ALGEBRA QUAL SOLUTIONS

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Note that I do not write full detail in here because of laziness. However, those details should be written down when you take the qual.

1. Groups

2. Rings

Problem 2.1. (Fall08, R2) Let A be a domain and $B = A[T, \frac{1}{T}]$ for an indeterminate T. Prove that the ring automorphism group $\mathrm{Aut}(B_{|A})$ of B inducing the identity on A is finite if and only if the group of invertible elements of A is finite.

Proof. ($\Leftarrow$) Suppose $A^\times$ is infinite, then we have infinitely many automorphisms defined for $a \in A^\times$:

$$f_a : B \longrightarrow B$$
$$T \mapsto aT$$

($\Rightarrow$) Suppose $A^\times$ is finite. Since A is an integral domain, $f \in A[T]$ with $f|T^n$ implies that $f = aT^k$ for some $a \in A^\times$ and $k \leq n$. Using this, we can characterize the unit group of B.

$$B^\times = \{aT^n | a \in A^\times, \text{ and } n \in \mathbb{Z}\}.$$

Any automorphism σ of B must satisfy $\sigma(T) \in B^\times$, since $\sigma(T)\sigma(\frac{1}{T}) = 1$. Then, we have $\sigma(T) = aT^n$ for some $a \in A^\times$ and $n \in \mathbb{Z}$. However σ is an automorphism if and only if $n = \pm 1$. It follows that,

$$\sigma \in \mathrm{Aut}(B_{|A}) \Longleftrightarrow \sigma(T) \in \{aT^n | a \in A^\times \text{ and } n = \pm 1\}$$

Hence, we obtain that $\mathrm{Aut}(B_{|A})$ is finite. □

Problem 2.2. (Fall08, R3) Consider the covariant functor $F : A \mapsto A^\times$ from the category ALG of commutative rings with identity to the category of sets. Here $A^\times$ is the group of invertible elements of A. Give an explicit form of a commutative ring R such that the functor F is isomorphic to the functor $A \mapsto \mathrm{Hom}_{\mathrm{ALG}}(R, A)$.

Proof. This problem is easy when you know the answer. The proof is obvious from the answer.

$$R = A\left[T, \frac{1}{T}\right].$$

□

Problem 2.3. (Spring08, R1) Let D be an associative ring with unit having no zero divisors. Assume that the center of D contains a field k such that $dim_k(D) < \infty$. Prove that D is a division algebra(i.e. every nonzero elements are invertible).

Proof. Let $a \in D$ be a nonzero element. Since $d = dim_k(D) < \infty$, the set $\{1, a, a^2, a^3, \cdots, a^d\}$ is linearly dependent over k. Thus, there exists a polynomial $p \in k[T]$ such that $p(a) = 0$. We take the minimal one so that p has nonzero constant term. Then, we obtain by dividing the constant term, $ag(a) + 1 = 0$ for some polynomial $g \in k[T]$. $-g(a)$ is the inverse of a. Hence, D is a division algebra. □

Problem 2.4. (Spring08, R3) Let R be a Noetherian ring and I any ideal of R. Prove that there exist prime ideals $P_1, \cdots, P_m$ such that

$$P_1 P_2 \cdots P_m \subset I$$

Hint: Show that if J is any non-prime ideal, then there exist $a, b \notin J$ such that $(J + a)(J + b) \subset J$. Then use the Noetherian property.

Proof. Suppose there exists an ideal I which does not contain any finite product of prime ideals. Then, we have a nonzero family of ideals

$$\mathcal{F} = \{J \lhd R | J \text{ does not contain any finite product of prime ideals.}\}.$$

Since R is Noetherian, the family $\mathcal{F}$ has a maximal element M. Note that $M \in \mathcal{F}$ is not a prime ideal. Thus, we can find $a, b \notin M$ such that $ab \in M$. This implies that $(M + a)(M + b) \subset M$. Maximality of M imply that $M + a$ and $M + b$ are not in $\mathcal{F}$, so they contain finite product of prime ideals. Then M contains a finite product of prime ideals, and this is a contradiction. □

Problem 2.5. (Fall07, R3) Determine all isomorphism classes of modules over the polynomial ring $\mathbb{F}_2[X]$ which are of dimension 2 over $\mathbb{F}_2$, and justify your answer. Here $\mathbb{F}_2$ is a field of two elements.

Problem 2.6. (Spring06, R3) Let R be a commutative ring with unit and $\mathfrak{m}$ a maximal ideal of R.
(a) Suppose $I_1, \cdots, I_n$ are ideals of R and that

$$\mathfrak{m} \supseteq I_1 \cdots I_n,$$

where $I_1, \cdots, I_n$ is the product of ideals. Show

$$\mathfrak{m} \supseteq I_k$$

for some k.
(b) Suppose that R satisfies the descending chain condition (dcc) on ideals, i.e. every strictly decreasing sequence of ideals is finite. Show R has only a finite number of maximal ideals. You may use part (a), but not theorems on the structure of rings satisfying dcc.

Proof. (a) Suppose not, there exists $x_k \in I_k$ with $x_k \notin \mathfrak{m}$ for all $k \leq n$. Since maximal ideal is a prime ideal, we must have $x_1 \cdots x_n \notin \mathfrak{m}$. This contradicts that $\mathfrak{m} \supseteq I_1 \cdots I_n$.
(b) Suppose we have infinitely many maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2 \cdots$. For each k, let

$$I_k = \mathfrak{m}_1 \cdots \mathfrak{m}_k$$

where $\mathfrak{m}_1 \cdots \mathfrak{m}_k$ is the product of ideals. The descending chain condition (dcc) implies that there is N such that

$$I_N = I_{N+r}$$

for all $r \geq 0$. From the definition of I_k, we have

$$\mathfrak{m}_{N+r} \supseteq I_{N+r} = I_N.$$

By (a), there is some $k \leq N$ such that $\mathfrak{m}_{N+r} \supseteq \mathfrak{m}_k$. Since both are maximal ideals, $\mathfrak{m}_{N+r} = \mathfrak{m}_k$. This is a contradiction since $\{\mathfrak{m}_1, \mathfrak{m}_2, \cdots\} \subseteq \{\mathfrak{m}_1, \cdots, \mathfrak{m}_N\}$. Hence, there are only a finite number of maximal ideals. □

Problem 2.7. (Fall05, R1) Let I and J be ideals of a commutative ring R with unit such that $I + J = R$. Prove that $IJ = I \cap J$.

Proof. Since $IJ \subset I$ and $IJ \subset J$, it follows $IJ \subset I \cap J$. For the reverse inclusion, we find $i \in I$ and $j \in J$ such that

$$i + j = 1.$$

Then, for any $k \in I \cap J$,

$$k = k \cdot 1 = k(i + j) = ki + kj \in IJ.$$

□

Problem 2.8. (Spring04, R3) Suppose we are given a collection of polynomials in r variables with rational coefficients:

$$f_1, \cdots, f_N \in \mathbb{Q}[T_1, \cdots, T_r]$$

We define the complex algebraic set $V_{\mathbb{C}} \subset \mathbb{C}^r$ by

$$V_{\mathbb{C}} = \{(a_1, \cdots, a_r) | f_i(a_1, \cdots, a_r) = 0 \text{ for all } i \text{ from } 1 \text{ to } N\}.$$

Suppose $V_{\mathbb{C}}$ is not empty. Show that there is a finite extension K of $\mathbb{Q}$ and a point

$$(a_1, \cdots, a_r) \in V_{\mathbb{C}}$$

with all $a_k \in K$.

Proof. We use Nullstellensatz.
(Hilbert's Nullstellensatz) Let K be an algebraically closed field, and let I be an ideal in $K[x_1, \ldots, x_n]$, the polynomial ring in n indeterminates. Define $V(I)$, the *zero set* of I, by

$$V(I) = \{(a_1, \ldots, a_n) \in K^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}$$

Then $\mathrm{Rad}(I) = I(V(I))$.

In the problem, we use this on the ideal $I = (f_1, \cdots, f_N)$.

$$\begin{aligned} V_{\mathbb{C}} \text{ is not empty } &\Rightarrow \mathrm{Rad}_{\mathbb{C}}(f_1, \cdots, f_N) \neq \mathbb{C}[T_1, \cdots, T_r] \\ &\Rightarrow \mathrm{Rad}_{\overline{\mathbb{Q}}}(f_1, \cdots, f_N) \neq \overline{\mathbb{Q}}[T_1, \cdots, T_r] \\ &\Rightarrow V_{\overline{\mathbb{Q}}} \text{ is not empty} \end{aligned}$$

where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. Taking any element $(a_1, \cdots, a_r) \in V_{\overline{\mathbb{Q}}}$ and letting $K = \mathbb{Q}(a_1, \cdots, a_r)$ gives the result. □

Problem 2.9. (Winter03, R1) Give an example of two integral domains A and B which contain a field F such that $A \otimes_F B$ is not an integral domain. Justiry your answer. Hint: Take A to be the field of rational functions $\mathbb{F}_p(X)$ for the field $\mathbb{F}_p$ with p elements.

Proof. Take $A = B = \mathbb{F}_p(X)$ and $F = \mathbb{F}_p(X^p)$. Then A and B are integral domains. However,

$$X \otimes 1 - 1 \otimes X \in A \otimes_F B$$

is a nonzero element in $A \otimes_F B$ satisfying

$$(X \otimes 1 - 1 \otimes X)^p = X^p \otimes 1 - 1 \otimes X^p = 0.$$

Hence, $A \otimes_F B$ is not an integral domain. □

3. Fields

4. Linear Algebra