MATH 450 **Advanced Mathematics for Engineers I** **Fall 2006**

Review Sheet #1 for Final Exam

Problem 1. Determine whether the following statements are True or False. Justify your answer.

(a) The transformation $w = \frac{1}{z}$ maps a circle centered at the origin into a line through the origin.

(b) The composition of two bi-linear transformations is a bi-linear transformation.

(c) The Laplace equation $\nabla^2\Phi = 0$ becomes $\Phi_{uu} - 2\Phi_{uv} + \Phi_{vv} = 0$ under the transformation $w = \frac{3iz-2}{z+5i}$.

(d) A complex valued function, $f(z) = u(x, y) + iv(x, y)$, is said to be analytic at $z = z_0$ if $u_x = v_y$ and $u_y = -v_x$ at $z = z_0$.

(e) If $f(z)$ satisfies $\int_C f(z)\,dz = 0$ where C is a positively oriented simple closed contour, then $f(z)$ is analytic on and inside C.

(f) The function $f(z) = \mathrm{Im}(z)$ is analytic in $\mathbb{C}$.

(g) $\sin(-z) = -\sin z$.

(h) $Log(z_1z_2) = Log(z_1) + Log(z_2)$. Here $Log(z)$ denotes the principal branch of the logarithmic function, $log(z)$.

(i) The image of the hyperbola $xy = 9$ under the transformation $w(z) = z^2$ is a line.

(j) The Inverse Fourier Transform of $\hat{u}(\omega, t)$ satisfies $F^{-1}\{\hat{u}\} = \frac{1}{i\,\omega}\, u_x(x, t)$.

Problem 2. Solve the diffusion equation with convection:

$$u_t = ku_{xx} + cu_x, \qquad -\infty < x < \infty, \qquad u(x, 0) = f(x).$$

Problem 3. Solve the wave equation, $u_{tt} - c^2u_{xx} = 0$, for $x \in \mathbb{R}$, $t > 0$, with initial conditions:

$$u(x, 0) = \begin{cases} 2 & \text{if} \quad x \in [0, 2] \\ 0 & \quad\text{otherwise} \end{cases}, \qquad u_t(x, 0) = \begin{cases} x & \text{if} \quad x \in [0, 1] \\ 0 & \quad\text{otherwise} \end{cases}$$

Problem 4. Use the method of characteristics to solve the following *quasi-linear* hyperbolic PDEs subject to the initial conditions $\rho(x, 0) = f(x)$

(a) $\rho_t + c\rho_x = e^{-3x}$

(b) $\rho_t + t\rho_x = 5$

(c) $\rho_t - t^2\rho_x = -\rho$

(d) $\rho_t + x\rho_x = t$

Problem 5. Use polar coordinates to solve Laplace equation, $\nabla^2 u = 0$ inside the circle $x^2 + y^2 = R^2$ with boundary conditions

$$|u(0,\theta)| < \infty, \quad u(R,\theta) = 12\cos 2\theta$$

$$\frac{\partial u}{\partial n}(r,0) = \frac{\partial u}{\partial n}(r,2\pi) = 0.$$

Problem 6. Use polar coordinates to solve Laplace equation, $\nabla^2 u = 0$, in the domain $1 < r < 3$, $0 < \theta < \pi$, with boundary conditions $u(1,\theta) = u_\theta(r,0) = u_\theta(r,\pi) = 0$, and $u(3,\theta) = 20$.

Problem 7. Solve Laplace equation, $\nabla^2 \Phi = 0$ in a domain consisting of the xy-plane with the circles $x^2 + (y-1)^2 = 9$ and $(x-5)^2 + (y-1)^2 = 1$ cut off, with $\Phi = 10$ on the first circle and $\Phi = 0$ on the second one. Give your answer in terms of x and y.

Problem 8. Find bi-linear transformations that map the given region in the left (D) onto that in the right (Ω):

(a) $D = \{z \in \mathbb{C} \ : \ |z| < 1\}, \quad \Omega = \{w \in \mathbb{C} \ : \ \mathrm{Im}(w) > 0\}$

(b) $D = \{z \in \mathbb{C} \ : \ |z-1| < 1\}, \quad \Omega = \{w \in \mathbb{C} \ : \ \mathrm{Im}(w) > 0\}$

(c) $D = \{z \in \mathbb{C} \ : \ |z| < 1\}, \quad \Omega = \{w \in \mathbb{C} \ : \ \mathrm{Im}(w) > 0\}$

(d) $D = \{z \in \mathbb{C} \ : \ |z+1| < 1\}, \quad \Omega = \{w \in \mathbb{C} \ : \ \mathrm{Im}(w) < 0\}$

(e) D : The region between the circle through the points $-2i, 2i$, and -4, and the circle through the poinst $-2i, 2i$, and 2, Ω: an infinite wedge.

Problem 9. Evaluate the following contour integrals (contours are assumed to traversed the given path only once):

(a) $\oint_{|z|=2} \sin z \, dz$

(b) $\oint_{|z|=2} \frac{\sin z}{(z-1)(z-3)} \, dz$

(c) $\oint_{|z|=4} \frac{\sin z}{(z-1)(z-3)} \, dz$

(d) $\oint_{|z|=2} \frac{e^z}{(z-1)(2z-3)} \, dz$

(e) $\oint_{|z|=1} \frac{\sin z}{(2z-1)z^2} \, dz$

(f) $\oint_{|z|=1} \frac{\cos z}{(z-4)(z+2)} \, dz$