^{(\mathbf{A} + \mathbf{B})t} &= [\mathbf{I} + (\mathbf{A} + \mathbf{B})t + (1/2!)(\mathbf{A} + \mathbf{B})^2t^2 + (1/3!)(\mathbf{A} + \mathbf{B})^3t^3 \dots + (1/k!) (\mathbf{A} + \mathbf{B})^{kt^k} + \dots] \\ &= \mathbf{I} + [\mathbf{A} + \mathbf{B}]t + [(1/2!)\mathbf{A}^2 + (1/2!)\mathbf{A}\mathbf{B} + (1/2!)\mathbf{B}\mathbf{A} + (1/2!)\mathbf{B}^2] t^2 + \\ &\quad [(1/3!)\mathbf{A}^3 + (1/3!)\mathbf{A}^2\mathbf{B} + (1/3!)\mathbf{A}\mathbf{B}\mathbf{A} + (1/3!)\mathbf{A}\mathbf{B}^2 + (1/3!)\mathbf{B}\mathbf{A}^2 + (1/3!)\mathbf{B}\mathbf{A}\mathbf{B} + \\ &\quad (1/3!)\mathbf{B}^2\mathbf{A} + (1/3!)\mathbf{B}^3] t^3 + \dots \end{aligned} \quad (5.65)$$

Comparing equations (5.64) and (5.65) implies that

$$e^{(\mathbf{A} + \mathbf{B})t} = e^{\mathbf{A}t} e^{\mathbf{B}t} \quad (5.66)$$

Only if $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$

Nonhomogeneous Solution

Nonhomogeneous matrix differential equation of equation (5.53) is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Pre-multiplying both sides of this equation by $e^{-\mathbf{A}t}$ gives

$$\begin{aligned} e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) &= e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \\ e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) &= e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \\ \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] &= e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \end{aligned}$$

Integrating both sides from 0 to t

$$e^{-\mathbf{A}t}\mathbf{x}(t)|_0^t = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Substituting upper and lower limit on the left hand side and using equation (5.61) results in

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Pre-multiplying both sides by $[e^{-\mathbf{A}t}]^{-1} = e^{\mathbf{A}t}$ gives

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (5.67)$$

Once $\mathbf{x}(t)$ is evaluated using equation (5.67), $\mathbf{y}(t)$ can be obtained by substituting $\mathbf{x}(t)$ in equation (5.54).

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \quad (5.68)$$

To find impulse response, we let $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{u}(t) = \delta(t)$ and hence $\mathbf{y}(t) = \mathbf{h}(t)$.

$$\mathbf{h}(t) = \mathbf{C} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)d\tau + \mathbf{D}\delta(t)$$

$$\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t) \quad t \geq 0 \quad (5.69)$$

The solution of state variable equations as described by equations (5.53) and (5.54) has been obtained. The system impulse response has been evaluated. Some useful properties of state transition matrix (STM) $\phi(t) = e^{\mathbf{A}t}$ have been introduced. Before considering some examples, STM must be evaluated. Here we consider two such methods, one in time-domain and one in s-domain. The s-domain procedure is discussed in Section 5.6. There are several methods that can be used in time domain. Even though possible, it is not advisable to use equation (5.59) to evaluate STM. The Cayley-Hamilton method is introduced in this section to determine STM. Cayley-Hamilton theorem simply states that every nxn matrix satisfies its own characteristic equation. Here the procedure is introduced and examples are given. The proof is left as an exercise.

$$\begin{aligned}
 f(\lambda_1) &= \beta_0 + \beta_1 \lambda_1 + \cdots + \beta_{n-1} \lambda_1^{n-1} \\
 f(\lambda_2) &= \beta_0 + \beta_1 \lambda_2 + \cdots + \beta_{n-1} \lambda_2^{n-1} \\
 &\dots\dots\dots \\
 f(\lambda_{n_1}) &= \beta_0 + \beta_1 \lambda_{n_1} + \cdots + \beta_{n-1} \lambda_{n_1}^{n-1}
 \end{aligned} \tag{5.74}$$

And for eigenvalue λ_i which is repeated n_2 times we use

$$\frac{d^m f(\lambda)}{d\lambda^m} \Big|_{\lambda_i} = \frac{d^m}{d\lambda^m} (\beta_0 + \beta_1 \lambda + \cdots + \beta_{n-1} \lambda^{n-1}) \Big|_{\lambda_i} \quad m = 1, 2, \dots, n_2 - 1 \tag{5.75}$$

Equations (5.74) and (5.75) provide the n independent equations required to solve for $\beta_0, \beta_1, \dots, \beta_{n-1}$. Then β 's are to be substituted in equation (5.71) to evaluate $\mathbf{f}(\mathbf{A}) = e^{\mathbf{A}t}$. It is important to note that the repeated eigenvalue λ_i is used one time in equation (5.74) and the remaining $n_2 - 1$ times in equation (5.75).

Now let us consider some examples.

Example 14: Evaluate state transition matrix $e^{\mathbf{A}t}$ for the given matrix \mathbf{A} .

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 3 & -5 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} -1 - \lambda & -1 \\ 3 & -5 - \lambda \end{vmatrix} = 0 \quad \lambda^2 + 6\lambda + 8 = 0 \quad \lambda = -2, -4$$

$$e^{\mathbf{A}t} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} \quad \text{Using equation (5.73)}$$

$$e^{\lambda t} = \beta_0 + \beta_1 \lambda$$

$$e^{-2t} = \beta_0 - 2\beta_1$$

$$e^{-4t} = \beta_0 - 4\beta_1$$

Solving for β_0 and β_1 gives

$$\beta_1 = \left(\frac{1}{2}\right)(e^{-2t} - e^{-4t})$$

$$\beta_0 = 2e^{-2t} - e^{-4t}$$

$$e^{\mathbf{A}t} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} = (2e^{-2t} - e^{-4t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{1}{2}\right)(e^{-2t} - e^{-4t}) \begin{bmatrix} -1 & -1 \\ 3 & -5 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} \frac{1}{2}(3e^{-2t} - e^{-4t}) & -\frac{1}{2}(e^{-2t} - e^{-4t}) \\ \frac{3}{2}(e^{-2t} - e^{-4t}) & \frac{1}{2}(-e^{-2t} + 3e^{-4t}) \end{bmatrix}$$

Example 15: Evaluate state transition matrix e^{At} for the given matrix **A**.

$$A = \begin{bmatrix} -2 & -1 \\ 5 & -4 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} -2 - \lambda & -1 \\ 5 & -4 - \lambda \end{vmatrix} = 0 \quad \lambda^2 + 6\lambda + 13 = 0 \quad \lambda = -3 \pm i2$$

$$e^{At} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} \quad \text{Using equation (5.73)}$$

$$e^{\lambda t} = \beta_0 + \beta_1 \lambda$$

$$e^{(-3+i2)t} = \beta_0 + (-3 + i2)\beta_1$$

$$e^{(-3-i2)t} = \beta_0 + (-3 - i2)\beta_1$$

Solving for β_0 and β_1 gives

$$\beta_1 = \left(\frac{1}{i4}\right) e^{-3t} (e^{i2t} - e^{-i2t}) = \left(\frac{1}{2}\right) e^{-3t} \sin 2t$$

$$\begin{aligned} \beta_0 &= e^{(-3+i2)t} - (-3 + i2) \left(\frac{1}{2}\right) e^{-3t} \sin 2t \\ &= e^{-3t} (\cos 2t + i \sin 2t) + \frac{3}{2} e^{-3t} \sin 2t - i e^{-3t} \sin 2t \\ &= e^{-3t} \cos 2t + \frac{3}{2} e^{-3t} \sin 2t \end{aligned}$$

$$e^{At} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} = (e^{-3t} \cos 2t + \frac{3}{2} e^{-3t} \sin 2t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{1}{2}\right) (e^{-3t} \sin 2t) \begin{bmatrix} -2 & -1 \\ 5 & -4 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} (e^{-3t} \cos 2t + \frac{1}{2} e^{-3t} \sin 2t) & -\frac{1}{2} e^{-3t} \sin 2t \\ \frac{5}{2} e^{-3t} \sin 2t & (e^{-3t} \cos 2t - \frac{1}{2} e^{-3t} \sin 2t) \end{bmatrix}$$

Example 16: Evaluate state transition matrix e^{At} for the given matrix **A**.

$$A = \begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} -3 - \lambda & -1 \\ 1 & -5 - \lambda \end{vmatrix} = 0 \quad \lambda^2 + 8\lambda + 16 = 0 \quad \lambda = -4, -4$$

$$e^{At} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} \quad \text{Using equations (5.74) and (5.75)}$$

$$e^{\lambda t} = \beta_0 + \beta_1 \lambda$$

$$e^{-4t} = \beta_0 - 4\beta_1$$

$$\frac{de^{\lambda t}}{d\lambda} \Big|_{\lambda=-4} = \frac{d}{d\lambda}(\beta_0 + \beta_1\lambda) \Big|_{\lambda=-4}$$

$$te^{-4t} = \beta_1 \quad \Rightarrow \quad \beta_0 = e^{-4t} + 4te^{-4t}$$

$$e^{At} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} = (e^{-4t} + 4te^{-4t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (te^{-4t}) \begin{bmatrix} -3 & -1 \\ 1 & -5 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} (e^{-4t} + te^{-4t}) & (-te^{-4t}) \\ (te^{-4t}) & (e^{-4t} - te^{-4t}) \end{bmatrix}$$

Example 17: Assume \mathbf{A} is a 5x5 matrix with eigenvalues $\lambda = -1, -1, -1, -2, -3$. Write only the necessary equations to solve for $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$. Evaluate state transition matrix e^{At} for the given matrix \mathbf{A} .

$$e^{At} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \beta_2 \mathbf{A}^2 + \beta_3 \mathbf{A}^3 + \beta_4 \mathbf{A}^4 \quad \text{Using equations (5.74) and (5.75)}$$

$$e^{\lambda t} = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2 + \beta_3 \lambda^3 + \beta_4 \lambda^4$$

$$e^{-t} = \beta_0 - \beta_1 + \beta_2 - \beta_3 + \beta_4$$

$$e^{-2t} = \beta_0 - 2\beta_1 + 4\beta_2 - 8\beta_3 + 16\beta_4$$

$$e^{-3t} = \beta_0 - 3\beta_1 + 9\beta_2 - 27\beta_3 + 81\beta_4$$

$$\frac{d^m e^{\lambda t}}{d\lambda^m} \Big|_{\lambda=-1} = \frac{d^m}{d\lambda^m} (\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2 + \beta_3 \lambda^3 + \beta_4 \lambda^4) \Big|_{\lambda=-1} \quad m = 1, 2$$

$$te^{\lambda t} \Big|_{\lambda=-1} = (\beta_1 + 2\beta_2 \lambda + 3\beta_3 \lambda^2 + 4\beta_4 \lambda^3) \Big|_{\lambda=-1}$$

$$te^{-t} = \beta_1 - 2\beta_2 + 3\beta_3 - 4\beta_4$$

$$t^2 e^{\lambda t} \Big|_{\lambda=-1} = (2\beta_2 + 6\beta_3 \lambda + 12\beta_4 \lambda^2) \Big|_{\lambda=-1}$$

$$t^2 e^{-t} = 2\beta_2 - 6\beta_3 + 12\beta_4$$

Hence the five equations required to solve for $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ are

$$e^{-t} = \beta_0 - \beta_1 + \beta_2 - \beta_3 + \beta_4$$

$$e^{-2t} = \beta_0 - 2\beta_1 + 4\beta_2 - 8\beta_3 + 16\beta_4$$

$$e^{-3t} = \beta_0 - 3\beta_1 + 9\beta_2 - 27\beta_3 + 81\beta_4$$

$$te^{-t} = \beta_1 - 2\beta_2 + 3\beta_3 - 4\beta_4$$

$$t^2 e^{-t} = 2\beta_2 - 6\beta_3 + 12\beta_4$$

once $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ are evaluated, e^{At} can be obtained using

$$e^{At} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \beta_2 \mathbf{A}^2 + \beta_3 \mathbf{A}^3 + \beta_4 \mathbf{A}^4$$

Example 18: A system is described by the following state variable equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad u(t) = \text{unit step function}$$

- Evaluate state transition matrix e^{At}
- Evaluate $\mathbf{x}(t)$ and $y(t)$
- Evaluate impulse response $h(t)$

a) e^{At}

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} 1 - \lambda & 2 \\ 0 & 2 - \lambda \end{vmatrix} = 0 \quad (1 - \lambda)(2 - \lambda) = 0 \quad \lambda = 1, 2$$

$$e^{At} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A}$$

$$e^{\lambda t} = \beta_0 + \beta_1 \lambda$$

$$e^t = \beta_0 + \beta_1$$

$$e^{2t} = \beta_0 + 2\beta_1$$

$$\beta_1 = -e^t + e^{2t}$$

$$\beta_0 = 2e^t - e^{2t}$$

$$e^{At} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} = (2e^t - e^{2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-e^t + e^{2t}) \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^t & -2e^t + 2e^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

b) $\mathbf{x}(t)$ and $y(t)$

$$\mathbf{x}(t) = e^{At} \mathbf{x}(0) + \int_0^t e^{A(t-\tau)} \mathbf{B} u(\tau) d\tau$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^t & -2e^t + 2e^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{(t-\tau)} & -2e^{(t-\tau)} + 2e^{2(t-\tau)} \\ 0 & e^{2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} (1) d\tau$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2e^t + 2e^{2t} \\ e^{2t} \end{bmatrix} + \int_0^t \begin{bmatrix} -3e^{(t-\tau)} + 4e^{2(t-\tau)} \\ 2e^{2(t-\tau)} \end{bmatrix} d\tau$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2e^t + 2e^{2t} \\ e^{2t} \end{bmatrix} + \begin{bmatrix} -3e^t \int_0^t e^{-\tau} d\tau + 4e^{2t} \int_0^t e^{-2\tau} d\tau \\ 2e^{2t} \int_0^t e^{-2\tau} d\tau \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2e^t + 2e^{2t} \\ e^{2t} \end{bmatrix} + \begin{bmatrix} -3e^t(1 - e^{-t}) + 2e^{2t}(1 - e^{-2t}) \\ e^{2t}(1 - e^{-2t}) \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2e^t + 2e^{2t} \\ e^{2t} \end{bmatrix} + \begin{bmatrix} -3e^t + 2e^{2t} + 1 \\ e^{2t} - 1 \end{bmatrix} = \begin{bmatrix} -5e^t + 4e^{2t} + 1 \\ 2e^{2t} - 1 \end{bmatrix}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

$$y(t) = [2 \quad -1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u(t) = 2(-5e^t + 4e^{2t} + 1) - (2e^{2t} - 1)$$

$$y(t) = -10e^t + 6e^{2t} + 3$$

c) $h(t)$

$$\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t)$$

$$h(t) = [2 \quad -1] \begin{bmatrix} e^t & -2e^t + 2e^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [2 \quad -1] \begin{bmatrix} -3e^t + 4e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$h(t) = -6e^t + 6e^{2t}$$

One can note the output consists of e^t , e^{2t} and a constant. This is expected considering the eigenvalue are $\lambda = 1$ and 2 and the input is a unit step. Similarly the terms e^t , e^{2t} in the impulse response $h(t)$ are expected as well.

5.6 – Solution of State Variable Equations – S-Domain

In the previous section the solution of state variable equations in time-domain were obtained. In this section the solution of state variable equations in s-domain is evaluated. We begin with the equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

Taking Laplace Transform of both equations and using the properties of linearity and derivative we have

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \quad (5.76)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s) \quad (5.77)$$

From equation (5.76) we write

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}U(s) \quad (5.78)$$

Pre-multiplying both sides of equation (5.78) by $(s\mathbf{I} - \mathbf{A})^{-1}$ results in

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{U}(s) \quad (5.79)$$

and

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(0) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \quad (5.80)$$

Finally the system transfer function can be obtained by letting $\mathbf{x}(0) = \mathbf{0}$ in equation (5.80).

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}] \mathbf{U}(s) \quad (5.81)$$

Hence

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D} \quad (5.82)$$

Comparing the equations obtained in this section with those in the previous section, the following observations can be made.

$$L[\boldsymbol{\phi}(t)] = L[e^{\mathbf{A}t}] = (s\mathbf{I} - \mathbf{A})^{-1} \quad (5.83)$$

or

$$\boldsymbol{\phi}(t) = e^{\mathbf{A}t} = L^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \quad (5.84)$$

$$\mathbf{H}(s) = L[\mathbf{h}(t)] \quad (5.85)$$

Equation (5.84) can be used to evaluate $\boldsymbol{\phi}(t) = e^{\mathbf{A}t}$ if so desired. However for an $n \times n$ matrix \mathbf{A} , n^2 partial fractions will be required to convert state transition matrix from s-domain to time-domain. The advantage of using equation (5.79) over equation (5.67) in evaluating state variables is avoiding integration. However n partial fraction will be necessary to convert $\mathbf{X}(s)$ to $\mathbf{x}(t)$ should equation (5.79) be used and state variables in time domain be desired.

Example 19: A system is described by the following state variable equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad u(t) = \text{unit step function}$$

- Evaluate $\mathbf{X}(s)$ and $\mathbf{x}(t) = L^{-1}[\mathbf{X}(s)]$
- Evaluate $\mathbf{Y}(s)$ and $y(t) = L^{-1}[\mathbf{Y}(s)]$
- Evaluate $\mathbf{H}(s)$ and $\mathbf{h}(t) = L^{-1}[\mathbf{H}(s)]$
- Evaluate state transition matrix $e^{\mathbf{A}t}$ using $L^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$

$$\text{a) } \mathbf{X}(s) \text{ and } \mathbf{x}(t) = L^{-1}[\mathbf{X}(s)]$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{U}(s)$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} s-1 & -2 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} s-1 & -2 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{1}{s}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-2 & 2 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{s(s-1)(s-2)} \begin{bmatrix} s-2 & 2 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{(s-1)(s-2)} \begin{bmatrix} 2 \\ s-1 \end{bmatrix} + \frac{1}{s(s-1)(s-2)} \begin{bmatrix} s+2 \\ 2s-2 \end{bmatrix} = \frac{1}{s(s-1)(s-2)} \begin{bmatrix} 3s+2 \\ s^2+s-2 \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{3s+2}{s(s-1)(s-2)} \\ \frac{s^2+s-2}{s(s-1)(s-2)} \end{bmatrix} = \begin{bmatrix} \frac{3s+2}{s(s-1)(s-2)} \\ \frac{s+2}{s(s-2)} \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s} - \frac{5}{s-1} + \frac{4}{s-2} \\ \frac{-1}{s} + \frac{2}{s-2} \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 - 5e^t + 4e^{2t} \\ -1 + 2e^{2t} \end{bmatrix}$$

b) $Y(s)$ and $y(t) = L^{-1}[Y(s)]$

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(0) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}U(s) + \mathbf{D}U(s)$$

$$Y(s) = \begin{bmatrix} 2 & -1 \end{bmatrix} \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-2 & 2 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \end{bmatrix} \frac{1}{s(s-1)(s-2)} \begin{bmatrix} s-2 & 2 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Y(s) = \frac{-s+5}{(s-1)(s-2)} + \frac{6}{s(s-1)(s-2)} = \frac{-s^2+5s+6}{s(s-1)(s-2)}$$

$$Y(s) = \frac{3}{s} - \frac{10}{s-1} + \frac{6}{s-2}$$

$$y(t) = 3 - 10e^t + 6e^{2t}$$

c) $H(s)$ and $h(t) = L^{-1}[H(s)]$

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

$$H(s) = \begin{bmatrix} 2 & -1 \end{bmatrix} \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-2 & 2 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$H(s) = \frac{6}{(s-1)(s-2)}$$

$$H(s) = \frac{-6}{(s-1)} + \frac{6}{(s-2)}$$

$$h(t) = -6e^t + 6e^{2t}$$

d) State transition matrix e^{At} using $L^{-1}[(sI - A)^{-1}]$

$$\Phi(s) = (sI - A)^{-1} = \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-2 & 2 \\ 0 & s-1 \end{bmatrix} = \begin{bmatrix} \frac{s-2}{(s-1)(s-2)} & \frac{2}{(s-1)(s-2)} \\ 0 & \frac{s-1}{(s-1)(s-2)} \end{bmatrix}$$

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} \frac{1}{(s-1)} & \frac{2}{(s-1)(s-2)} \\ 0 & \frac{1}{(s-2)} \end{bmatrix}$$

$$\phi(t) = e^{At} = L^{-1}[(sI - A)^{-1}] = L^{-1} \begin{bmatrix} \frac{1}{(s-1)} & -\frac{2}{(s-1)} + \frac{2}{(s-2)} \\ 0 & \frac{1}{(s-2)} \end{bmatrix}$$

$$\phi(t) = e^{At} = \begin{bmatrix} e^t & -2e^t + 2e^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

It can be observed the results obtained in example 19 using s-domain method are the same as those obtained in example 18 using time-domain.

5.7 – Linear Transformation and Diagonalization

In Section 5.2 it was demonstrated that given an $n \times n$ nonsingular matrix \mathbf{T} and an $n \times n$ matrix \mathbf{A} the eigenvalues of the similar matrix $\mathbf{A}_T = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is the same as \mathbf{A} and the corresponding eigenvectors of \mathbf{A}_T is $\mathbf{X}_T = \mathbf{T}^{-1}\mathbf{X}$, where \mathbf{X} is the eigenvector of matrix \mathbf{A} . Furthermore if an $n \times n$ matrix \mathbf{A} has n distinct eigenvalues and n linearly independent eigenvectors and the columns of the transformation matrix in the equation $\mathbf{A}_T = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ are made up of the eigenvectors of matrix \mathbf{A} , then the resulting \mathbf{A}_T is a diagonal matrix whose diagonal elements are the eigenvalues of \mathbf{A} . Once again we begin with

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

We let \mathbf{T} be a nonsingular $n \times n$ matrix and define state $\mathbf{z}(t)$ as

$$\mathbf{x}(t) = \mathbf{T}\mathbf{z}(t) \tag{5.86}$$

or

$$\mathbf{z}(t) = \mathbf{T}^{-1}\mathbf{x}(t) \tag{5.87}$$

Substituting equation (5.86) in state variable equations shown above results in

$$\mathbf{T}\dot{\mathbf{z}}(t) = \mathbf{ATz}(t) + \mathbf{Bu}(t) \quad (5.88)$$

$$\mathbf{y}(t) = \mathbf{CTz}(t) + \mathbf{Du}(t) \quad (5.89)$$

Pre-multiplying both sides of equation (5.88) with \mathbf{T}^{-1} gives

$$\dot{\mathbf{z}}(t) = \mathbf{T}^{-1}\mathbf{ATz}(t) + \mathbf{T}^{-1}\mathbf{Bu}(t) \quad (5.90)$$

We now define

$$\begin{aligned} \mathbf{A}_T &= \mathbf{T}^{-1}\mathbf{AT} \\ \mathbf{B}_T &= \mathbf{T}^{-1}\mathbf{B} \\ \mathbf{C}_T &= \mathbf{CT} \\ \mathbf{D}_T &= \mathbf{D} \end{aligned} \quad (5.91)$$

Using transformations defined in equation (5.91), state variable equations become

$$\dot{\mathbf{z}}(t) = \mathbf{A}_T\mathbf{z}(t) + \mathbf{B}_T\mathbf{u}(t) \quad (5.92)$$

$$\mathbf{y}(t) = \mathbf{C}_T\mathbf{z}(t) + \mathbf{D}_T\mathbf{u}(t) \quad (5.93)$$

and from equation (5.87)

$$\mathbf{z}(0) = \mathbf{T}^{-1}\mathbf{x}(0) \quad (5.94)$$

The equations already obtained to solve for state transition matrix, state variables, outputs, and impulse responses do not change. Same equations in time-domain and s-domain apply. However \mathbf{A}_T , \mathbf{B}_T , \mathbf{C}_T , and \mathbf{D}_T must now be used. Clearly the state variables $\mathbf{x}(t)$ is different from state variables $\mathbf{z}(t)$ unless $\mathbf{T} = \mathbf{I}$. State variables are in general different depending on what modeling form is utilized. However for a given input the system outputs and impulse responses remain the same regardless of modeling or transformation matrix \mathbf{T} utilized. This is shown in the following theorem.

Theorem 5.3

Given a linear time invariant state variable equations

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t)$$

$$\mathbf{y}(t) = \mathbf{Cx}(t) + \mathbf{Du}(t)$$

The system characteristics are not changed after transforming the it to

$$\dot{\mathbf{z}}(t) = \mathbf{A}_T\mathbf{z}(t) + \mathbf{B}_T\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}_T\mathbf{z}(t) + \mathbf{D}_T\mathbf{u}(t)$$

where

$$\mathbf{A}_T = \mathbf{T}^{-1}\mathbf{AT}$$

$$\mathbf{B}_T = \mathbf{T}^{-1}\mathbf{B}$$

$$\mathbf{C}_T = \mathbf{CT}$$

$$\mathbf{D}_T = \mathbf{D}$$

and \mathbf{T} is a nonsingular $n \times n$ matrix.

Proof:

The proof is complete once it is shown that $\mathbf{h}_T(t) = \mathbf{h}(t)$. Using equation (5.69), we have

$$\mathbf{h}_T(t) = \mathbf{C}_T e^{\mathbf{A}_T t} \mathbf{B}_T + \mathbf{D}_T \delta(t) \quad t \geq 0 \quad (5.95)$$

After the transformation, state transition matrix is

$$e^{\mathbf{A}_T t} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A}_T + \cdots + \beta_{n-1} \mathbf{A}_T^{n-1} \quad (5.96)$$

using equations (5.91) and (5.96)

$$\mathbf{h}_T(t) = \mathbf{C}_T [\beta_0 \mathbf{I} + \beta_1 \mathbf{A}_T + \cdots + \beta_{n-1} \mathbf{A}_T^{n-1}] \mathbf{T}^{-1} \mathbf{B} + \mathbf{D} \delta(t) \quad (5.97)$$

Substituting \mathbf{A}_T from equation (5.91) gives

$$\begin{aligned} \mathbf{h}_T(t) &= \mathbf{C}_T [\beta_0 \mathbf{I} + \beta_1 \mathbf{A}_T + \cdots + \beta_{n-1} \mathbf{A}_T^{n-1}] \mathbf{T}^{-1} \mathbf{B} + \mathbf{D} \delta(t) \\ \mathbf{h}_T(t) &= \mathbf{C}_T [\beta_0 \mathbf{I} + \beta_1 (\mathbf{T}^{-1} \mathbf{A} \mathbf{T}) + \cdots + \beta_{n-1} (\mathbf{T}^{-1} \mathbf{A} \mathbf{T})^{n-1}] \mathbf{T}^{-1} \mathbf{B} + \mathbf{D} \delta(t) \end{aligned} \quad (5.98)$$

Pre-multiplying \mathbf{T} from the left hand side and \mathbf{T}^{-1} from the right hand side results in

$$\begin{aligned} \mathbf{h}_T(t) &= \mathbf{C} [\beta_0 \mathbf{T} \mathbf{T}^{-1} + \beta_1 \mathbf{T} (\mathbf{T}^{-1} \mathbf{A} \mathbf{T}) \mathbf{T}^{-1} + \cdots + \beta_{n-1} \mathbf{T} (\mathbf{T}^{-1} \mathbf{A} \mathbf{T})^{n-1} \mathbf{T}^{-1}] \mathbf{B} + \mathbf{D} \delta(t) \\ \mathbf{h}_T(t) &= \mathbf{C} [\beta_0 \mathbf{I} + \beta_1 \mathbf{I} \mathbf{A} \mathbf{I} + \cdots + \beta_{n-1} \mathbf{T} (\mathbf{T}^{-1} \mathbf{A} \mathbf{T}) (\mathbf{T}^{-1} \mathbf{A} \mathbf{T}) \dots (\mathbf{T}^{-1} \mathbf{A} \mathbf{T}) \mathbf{T}^{-1}] \mathbf{B} + \mathbf{D} \delta(t) \\ \mathbf{h}_T(t) &= \mathbf{C} [\beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \cdots + \beta_{n-1} (\mathbf{I} \mathbf{A} (\mathbf{I} \mathbf{A}) \dots (\mathbf{I} \mathbf{A} (\mathbf{I} \mathbf{A})) \dots (\mathbf{I} \mathbf{A} (\mathbf{I} \mathbf{A})) \mathbf{I}] \mathbf{B} + \mathbf{D} \delta(t) \\ \mathbf{h}_T(t) &= \mathbf{C} [\beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \cdots + \beta_{n-1} \mathbf{A}^{n-1}] \mathbf{B} + \mathbf{D} \delta(t) = \mathbf{C} e^{\mathbf{A} t} \mathbf{B} + \mathbf{D} \delta(t) = \mathbf{h}(t) \end{aligned} \quad (5.99)$$

As shown in Theorem (5.3), system response is unaltered after linear transformation using matrix \mathbf{T} . In general such a transformation allows one to convert from one form of modeling to another. For example to convert from controllable canonical form to observable canonical form or vice versa. However one of the most useful transformation is diagonalization where matrix \mathbf{A} is converted to a diagonal form. As indicated earlier, such a form is obtained when the columns of the transformation matrix \mathbf{T} are made up of the eigenvectors of matrix \mathbf{A} . This is demonstrated in the following example.

Example 20: A system is described by the following state variable equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y(t) = [1 \quad 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad u(t) = \text{unit step function}$$

- Evaluate $H(s)$
- Evaluate $Y(s)$
- Transform the system to a diagonal form
- Evaluate $H_T(s)$
- Evaluate $Y_T(s)$

$$\text{a) } \mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

$$H(s) = [1 \quad 2] \begin{bmatrix} s+1 & 1 \\ -3 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$H(s) = [1 \quad 2] \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+5 & -1 \\ 3 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$H(s) = \frac{-s+10}{(s+2)(s+4)}$$

$$\text{b) } \mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}(0) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

$$Y(s) = [1 \quad 2] \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+5 & -1 \\ 3 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [1 \quad 2] \frac{1}{s(s+2)(s+4)} \begin{bmatrix} s+5 & -1 \\ 3 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Y(s) = \frac{3s+12}{(s+2)(s+4)} + \frac{-s+10}{s(s+2)(s+4)} = \frac{3s^2+11s+10}{s(s+2)(s+4)}$$

c) Transformation Matrix

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} -1-\lambda & -1 \\ 3 & -5-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 6\lambda + 8 = 0 \quad \Rightarrow \quad \lambda = -2, -4$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{0}$$

$$\begin{bmatrix} -1-\lambda & -1 \\ 3 & -5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} (-1-\lambda)x_1 - x_2 &= 0 \\ 3x_1 + (-5-\lambda)x_2 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= -2 \\ x_1 - x_2 &= 0 \\ 3x_1 - 3x_2 &= 0 \end{aligned}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -4$$

$$3x_1 - x_2 = 0$$

$$3x_1 - x_2 = 0$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\mathbf{A}_T = \mathbf{T}^{-1} \mathbf{A} \mathbf{T} = \left(\frac{1}{2}\right) \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\mathbf{B}_T = \mathbf{T}^{-1} \mathbf{B} = \left(\frac{1}{2}\right) \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\mathbf{C}_T = \mathbf{C} \mathbf{T} = [1 \quad 2] \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = [3 \quad 7]$$

$$\mathbf{D}_T = \mathbf{D} = 0$$

and

$$\mathbf{z}(0) = \mathbf{T}^{-1} \mathbf{x}(0) = \left(\frac{1}{2}\right) \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$d) \mathbf{H}_T(s) = \mathbf{C}_T (s\mathbf{I} - \mathbf{A}_T)^{-1} \mathbf{B}_T + \mathbf{D}_T$$

$$H_T(s) = [3 \quad 7] \begin{bmatrix} s+2 & 0 \\ 0 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$H_T(s) = [3 \quad 7] \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+4 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$H_T(s) = \frac{-s+10}{(s+2)(s+4)}$$

$$e) \mathbf{Y}_T(s) = \mathbf{C}_T (s\mathbf{I} - \mathbf{A}_T)^{-1} \mathbf{z}(0) + \mathbf{C}_T (s\mathbf{I} - \mathbf{A}_T)^{-1} \mathbf{B}_T \mathbf{U}(s) + \mathbf{D}_T \mathbf{U}(s)$$

$$Y_T(s) = [3 \quad 7] \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+4 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [3 \quad 7] \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+4 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$Y_T(s) = [3 \quad 7] \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+4 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [3 \quad 7] \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+4 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$Y_T(s) = \frac{3s+12}{(s+2)(s+4)} + \frac{-s+10}{s(s+2)(s+4)} = \frac{3s^2+11s+10}{s(s+2)(s+4)}$$

As can be observed, $H(s)$ and $Y(s)$ are the same before and after the transformation.

CHAPTER 6

Partial Differential Equations

Overview

Section 6.1 starts with the discussion of partial differential equations (PDE) as compared to ordinary differential equations (ODE) and some fundamental definitions such as order of PDE, linear and nonlinear PDE, initial and boundary conditions, homogeneous and nonhomogeneous PDE, and hyperbolic, parabolic, and elliptic forms. and eigenvectors. Soling PDE using ODE is presented in Section 6.2. Section 6.3 discusses the solution of one-dimensional wave equation and D'Alembert's solution. The solution of one-dimensional diffusion or heat equation is discussed in Section 6.4 and the solution of two-dimensional Laplace's equation is presented in Section 6.5.

6.1 – Introduction

Ordinary differential equations (ODEs) are used in many application to model systems for single input-single output as well as for multiple input-multiple output. Such modeling works when there is one independent variable such at time (t) is involved. However in many other applications there is more than one independent variable defines the system. As a result this category of problems cannot be modeled by ordinary differential equations and in such cases partial differential equations (PDEs) are utilized. Wave propagation, solid state theory, solid mechanics, and electromagnetic theory are just a few to mention that are governed by partial differential equations. Partial differential equations involve one or more partial derivatives of order one or higher of an unknown function depending on two or more independent variables. The unknown function can be voltage, temperature, displacement, etc. and the independent variable can be time t as well as displacements x, y, and z. In general the more independent variables the more complex the PDE becomes.

We begin with some definitions involving partial differential equations.

Definitions

Order of Partial Differential Equation

The order of highest partial derivative present in the partial differential equation. Here are some examples with u as the unknown function and x, y, z, and t as independent variables.

$u(x, t)$	$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t} = 0$	second order
$u(x, y)$	$\frac{\partial^3 u}{\partial x \partial^2 y} + 2 \frac{\partial^2 u}{\partial^2 x} + 10 \frac{\partial u}{\partial y} = 10e^{-x} \sin 2y$	third order
$u(x, y)$	$\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial y} = 0$	first order

Initial Conditions and Boundary Conditions

In the case of ordinary differential equation with unknown function as y and independent variable as t, the solution involved coefficient to be determined using what was referred to as initial conditions such as $y(0)$, $y'(0)$, etc. The number of such initial conditions required were the same as the order of the given differential equation. In partial differential equation with unknown function as u (t, x, y) initial condition are known functions such as $u(0, x, y)$, $\frac{\partial u}{\partial t}(0, x, y)$, etc. and boundary conditions are known functions such as $u(t, 0, y)$, $u(t, x, 2)$, $\frac{\partial u}{\partial y}(t, x, 0)$, $\frac{\partial u}{\partial x}(t, 5, y)$, etc. Here are some given initial and boundary conditions.

$u(0, x, y) = x^2 + y^2 + 10$	Initial Condition
$\frac{\partial u}{\partial t}(0, x, y) = xy$	Initial Condition
$u(t, 0, y) = 10 \sin t + y + 2$	Boundary Condition
$\frac{\partial u}{\partial y}(t, x, 0) = t^2 + \sin 4x$	Boundary Condition

Linear and Nonlinear Partial Differential Equations

In system analysis, it was observed that a linear system must satisfy the homogeneity and additive property. In other words a system is linear if it satisfy property of superposition. Based on the same concept, a partial differential equation is linear if the unknown function and all its partial derivatives present in the equation are of first degree, otherwise the partial differential equation is nonlinear. Below are examples of some linear and nonlinear partial differential equations.

$u(x,t)$	$\frac{\partial^2 u}{\partial x^2} + 10 \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial t} = 4$	Linear
$u(x,y)$	$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial u}{\partial y} = 4 \sin x \cos y$	Linear
$u(x,y)$	$\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial y} + u = 0$	Nonlinear
$u(x,t)$	$\left(\frac{\partial^2 u}{\partial x \partial t}\right)^2 + \frac{\partial u}{\partial t} = 10u^2$	Nonlinear

Homogeneous and Nonhomogeneous Partial Differential Equations

A homogeneous partial differential equation is one which contains no functions other than the unknown function and its partial derivatives with respect to the independent variables. A nonhomogeneous partial differential equation is one which contains known function or functions along with the unknown function and its partial derivatives with respect to the independent variables. Below are some examples of homogeneous and nonhomogeneous partial differential equations.

$u(x,t)$	$\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$	Homogeneous
$u(x,y)$	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4u^2$	Homogeneous
$u(x,t)$	$\frac{\partial^2 u}{\partial x^2} + 10 \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial t} = 4$	Nonhomogeneous
$u(x,y)$	$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial y^3} + 3 \frac{\partial u}{\partial y} = 4 \sin x \cos y + 2x$	Nonhomogeneous

In this chapter the solutions of the following important partial differential equations will be presented.

$u(x,t)$	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	One Dimensional Wave Equation
$u(x,y,t)$	$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$	Two Dimensional Wave Equation
$u(x,t)$	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	One Dimensional Diffusion Equation
$u(x,y)$	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	Two Dimensional Laplace Equation

Hyperbolic, Parabolic, and Elliptic Forms

Consider a linear, constant coefficient, second order partial differential equation.

$$A_1 \frac{\partial^2 u}{\partial x^2} + A_2 \frac{\partial^2 u}{\partial x \partial y} + A_3 \frac{\partial^2 u}{\partial y^2} + A_4 \frac{\partial u}{\partial x} + A_5 \frac{\partial u}{\partial y} + A_6 u + A_7 = 0$$

If $A_2^2 - 4A_1A_3 > 0$ partial differential equation is Hyperbolic

If $A_2^2 - 4A_1A_3 = 0$ partial differential equation is Parabolic

If $A_2^2 - 4A_1A_3 < 0$ partial differential equation is Elliptic

It can be observed that the wave partial differential equation is Hyperbolic, diffusion partial differential equation is parabolic, and Laplace partial differential equation is elliptic.

Similar to ordinary differential equation, it can be stated that if u_1, u_2, \dots, u_n are the solutions of a linear homogeneous partial differential equation, then

$$u = C_1 u_1 + C_2 u_2 + \dots + C_n u_n \quad (6.1)$$

is also the solution. In equation (6.1), C_1, C_2, \dots, C_n are constant coefficients. The proof is rather simple and it is based on the properties of linearity (homogeneity and additivity).

To verify a solution of a partial differential equation, the solution must satisfy the partial differential equation as well as given boundary or initial conditions.

Example 1: Verify $u(x,y) = 10 \sin 5x \cosh 5y$ is a solution of Laplace's partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = 50 \cos 5x \cosh 5y \quad \frac{\partial^2 u}{\partial x^2} = -250 \sin 5x \cosh 5y$$

$$\frac{\partial u}{\partial y} = 50 \sin 5x \sinh 5y \quad \frac{\partial^2 u}{\partial y^2} = 250 \sin 5x \cosh 5y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -250 \sin 5x \cosh 5y + 250 \sin 5x \cosh 5y = 0$$

$$u(x,y) = 3ye^{3y}$$

Complete or total solution: $u(x,y) = f_1(x)e^{3y} + f_2(x)e^{-3y} + 3ye^{3y}$

6.3 – Wave Equation

In this section we consider the solution of one dimensional wave equation. Vibration in different structural elements and strings are just a couple of examples. We assume a string vibrates only in vertical direction and its motion is a function of horizontal one dimensional displacement x and time t . Hence tension in the horizontal direction is constant and motion is in vertical direction. The partial differential equation of wave equation is derived by applying Newton's law ($\sum F = ma$), where F is the algebraic sum of forces in the vertical direction and acceleration is the vertical acceleration. Here $u(x,t)$ designates the vertical displacement of the string.

$$u(x,t) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (6.2)$$

Where

$$c^2 = \frac{T}{\rho} \quad \text{and} \quad a = \frac{\partial^2 u}{\partial t^2} \quad (6.3)$$

In equation (6.3), ρ is mass per unit length. The derivation of the wave equation is achieved by following the procedure outlined above and is left as an exercise.

We assume the string is of length L with the following boundary and initial conditions.

$$\text{Boundary Conditions:} \quad u(0,t) = 0 \quad u(L,t) = 0 \quad (6.4)$$

$$\text{Initial Conditions:} \quad u(x,0) = f_1(x) \quad \frac{\partial u}{\partial t}(x,0) = f_2(x) \quad (6.5)$$

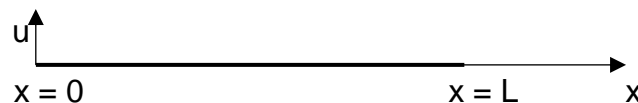


Figure 6.1 Vibrating String of Length L with Displacement $u(x,t)$

The string is fixed at both ends as described by equation (6.4) and equation (6.5) indicates the initial displacement and initial velocity of the string as $f_1(x)$ and $f_2(x)$ respectively. Clearly $f_1(0)$ and $f_1(L)$ must be both zero.

Solution of wave partial differential equation with boundary and initial conditions described above is obtained using separation of variables as shown below.

$$u(x,t) = F_1(x)F_2(t) \quad (6.6)$$

Substituting equation (6.6) in equation (6.2) results in

$$F_1(x) \frac{d^2 F_2(t)}{dt^2} = c^2 \frac{d^2 F_1(x)}{dx^2} F_2(t) \quad (6.7)$$

Dividing both sides of equation (6.6) by $c^2 F_1(x) F_2(t)$ we have

$$\frac{1}{c^2 F_2(t)} \left[\frac{d^2 F_2(t)}{dt^2} \right] = \frac{1}{F_1(x)} \left[\frac{d^2 F_1(x)}{dx^2} \right] \quad (6.8)$$

The left hand side of equation (6.8) is mainly a function of t and the right hand side of equation (6.8) is a function of x . Therefore the only way a function of t is equal to a function of x is if they are both equal to a constant. This constant is designated as K .

$$\frac{1}{c^2 F_2(t)} \left[\frac{d^2 F_2(t)}{dt^2} \right] = \frac{1}{F_1(x)} \left[\frac{d^2 F_1(x)}{dx^2} \right] = K \quad (6.9)$$

Equation (6.9) results in two homogeneous differential equations as follows.

$$\frac{d^2 F_1(x)}{dx^2} - K F_1(x) = 0 \quad (6.10)$$

$$\frac{d^2 F_2(t)}{dt^2} - K c^2 F_2(t) = 0 \quad (6.11)$$

Consider the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Since $u(x,t) = F_1(x)F_2(t)$, we have

$$u(0,t) = F_1(0)F_2(t) = 0$$

$$u(L,t) = F_1(L)F_2(t) = 0$$

Hence

$$F_1(0) = 0 \quad (6.12)$$

$$F_1(L) = 0 \quad (6.13)$$

We begin with solving differential equation (6.10). The constant K can be zero, positive, or negative. Hence we consider all the three cases.

Case I: $K = 0$

From equation (6.10), we have

$$\frac{d^2 F_1(x)}{dx^2} = 0 \quad \Rightarrow \quad F_1(x) = C_1 x + C_2$$

Applying equation (6.12) and (6.13)

$$\begin{aligned} C_1(0) + C_2 &= 0 \\ C_1(L) + C_2 &= 0 \quad \Rightarrow \quad C_1 = C_2 = 0 \quad \Rightarrow \quad F_1(x) = 0 \\ u(x,t) &= F_1(x)F_2(t) = 0 \end{aligned}$$

This is a trivial solution.

Case II: $K > 0, K = \lambda^2$

From equation (6.10), we have

$$\frac{d^2 F_1(x)}{dx^2} - \lambda^2 F_1(x) = 0 \quad \Rightarrow \quad F_1(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

Applying equation (6.12) and (6.13)

$$\begin{aligned} C_1 + C_2 &= 0 \\ C_1 e^{\lambda L} + C_2 e^{-\lambda L} &= 0 \quad \Rightarrow \quad C_1 = C_2 = 0 \quad \Rightarrow \quad F_1(x) = 0 \\ u(x,t) &= F_1(x)F_2(t) = 0 \end{aligned}$$

This is also a trivial solution.

Case III: $K < 0, K = -\lambda^2$

From equation (6.10), we have

$$\frac{d^2 F_1(x)}{dx^2} + \lambda^2 F_1(x) = 0 \quad \Rightarrow \quad F_1(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

Applying equation (6.12) and (6.13)

$$\begin{aligned} C_1 &= 0 \\ C_2 \sin \lambda L &= 0 \quad \Rightarrow \quad C_2 \neq 0, \text{ otherwise } F_1(x) = 0 \text{ and } u(x,t) = F_1(x)F_2(t) = 0 \end{aligned}$$

Hence

$$\begin{aligned} \sin \lambda L &= 0 & \lambda L &= n\pi & \lambda &= \frac{n\pi}{L} & \Rightarrow \\ F_1(x) &= C_2 \sin\left(\frac{n\pi}{L}x\right) & n &= 1, 2, \dots & & & (6.14) \end{aligned}$$

Using equation (6.11), we have

$$\frac{d^2 F_2(t)}{dt^2} + \lambda^2 c^2 F_2(t) = 0$$

$$\frac{d^2 F_2(t)}{dt^2} + \left(\frac{n\pi}{L}\right)^2 c^2 F_2(t) = 0 \quad (6.15)$$

Solution of this differential equation is given by

$$F_2(t) = A_n \cos\left(\frac{cn\pi}{L}\right)t + B_n \sin\left(\frac{cn\pi}{L}\right)t \quad (6.16)$$

With $u(x,t) = F_1(x)F_2(t)$ and using equations (6.14) and (6.16) we have

$$u_n(x,t) = C_2 \sin\left(\frac{n\pi}{L}\right)x \left[A_n \cos\left(\frac{cn\pi}{L}\right)t + B_n \sin\left(\frac{cn\pi}{L}\right)t \right]$$

Without loss of generality we can assume $C_2 = 1$ and using equation (6.1) write the solution for $u(x,t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi}{L}\right)t + B_n \sin\left(\frac{cn\pi}{L}\right)t \right] \sin\left(\frac{n\pi}{L}\right)x \quad (6.17)$$

Equation (6.17) is the solution of the wave equation with boundary conditions as specified by equation (6.4). Now the initial conditions of equation (6.5) can be applied to determine A_n and B_n .

$$u(x,0) = f_1(x)$$

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)x = f_1(x)$$

Using Fourier sine expansion we have

$$A_n = \frac{2}{L} \int_0^L f_1(x) \sin\left(\frac{n\pi}{L}\right)x dx \quad (6.18)$$

Applying initial velocity

$$\frac{\partial u}{\partial t}(x,0) = f_2(x)$$

$$\sum_{n=1}^{\infty} \left[-A_n \left(\frac{cn\pi}{L}\right) \sin\left(\frac{cn\pi}{L}\right)t + B_n \left(\frac{cn\pi}{L}\right) \cos\left(\frac{cn\pi}{L}\right)t \right] \sin\left(\frac{n\pi}{L}\right)x \Big|_{t=0} = f_2(x)$$

$$\sum_{n=1}^{\infty} B_n \left(\frac{cn\pi}{L}\right) \sin\left(\frac{n\pi}{L}\right)x = f_2(x)$$

Using Fourier sine expansion we have

$$B_n \left(\frac{cn\pi}{L}\right) = \frac{2}{L} \int_0^L f_2(x) \sin\left(\frac{n\pi}{L}\right)x dx$$

$$B_n = \frac{2}{c n \pi} \int_0^L f_2(x) \sin\left(\frac{n\pi}{L}\right) x dx \quad (6.19)$$

In summary, equation (6.17) is the solution of the wave equation with the boundary and initial conditions as specified by equations (6.4) and (6.5). Equations (6.18) and (6.19) are used to determine the unknown coefficients A_n and B_n .

If the initial velocity $\frac{\partial u}{\partial t}(x, 0) = f_2(x) = 0$, $B_n = 0$ and the solution reduces to

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{c n \pi}{L}\right) t \sin\left(\frac{n\pi}{L}\right) x \quad (6.20)$$

Example 3: Solve the wave equation for a string with length $L = 1$, $c = 1$, and given boundary and initial conditions.

$$u(0,t) = 0, u(1,t) = 0$$

$$u(x,0) = \begin{cases} x & 0 < x < 0.5 \\ 1 - x & 0.5 < x < 1 \end{cases} \quad \frac{\partial u}{\partial t}(x, 0) = 0$$

$$B_n = 0, \text{ since } \frac{\partial u}{\partial t}(x, 0) = f_2(x) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n\pi t) \sin(n\pi x)$$

$$A_n = \frac{2}{L} \int_0^L f_1(x) \sin\left(\frac{n\pi}{L}\right) x dx = 2 \left[\int_0^{0.5} x \sin n\pi x dx + \int_{0.5}^1 (1-x) \sin n\pi x dx \right]$$

$$A_n = 2 \left[\int_0^{0.5} x \sin n\pi x dx + \int_{0.5}^1 \sin n\pi x dx - \int_{0.5}^1 x \sin n\pi x dx \right]$$

$$A_n = 2 \left[-\frac{x}{n\pi} \cos n\pi x + \frac{1}{(n\pi)^2} \sin n\pi x \Big|_0^{0.5} - \frac{1}{n\pi} \cos n\pi x \Big|_{0.5}^1 + \frac{x}{n\pi} \cos n\pi x - \frac{1}{(n\pi)^2} \sin n\pi x \Big|_{0.5}^1 \right]$$

$$A_n = 2 \left[-\frac{1}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{1}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi} \cos n\pi + \frac{1}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi} \cos n\pi - \frac{1}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{1}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$A_n = \frac{4}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \cos(n\pi t) \sin(n\pi x)$$

$$u(x,t) = \frac{4}{\pi^2} \left(\cos \pi t \sin \pi x - \frac{1}{9} \cos 3\pi t \sin 3\pi x + \frac{1}{25} \cos 5\pi t \sin 5\pi x + \dots \right)$$

Example 4: Solve the wave equation for a string with length $L = \pi$, $c = 1$, and given boundary and initial conditions.

$$u(0,t) = 0, u(\pi,t) = 0$$

$$u(x,0) = 5 \sin x + 3 \sin 3x - 10 \sin 4x \quad \frac{\partial u}{\partial t}(x,0) = 0$$

$$B_n = 0, \text{ since } \frac{\partial u}{\partial t}(x,0) = f_2(x) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(nt) \sin(nx)$$

Before evaluating A_n using equation (6.18), consider $u(x,0)$

$$u(x,0) = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + A_4 \sin 4x + A_5 \sin 5x + \dots$$

Comparing with the given initial condition results in

$$A_1 = 5, A_3 = 3, A_4 = -10, \text{ and } A_n = 0 \text{ for } n \neq 1, 3, 4$$

$$u(x,t) = 5 \cos t \sin x + 3 \cos 3t \sin 3x - 10 \cos 4t \sin 4x$$

Example 5: Solve the wave equation for a string with length $L = \pi$, $c = 1$, and given boundary and initial conditions.

$$u(0,t) = 0, u(\pi,t) = 0$$

$$u(x,0) = \begin{cases} x & 0 < x < 0.5\pi \\ \pi - x & 0.5\pi < x < \pi \end{cases} \quad \frac{\partial u}{\partial t}(x,0) = x(\pi - x)$$

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos nt + B_n \sin nt] \sin nx$$

$$A_n = \frac{2}{L} \int_0^L f_1(x) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{2}{\pi} \left[\int_0^{0.5\pi} x \sin nx dx + \int_{0.5\pi}^{\pi} (\pi - x) \sin nx dx \right]$$

$$A_n = \frac{2}{\pi} \left[\int_0^{0.5\pi} x \sin nx dx + \pi \int_{0.5\pi}^{\pi} \sin nx dx - \int_{0.5\pi}^{\pi} x \sin nx dx \right]$$

$$A_n = \frac{2}{\pi} \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \Big|_0^{0.5\pi} - \frac{\pi}{n} \cos nx \Big|_{0.5\pi}^{\pi} + \frac{x}{n} \cos nx - \frac{1}{n^2} \sin nx \Big|_{0.5\pi}^{\pi} \right]$$

$$A_n = \frac{2}{\pi} \left[-\frac{\pi}{2n} \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) - \frac{\pi}{n} \cos n\pi + \frac{\pi}{n} \cos\left(\frac{n\pi}{2}\right) + \frac{\pi}{n} \cos n\pi - \frac{\pi}{2n} \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$A_n = \frac{4}{\pi n^2} \sin\left(\frac{n\pi}{2}\right)$$

$$B_n = \frac{2}{c n \pi} \int_0^L f_2(x) \sin\left(\frac{n\pi}{L}\right) x dx = \frac{2}{n\pi} \int_0^\pi x(\pi - x) \sin nx dx$$

$$B_n = \frac{2}{n\pi} \left[\int_0^\pi \pi x \sin nx dx - \int_0^\pi x^2 \sin nx dx \right]$$

$$B_n = \frac{2}{n} \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \Big|_0^\pi \right] - \frac{2}{n\pi} \left[\frac{2-n^2 x^2}{n^3} \cos nx + \frac{2x}{n^2} \sin nx \Big|_0^\pi \right]$$

$$B_n = -\frac{2\pi}{n^2} \cos n\pi - \frac{2}{n\pi} \left[\frac{2-n^2 \pi^2}{n^3} \cos n\pi - \frac{2}{n^3} \right] = \left[-\frac{2\pi}{n^2} \cos n\pi - \frac{4-2n^2 \pi^2}{\pi n^4} \cos n\pi + \frac{4}{\pi n^4} \right]$$

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos nt + B_n \sin nt] \sin nx$$

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[\frac{4}{\pi n^2} \sin\left(\frac{n\pi}{2}\right) \right] \cos nt + \left[-\frac{2\pi}{n^2} \cos n\pi - \frac{4-2n^2 \pi^2}{\pi n^4} \cos n\pi + \frac{4}{\pi n^4} \right] \sin nt \right\} \sin nx$$

D'Alembert's Solution

Let us consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with initial conditions as

$$u(x,0) = f_1(x) \quad \frac{\partial u}{\partial t}(x,0) = f_2(x)$$

D'Alembert's solution of the wave equation consists of a substitution process of variables x and t to alternate variables X and T and transforming the wave partial differential equation into a form whose solution is already evaluated in Section 6.1. This process is presented here.

$$X = x + ct \tag{6.21}$$

$$T = x - ct \tag{6.22}$$

Now $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ must be transformed in terms of the new variables X and T .

$$\begin{aligned} u_t &= u_X X_t + u_T T_t = cu_X - cu_T \\ u_{tt} &= (cu_X - cu_T)_t = (cu_X - cu_T)_X X_t + (cu_X - cu_T)_T T_t \\ u_{tt} &= c(cu_X - cu_T)_X - c(cu_X - cu_T)_T = c^2(u_{XX} - 2u_{TX} + u_{TT}) \end{aligned} \tag{6.23}$$

$$\begin{aligned}
u_x &= u_X X_x + u_T T_x = u_X + u_T \\
u_{xx} &= (u_X + u_T)_x = (u_X + u_T)_X X_x + (u_X + u_T)_T T_x \\
u_{xx} &= (u_X + u_T)_X + (u_X + u_T)_T = u_{XX} + 2u_{TX} + u_{TT}
\end{aligned} \tag{6.24}$$

Substituting equations (6.23) and (6.24) in the wave partial differential equation gives

$$\begin{aligned}
c^2(u_{XX} - 2u_{TX} + u_{TT}) &= c^2(u_{XX} + 2u_{TX} + u_{TT}) \\
2c^2 u_{TX} &= 0
\end{aligned}$$

Using example 2c in Section 6.1 we write

$$u(X, T) = f(X) + g(T)$$

Using equations (6.21) and (6.22) we can write

$$u(x, t) = f(x + ct) + g(x - ct) \tag{6.25}$$

Applying initial conditions $u(x, 0) = f_1(x)$ and $\frac{\partial u}{\partial t}(x, 0) = f_2(x)$, we have

$$f_1(x) = f(x) + g(x) \tag{6.26}$$

$$f_2(x) = cf'(x) - cg'(x) \tag{6.27}$$

Integrating equation (6.27) from 0 to x results in

$$\begin{aligned}
\int_0^x f_2(\alpha) d\alpha &= c[f(x) - f(0)] - c[g(x) - g(0)] \\
\frac{1}{c} \int_0^x f_2(\alpha) d\alpha + f(0) - g(0) &= f(x) - g(x)
\end{aligned} \tag{6.28}$$

After adding and subtracting equations (6.26) and (6.28) to evaluate $f(x)$ and $g(x)$ we obtain

$$f(x) = \frac{1}{2} f_1(x) + \frac{1}{2c} \int_0^x f_2(\alpha) d\alpha + \frac{1}{2} [f(0) - g(0)] \tag{6.29}$$

$$g(x) = \frac{1}{2} f_1(x) - \frac{1}{2c} \int_0^x f_2(\alpha) d\alpha - \frac{1}{2} [f(0) - g(0)] \tag{6.30}$$

Using equations (6.25), (6.29), and (6.30) we have

$$\begin{aligned}
u(x, t) &= \frac{1}{2} f_1(x + ct) + \frac{1}{2c} \int_0^{x+ct} f_2(\alpha) d\alpha + \frac{1}{2} [f(0) - g(0)] \\
&\quad + \frac{1}{2} f_1(x - ct) - \frac{1}{2c} \int_0^{x-ct} f_2(\alpha) d\alpha - \frac{1}{2} [f(0) - g(0)] \\
u(x, t) &= \frac{1}{2} f_1(x + ct) + \frac{1}{2} f_1(x - ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} f_2(\alpha) d\alpha
\end{aligned} \tag{6.31}$$

If $\frac{\partial u}{\partial t}(x, 0) = f_2(x) = 0$, then equation (6.31) reduces to

$$u(x,t) = \frac{1}{2}f_1(x + ct) + \frac{1}{2}f_1(x - ct) \quad (6.32)$$

Example 6: Solve the wave equation for a string with length $c = 4$, and given initial conditions.

$$u(x,0) = e^{-x^2} \quad \frac{\partial u}{\partial t}(x,0) = 10$$

$$u(x,t) = \frac{1}{2}f_1(x + ct) + \frac{1}{2}f_1(x - ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} f_2(\alpha) d\alpha$$

$$u(x,t) = \frac{1}{2}e^{-(x+4t)^2} + \frac{1}{2}e^{-(x-4t)^2} + \frac{1}{8} \int_{x-4t}^{x+4t} 10 d\alpha$$

$$u(x,t) = \frac{1}{2}e^{-(x+4t)^2} + \frac{1}{2}e^{-(x-4t)^2} + \frac{5}{4} \alpha \Big|_{x-4t}^{x+4t}$$

$$u(x,t) = \frac{1}{2}e^{-(x+4t)^2} + \frac{1}{2}e^{-(x-4t)^2} + 10t$$

6.4 – Diffusion or Heat Equation

In this section we consider the solution of one dimensional diffusion or heat equation. Here $u(x,t)$ designates the temperature of a homogeneous thin rod with length L that is insulated completely except at the ends ($x = 0$ and $x = L$). The temperature of the rod is assumed to be a function of one dimensional displacement x and time t . We also assume the temperature of the rod at both ends is kept at zero and the initial temperature of the bar to be $f_1(x)$. The partial differential equation of the heat equation is given by

$$u(x,t) \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (6.33)$$

with temperature kept at zero at both end and initial temperature as $f_1(x)$, the boundary conditions and the initial condition are

$$\text{Boundary Conditions:} \quad u(0,t) = 0 \quad u(L,t) = 0 \quad (6.34)$$

$$\text{Initial Condition:} \quad u(x,0) = f_1(x) \quad (6.35)$$



Figure 6.2 Temperature $u(x,t)$ in a Thin Rod of Length L

Clearly $f_1(0)$ and $f_1(L)$ must be both zero.

The constant term c^2 in equation (6.33) represents thermal diffusivity. Thermal diffusivity depends on thermal conductivity, density, and specific heat of the rod.

Solution of wave partial differential equation with boundary conditions and initial condition described above is obtained using separation of variables as shown below.

$$u(x,t) = F_1(x)F_2(t) \quad (6.36)$$

Substituting equation (6.36) in equation (6.33) results in

$$F_1(x) \frac{dF_2(t)}{dt} = c^2 \frac{d^2F_1(x)}{dx^2} F_2(t) \quad (6.37)$$

Dividing both sides of equation (6.37) by $c^2F_1(x)F_2(t)$ we have

$$\frac{1}{c^2F_2(t)} \left[\frac{dF_2(t)}{dt} \right] = \frac{1}{F_1(x)} \left[\frac{d^2F_1(x)}{dx^2} \right] \quad (6.38)$$

The left hand side of equation (6.38) is mainly a function of t and the right hand side of equation (6.37) is a function of x . Therefore the only way a function of t is equal to a function of x is if they are both equal to a constant. This constant is designated as K .

$$\frac{1}{c^2F_2(t)} \left[\frac{d^2F_2(t)}{dt^2} \right] = \frac{1}{F_1(x)} \left[\frac{d^2F_1(x)}{dx^2} \right] = K \quad (6.39)$$

Equation (6.39) results in two homogeneous differential equations as follows.

$$\frac{d^2F_1(x)}{dx^2} - KF_1(x) = 0 \quad (6.40)$$

$$\frac{dF_2(t)}{dt} - Kc^2F_2(t) = 0 \quad (6.41)$$

Consider the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Since $u(x,t) = F_1(x)F_2(t)$, we have

$$u(0,t) = F_1(0)F_2(t) = 0$$

$$u(L,t) = F_1(L)F_2(t) = 0$$

Hence

$$F_1(0) = 0 \quad (6.42)$$

$$F_1(L) = 0 \quad (6.43)$$

We begin with solving differential equation (6.40). The constant K can be zero, positive, or negative. Hence we consider all the three cases.

Case I: $K = 0$

From equation (6.40), we have

$$\frac{d^2 F_1(x)}{dx^2} = 0 \quad \Rightarrow \quad F_1(x) = C_1 x + C_2$$

Applying equation (6.42) and (6.43)

$$C_1(0) + C_2 = 0$$

$$C_1(L) + C_2 = 0 \quad \Rightarrow \quad C_1 = C_2 = 0 \quad \Rightarrow \quad F_1(x) = 0$$

$$u(x,t) = F_1(x)F_2(t) = 0$$

This is a trivial solution.

Case II: $K > 0, K = \lambda^2$

From equation (6.40), we have

$$\frac{d^2 F_1(x)}{dx^2} - \lambda^2 F_1(x) = 0 \quad \Rightarrow \quad F_1(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

Applying equation (6.42) and (6.43)

$$C_1 + C_2 = 0$$

$$C_1 e^{\lambda L} + C_2 e^{-\lambda L} = 0 \quad \Rightarrow \quad C_1 = C_2 = 0 \quad \Rightarrow \quad F_1(x) = 0$$

$$u(x,t) = F_1(x)F_2(t) = 0$$

This is also a trivial solution.

Case III: $K < 0, K = -\lambda^2$

From equation (6.40), we have

$$\frac{d^2 F_1(x)}{dx^2} + \lambda^2 F_1(x) = 0 \quad \Rightarrow \quad F_1(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

Applying equation (6.42) and (6.43)

$$\begin{aligned} C_1 &= 0 \\ C_2 \sin \lambda L &= 0 \quad \Rightarrow \quad C_2 \neq 0, \text{ otherwise } F_1(x) = 0 \text{ and } u(x,t) = F_1(x)F_2(t) = 0 \end{aligned}$$

Hence

$$\begin{aligned} \sin \lambda L &= 0 & \lambda L &= n\pi & \lambda &= \frac{n\pi}{L} & \Rightarrow \\ F_1(x) &= C_2 \sin\left(\frac{n\pi}{L}x\right) & n &= 1, 2, \dots \end{aligned} \quad (6.44)$$

Using equation (6.41), we have

$$\begin{aligned} \frac{dF_2(t)}{dt} + \lambda^2 c^2 F_2(t) &= 0 \\ \frac{dF_2(t)}{dt} + \left(\frac{n\pi}{L}\right)^2 c^2 F_2(t) &= 0 \end{aligned} \quad (6.45)$$

Solution of this differential equation is given by

$$F_2(t) = A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \quad (6.46)$$

With $u(x,t) = F_1(x)F_2(t)$ and using equations (6.44) and (6.46) we have

$$u_n(x,t) = C_2 \sin\left(\frac{n\pi}{L}x\right) x [A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}]$$

Without loss of generality we can assume $C_2 = 1$ and using equation (6.1) write the solution for $u(x,t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} A_n [e^{-\left(\frac{cn\pi}{L}\right)^2 t}] \sin\left(\frac{n\pi x}{L}\right) \quad (6.47)$$

Equation (6.47) is the solution of the heat equation with boundary conditions as specified by equation (6.34). Now the initial condition of equation (6.35) can be applied to determine A_n .

$$\begin{aligned} u(x,0) &= f_1(x) \\ \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) &= f_1(x) \end{aligned}$$

Using Fourier sine expansion we have

$$A_n = \frac{2}{L} \int_0^L f_1(x) \sin\left(\frac{n\pi}{L}x\right) dx \quad (6.48)$$

In summary, equation (6.47) is the solution of the diffusion or heat equation with the boundary condition and initial condition as specified by equations (6.34) and (6.35). Equation (6.48) is used to determine the unknown coefficients A_n .

Example 7: Evaluate the temperature of a thin silver rod with $L = 10\text{cm}$, $c^2 = 1.74\text{cm}^2/\text{sec}$, and given boundary and initial conditions.

$$u(0,t) = 0, u(10,t) = 0$$

$$u(x,0) = \begin{cases} x & 0 < x < 5 \\ 0 & 5 < x < 10 \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n [e^{-(\frac{cn\pi}{L})^2 t}] \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n [e^{-0.0174(n\pi)^2 t}] \sin\left(\frac{n\pi x}{10}\right)$$

$$A_n = \frac{2}{L} \int_0^L f_1(x) \sin\left(\frac{n\pi}{L}x\right) dx = 0.2 \int_0^5 x \sin 0.1n\pi x dx$$

$$A_n = 0.2 \left[-\frac{x}{0.1n\pi} \cos 0.1n\pi x + \frac{1}{(0.1n\pi)^2} \sin 0.1n\pi x \right] \Big|_0^5$$

$$A_n = 0.2 \left[-\frac{5}{0.1n\pi} \cos 0.5n\pi + \frac{1}{(0.1n\pi)^2} \sin 0.5n\pi \right]$$

$$A_n = \left[-\frac{10}{n\pi} \cos 0.5n\pi + \frac{20}{(n\pi)^2} \sin 0.5n\pi \right]$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[-\frac{10}{n\pi} \cos 0.5n\pi + \frac{20}{(n\pi)^2} \sin 0.5n\pi \right] [e^{-0.0174(n\pi)^2 t}] \sin\left(\frac{n\pi x}{10}\right)$$

$$u(x,t) = \frac{20}{\pi^2} [e^{-0.0174\pi^2 t}] \sin\left(\frac{\pi x}{10}\right) + \frac{5}{\pi} [e^{-0.0696\pi^2 t}] \sin\left(\frac{\pi x}{5}\right) - \frac{20}{9\pi^2} [e^{-0.1566\pi^2 t}] \sin\left(\frac{3\pi x}{10}\right) - \dots$$

Example 8: Evaluate the temperature of a thin silver rod with $L = 10\text{cm}$, $c^2 = 1\text{cm}^2/\text{sec}$, and given boundary and initial conditions.

$$u(0,t) = 0, u(10,t) = 0$$

$$u(x,0) = 8 \sin 0.2\pi x$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n [e^{-(\frac{cn\pi}{L})^2 t}] \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n [e^{-0.01(n\pi)^2 t}] \sin\left(\frac{n\pi x}{10}\right)$$

Before evaluating A_n using equation (6.48), consider $u(x,0)$

$$u(x,0) = A_1 \sin 0.1\pi x + A_2 \sin 0.2\pi x + A_3 \sin 0.3\pi x + A_4 \sin 0.4\pi x + \dots$$

Comparing with the given initial condition results in

$A_2 = 8$, and $A_n = 0$ for all $n \neq 2$

$$u(x,t) = 8e^{-0.04\pi^2 t} \sin\left(\frac{\pi x}{5}\right)$$

6.5 – Laplace’s Equation

In this section we consider the solution of two dimensional Laplace’s partial differential equation. Here u is assumed to be a function of coordinates x and y . The applications of Laplace’s partial differential solution is in electromagnetics, fluid mechanics, heat, etc. The Laplace’s two dimensional partial differential equation is given by

$$u(x,y) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (6.49)$$

With the operator ∇^2 defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (6.50)$$

Equation (6.49) is simply written as

$$\nabla^2 u = 0 \quad (6.51)$$

The boundary conditions assumed here are

$$\text{Boundary Conditions:} \quad u(0,y) = 0 \quad u(L_1,y) = 0 \quad (6.52)$$

$$u(x,0) = 0 \quad u(x,L_2) = f_1(x) \quad (6.53)$$



Figure 6.3 Boundary Condition – Laplace’s Partial Differential Equation

Clearly $f_1(0)$ and $f_1(L_1)$ must be both zero.

Solution of Laplace's partial differential equation with boundary described above is obtained using separation of variables as shown below.

$$u(x,y) = F_1(x)F_2(y) \quad (6.54)$$

Substituting equation (6.54) in equation (6.49) results in

$$\frac{d^2 F_1(x)}{dx^2} F_2(y) + F_1(x) \frac{d^2 F_2(y)}{dy^2} = 0 \quad (6.55)$$

Dividing both sides of equation (6.55) by $F_1(x)F_2(y)$ we have

$$\begin{aligned} \frac{1}{F_1(x)} \left[\frac{d^2 F_1(x)}{dx^2} \right] + \frac{1}{F_2(y)} \left[\frac{d^2 F_2(y)}{dy^2} \right] &= 0 \\ \frac{1}{F_1(x)} \left[\frac{d^2 F_1(x)}{dx^2} \right] &= - \frac{1}{F_2(y)} \left[\frac{d^2 F_2(y)}{dy^2} \right] \end{aligned} \quad (6.56)$$

The left hand side of equation (6.56) is mainly a function of x and the right hand side of equation (6.56) is a function of y . Therefore the only way a function of x is equal to a function of y is if they are both equal to a constant. This constant is designated as K .

$$\frac{1}{F_1(x)} \left[\frac{d^2 F_1(x)}{dx^2} \right] = - \frac{1}{F_2(y)} \left[\frac{d^2 F_2(y)}{dy^2} \right] = K \quad (6.57)$$

Equation (6.57) results in two homogeneous differential equations as follows.

$$\frac{d^2 F_1(x)}{dx^2} - K F_1(x) = 0 \quad (6.58)$$

$$\frac{d^2 F_2(y)}{dy^2} + K F_2(y) = 0 \quad (6.59)$$

Consider the boundary conditions $u(0,y) = 0$ and $u(L_1,y) = 0$. Since $u(x,y) = F_1(x)F_2(y)$, we have

$$\begin{aligned} u(0,y) &= F_1(0)F_2(y) = 0 \\ u(L_1,y) &= F_1(L_1)F_2(y) = 0 \end{aligned}$$

Hence

$$F_1(0) = 0 \quad (6.60)$$

$$F_1(L_1) = 0 \quad (6.61)$$

We begin with solving differential equation (6.58). The constant K can be zero, positive, or negative. Hence we consider all the three cases.

Case I: $K = 0$

From equation (6.58), we have

$$\frac{d^2 F_1(x)}{dx^2} = 0 \quad \Rightarrow \quad F_1(x) = C_1 x + C_2$$

Applying equation (6.60) and (6.61)

$$\begin{aligned} C_1(0) + C_2 &= 0 \\ C_1(L_1) + C_2 &= 0 \quad \Rightarrow \quad C_1 = C_2 = 0 \quad \Rightarrow \quad F_1(x) = 0 \\ u(x,y) &= F_1(x)F_2(y) = 0 \end{aligned}$$

This is a trivial solution.

Case II: $K > 0, K = \lambda^2$

From equation (6.58), we have

$$\frac{d^2 F_1(x)}{dx^2} - \lambda^2 F_1(x) = 0 \quad \Rightarrow \quad F_1(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

Applying equation (6.60) and (6.61)

$$\begin{aligned} C_1 + C_2 &= 0 \\ C_1 e^{\lambda L_1} + C_2 e^{-\lambda L_1} &= 0 \quad \Rightarrow \quad C_1 = C_2 = 0 \quad \Rightarrow \quad F_1(x) = 0 \\ u(x,y) &= F_1(x)F_2(y) = 0 \end{aligned}$$

This is also a trivial solution.

Case III: $K < 0, K = -\lambda^2$

From equation (6.58), we have

$$\frac{d^2 F_1(x)}{dx^2} + \lambda^2 F_1(x) = 0 \quad \Rightarrow \quad F_1(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

Applying equation (6.60) and (6.61)

$$\begin{aligned} C_1 &= 0 \\ C_2 \sin \lambda L_1 &= 0 \quad \Rightarrow \quad C_2 \neq 0, \text{ otherwise } F_1(x) = 0 \text{ and } u(x,y) = F_1(x)F_2(y) = 0 \end{aligned}$$

Hence

$$\sin \lambda L_1 = 0 \quad \lambda L_1 = n\pi \quad \lambda = \frac{n\pi}{L_1} \quad \Rightarrow$$

$$F_1(x) = C_2 \sin\left(\frac{n\pi}{L_1}x\right) \quad n = 1, 2, \dots \quad (6.62)$$

Using equation (6.59), we have

$$\begin{aligned} \frac{d^2 F_2(y)}{dy^2} - \lambda^2 F_2(y) &= 0 \\ \frac{d^2 F_2(y)}{dy^2} - \left(\frac{n\pi}{L_1}\right)^2 F_2(y) &= 0 \end{aligned} \quad (6.63)$$

Solution of this differential equation is given by

$$F_2(y) = A_n \cosh\left(\frac{n\pi}{L_1}y\right) + B_n \sinh\left(\frac{n\pi}{L_1}y\right) \quad (6.64)$$

With $u(x,y) = F_1(x)F_2(y)$ and using equations (6.62) and (6.64) we have

$$u_n(x,y) = C_2 \sin\left(\frac{n\pi}{L_1}x\right) [A_n \cosh\left(\frac{n\pi}{L_1}y\right) + B_n \sinh\left(\frac{n\pi}{L_1}y\right)]$$

Without loss of generality we can assume $C_2 = 1$ and using equation (6.1) write the solution for $u(x,y)$ as

$$u(x,y) = \sum_{n=1}^{\infty} [A_n \cosh\left(\frac{n\pi}{L_1}y\right) + B_n \sinh\left(\frac{n\pi}{L_1}y\right)] \sin\left(\frac{n\pi}{L_1}x\right) \quad (6.65)$$

Equation (6.65) is the solution of the Laplace's equation with boundary conditions as specified by equation (6.52). Now the boundary conditions of equation (6.53) can be applied to determine A_n and B_n .

$$\begin{aligned} u(x,0) &= 0 \\ \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L_1}x\right) &= 0 \quad \Rightarrow \quad A_n = 0 \end{aligned}$$

Hence we have

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L_1}y\right) \sin\left(\frac{n\pi}{L_1}x\right) \quad (6.66)$$

We now apply the final boundary condition

$$\begin{aligned} u(x,L_2) &= f_1(x) \\ u(x,L_2) &= \sum_{n=1}^{\infty} [B_n \sinh\left(\frac{n\pi}{L_1}L_2\right)] \sin\left(\frac{n\pi}{L_1}x\right) = f_1(x) \end{aligned}$$

Using Fourier sine expansion we have

$$B_n \sinh\left(\frac{n\pi L_2}{L_1}\right) = \frac{2}{L_1} \int_0^{L_1} f_1(x) \sin\left(\frac{n\pi}{L_1}x\right) dx$$

$$B_n = \frac{2}{L_1 \sinh\left(\frac{n\pi L_2}{L_1}\right)} \int_0^{L_1} f_1(x) \sin\left(\frac{n\pi}{L_1}x\right) dx \quad (6.67)$$

In summary, equation (6.66) is the solution of the Laplace's partial differential equation with the boundary conditions as specified by equations (6.52) and (6.53). Equation (6.67) is used to determine the unknown coefficient B_n . Should the boundary conditions be different than those specified by equations (6.52) and (6.53), a new solution of Laplace's partial differential equation needs to be evaluated following similar procedure as done above.

Example 9: Solve the Laplace's partial differential equation with length $L_1 = \pi$, $L_2 = 1$, and given boundary conditions.

$$u(0,y) = 0, u(\pi,y) = 0$$

$$u(x,0) = 0, u(x,1) = 1 - \cos 2x$$

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L_1}y\right) \sin\left(\frac{n\pi}{L_1}x\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh(ny) \sin(nx)$$

$$B_n = \frac{2}{L_1 \sinh\left(\frac{n\pi L_2}{L_1}\right)} \int_0^{L_1} f_1(x) \sin\left(\frac{n\pi}{L_1}x\right) dx$$

$$B_n = \frac{2}{\pi \sinh(n)} \int_0^{\pi} (1 - \cos 2x) \sin(nx) dx = \frac{2}{\pi \sinh(n)} \int_0^{\pi} [\sin(nx) - \sin nx \cos 2x] dx$$

$$\text{Note: } \sin(nx)\cos(2x) = (1/2)[\sin(n+2)x + \sin(n-2)x]$$

$$B_n = \frac{2}{\pi \sinh(n)} \left[-\frac{1}{n} \cos nx + \frac{0.5}{n+2} \cos(n+2)x + \frac{0.5}{n-2} \cos(n-2)x \right] \Big|_0^{\pi}$$

$$B_n = \frac{2}{\pi \sinh(n)} \left[-\frac{1}{n} \cos(n\pi) + \frac{0.5}{n+2} \cos(n+2)\pi + \frac{0.5}{n-2} \cos(n-2)\pi + \frac{1}{n} - \frac{0.5}{n+2} - \frac{0.5}{n-2} \right]$$

B_n evaluated here cannot be used to compute B_2 due to the term $(n-2)$ in the denominator. The coefficient B_2 needs to be evaluated separately as shown below.

$$B_2 = \frac{2}{\pi \sinh(2)} \int_0^{\pi} (1 - \cos 2x) \sin(2x) dx = \frac{2}{\pi \sinh(2)} \int_0^{\pi} [\sin(2x) - 0.5 \sin 4x] dx$$

$$B_1 = \frac{2}{\pi \sinh(2)} \left[-\frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right] \Big|_0^{\pi}$$

$$B_1 = 0$$

B_n can Now be substituted in $u(x,y)$ and the series solution can be expanded for as many terms as desired.

Example 10: Solve the Laplace's partial differential equation with length $L_1 = 4$, $L_2 = 4$, and given boundary conditions.

$$u(0,y) = 0, u(4,y) = 0$$

$$u(x,0) = 0, u(x,4) = 3\sin\left(\frac{\pi}{4}\right)x - 2\sin\left(\frac{3\pi}{4}\right)x$$

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L_1}\right)y \sin\left(\frac{n\pi}{L_1}\right)x$$

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi y}{4}\right) \sin\left(\frac{n\pi x}{4}\right)$$

Before evaluating B_n using equation (6.67), consider $u(x,4)$

$$u(x,4) = \sum_{n=1}^{\infty} B_n \sinh(n\pi) \sin\left(\frac{n\pi x}{4}\right)$$

$$u(x,4) = B_1 \sinh(\pi) \sin\left(\frac{\pi}{4}\right)x + B_2 \sinh(2\pi) \sin\left(\frac{\pi}{2}\right)x + B_3 \sinh(3\pi) \sin\left(\frac{3\pi}{4}\right)x + \dots$$

Comparing with the given boundary condition $u(x,4)$ results in

$$B_1 \sinh(\pi) = 3 \qquad B_1 = \frac{3}{\sinh \pi}$$

$$B_3 \sinh(3\pi) = -2 \qquad B_3 = \frac{-2}{\sinh 3\pi}$$

$$B_n = 0 \qquad n \neq 1, 3$$

$$u(x,t) = \frac{3}{\sinh \pi} \sinh\left(\frac{\pi y}{4}\right) \sinh\left(\frac{\pi x}{4}\right) - \frac{2}{\sinh 3\pi} \sinh\left(\frac{3\pi y}{4}\right) \sinh\left(\frac{3\pi x}{4}\right)$$