Midterm Review Sheet I – Linear Algebra

[*Sources: Linear Algebra and its Applications*, 4th Edition, Gilbert Strang, Brooks/Cole; and *Linear Algebra, Geodesy, and GPS*, Gilbert Strang and Kai Borre, Wellesley-Cambridge Press]

Exam information:

- **Date and Location:** The midterm exam will take place on Thursday, October 28, from 10 to 11.50AM at LO 1322
- **Topics:** Linear Algebra (Chapter 2) and Probability (Chapter 3)
- **Notes:** You can bring one page of notes, one side of an 8.5 in. × 11 in. sheet.

Additional Notes in Linear Algebra

Some Examples and Definitions:

– **Examples of Vector Spaces:**

- $\mathbb{R}^n$ Spaces: $\mathbb{R}^2$ – *all vectors* of two real components (including the zero vector), *i.e.*, the plane; $\mathbb{R}^3$ – all vectors with three real components; ...; $\mathbb{R}^n$ – all vectors of n real components.
- Function Spaces: **F** – the space of *all real functions* $f(x)$; C^2 – the space of all real continuous functions with continuous first and second derivatives.
- Matrix Spaces: **M** – The space of *all real 2 by 2 matrices.*
- Other Vector Spaces: **Z** – The vector space consisting only of a *zero vector.*

Definition. A *subspace* or a vector space is a set of vectors (including the zeros vector) of a vector space that satisfies two conditions: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the subspace, and c is any scalar, then

- $\mathbf{u} + \mathbf{w}$ is also in the subspace
- $c\mathbf{u}$ is also in the subspace

Examples: (1) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$, $\mathbb{R}^2$ consists of all vectors in $\mathbb{R}^3$ whose third component is zero. (2) Any line on the plane through $(0, 0)$ is a subspace of $\mathbb{R}^2$. (3) All upper triangular matrices in **M** form a subspace of **M**.

Definition. The *column space of a matrix A* consists of all linear combinations of the columns of of A. These combinations are the vectors $A\mathbf{x}$.

Remarks: (1) If $A \in \mathbb{R}^{m\times n}$ its columns have m entries, so the columns belong to $\mathbb{R}^m$ and the column space of A is a subspace of $\mathbb{R}^m$. (2) The system $A\mathbf{x} = \mathbf{b}$ has (at least) a solution if $\mathbf{b}$

is in the column space of A. The solution $\mathbf{x}$ is the vector whose entries are the coefficients of the linear combination.

Definition. The column space of A is called the *range of* A and it is denoted by $R(A)$.

Example: The column space of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

consist of all linear combinations

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \quad x_1, x_2 \in \mathbb{R}$$

These linear combinations define a plane through the origin in $\mathbb{R}^3$, a subspace of $\mathbb{R}^3$.

Definition. The *nullspace* (or *kernel*) of a matrix $A \in \mathbb{R}^{m \times n}$ consists of *all solutions* to $A\mathbf{x} = \mathbf{0}$ and it is denoted by $N(A)$. These vectors $\mathbf{x}$ are in $\mathbb{R}^n$.

Remark: The system $A\mathbf{x} = \mathbf{b}$ has a solution if $\mathbf{b}$ is in the range of A, and the solution is unique if the only element in the nullspace of A is $\mathbf{0}$.

Definition. The row space of a matrix A is the vector space spanned by its rows. It coincides with the range of A^T.

Definition. The *rank*, r, of a matrix A is the number of linearly independent rows/columns of A; $r \leq \min\{m, n\}$.

Definition. A basis for a vector space is a collection of linearly independent vectors that span the space. The dimension of a space is the number of vectors in any basis of the space.

Definition. The *nullity* of a matrix A is the dimension of its nullspace

Fundamental Theorem of Linear Algebra $A \in \mathbb{R}^{m \times n}$

1. $R(A)$ – column space of A; dimension r.
2. $N(A)$ – nullspace of A; dimension $n - r$.
3. $R(A^T)$ – row space of A^T; dimension r.
4. $N(A^T)$ – nullspace of A^T; dimension $m - r$.

Singular Value Decomposition. (SVD) of a matrix $A \in \mathbb{R}^{m \times n}$, $A = UDV^T$, where:

- $U \in \mathbb{R}^{m \times m}$ – orthogonal matrix whose columns are (1) the eigenvectors of AA^T, (2) its first r columns form a basis for $R(A)$, and (3) its last $m - r$ columns form a basis for $N(A^T)$.
- $D \in \mathbb{R}^{m \times n}$ – diagonal matrix with the r non-singular values of A which are the square root of the nonzero eigenvalues of both AA^T and A^TA.
- $V \in \mathbb{R}^{n \times n}$ – orthogonal matrix whose columns are (1) the eigenvectors of A^TA, (2) its first r columns a basis for $R(A^T)$, and (3) its las $n - r$ columns form a basis for $N(A)$.

- **Note:** $AV = UD$. That is, when A multiplies column v_j of V, it produces d_j times a column of U.

Problems in Linear Algebra

Problem 1. Determine whether the following statements are **T**rue or **F**alse. Justify your answer.

(a) If $A \in \mathbb{R}^{m \times n}$, then A and A^T have the same nullspaces.

(b) If the columns of a matrix are linearly dependent so are the rows.

(c) The column space of a 2 by 2 matrix is the same as its row space.

(d) The columns of a matrix are a basis for its column space.

(e) Suppose the columns of a 4 by 4 matrix are a basis for $\mathbb{R}^4$, then the equation $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$

Problem 2. Describe the column spaces and nullspaces of the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Problem 3. For which numbers c and d do this matrices have rank $r = 2$?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

Problem 4. Consider the system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 2 & \alpha \\ 4 & 8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 16 \\ \beta \end{bmatrix}.$$

For what values of α and β does the system have:

(a) No solution?

(b) Infinitely many solutions?

(c) Exactly one solution?

Problem 5. The matrix B has eigenvalues 0, 1, and 2. Find:

(a) The rank of B.

(b) The determinant of $B^T B$.

(c) The eigenvalues of $B^T B$.

(d) The eigenvalues of $(B + I)^{-1}$

Problem 6. Suppose $A = \mathbf{u}\mathbf{v}^T$ is a column times a row (a *rank-1* matrix)

(a) Show that $\mathbf{u}$ is an eigenvector of A by multiplying A times $\mathbf{u}$. What is the corresponding eigenvalue?

(b) What are the other eigenvalues of A? Why?

(c) Compute the trace of A by *(i)* adding its diagonal entries, and *(ii)* adding its eigenvalues

Problem 7. Singular Value Decomposition: $A = UDV^T$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, are two orthonormal bases for $\mathbb{R}^n$. Construct the matrix that transforms each $\mathbf{v}_j$ into $\mathbf{u}_j$ to give $A\mathbf{v}_1 = \mathbf{u}_1, \ldots, A\mathbf{v}_n = \mathbf{u}_n$.

Problem 8. Find UDV^T if A has orthogonal columns $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of lengths $d_1, \ldots, d_n$.