A DISJOINTNESS CRITERION FOR BEATTY'S SEQUENCES

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1. Introduction

Let α be an irrational number with $\alpha > 1$. We denote $S(\alpha)$ by

$$S(\alpha) = \{\lfloor n\alpha \rfloor | n \in \mathbb{N}\}.$$

In 1926, Sam Beatty [1] proved that if α, β are positive irrational numbers, then the disjoint union of $S(\alpha)$ and $S(\beta)$ is $\mathbb{N}$ if and only if $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. It is easy to see that if α, β are positive irrational numbers and

$$\frac{k}{\alpha} + \frac{l}{\beta} = 1$$

for some positive integers k, l, then $S(\alpha) \cap S(\beta) = \phi$.

In this note, we shall show that the converse of this statement is also valid. In addition, we deduce that $S(\alpha) \cap S(\beta)$ is either empty or infinite.

Theorem. *For irrational numbers α, $\beta > 1$,*

$$S(\alpha) \cap S(\beta) = \phi$$

if and only if

$$\frac{k}{\alpha} + \frac{l}{\beta} = 1$$

for some positive integers k, l.

2. Lemmas

We denote $B(x, y, r)$ by an open ball in $\mathbb{R}^2$ centered at (x, y) with radius r. For a real number x, (x) denotes the fractional part of x. For real numbers α and β, we define $S(\alpha, \beta)$ by

$$S(\alpha, \beta) = \{\big((n\alpha), (n\beta)\big) | n \in \mathbb{N}\}.$$

Lemma 1. *Let m, n, r be positive integers. Then, $\lfloor n\alpha \rfloor = \lfloor m\beta \rfloor = r - 1$ if and only if $\left(\frac{r}{\alpha}\right) \le \frac{1}{\alpha}$ and $\left(\frac{r}{\beta}\right) \le \frac{1}{\beta}$.*

Proof. We observe that $\lfloor n\alpha \rfloor = \lfloor m\beta \rfloor = r - 1$ is equivalent to any of the following statements:

$$n\alpha < r \le n\alpha + 1,\ m\beta < r \le m\beta + 1;$$
$$n < \frac{r}{\alpha} \le n + \frac{1}{\alpha},\ m < \frac{r}{\beta} \le m + \frac{1}{\beta};$$
$$0 < \left(\frac{r}{\alpha}\right) \le \frac{1}{\alpha},\ 0 < \left(\frac{r}{\beta}\right) \le \frac{1}{\beta}.$$

Thus, Lemma 1 follows. □

Lemma 2 (Kronecker's Theorem). *If* $1,\ \alpha,\ \beta$ *are linearly independent over* $\mathbb{Q}$, *then the set* $S(\alpha,\beta)$ *is dense in* $[0,1]^2$.

Proof. See [2], p382. □

Lemma 3. *Let* $\alpha > 1,\ \beta > 1$ *be irrational numbers satisfying* $\frac{k}{\alpha} + \frac{l}{\beta} = m$ *with* $k,\ l,\ m$ *relatively prime integers and* $l > 0$. *Then* $S\big(\frac{1}{\alpha},\frac{1}{\beta}\big)$ *is dense in*

$$[0,1]^2 \cap \{(x,y)|kx+ly \in \mathbb{Z}\}.$$

Proof. Let $(x,y) \in [0,1]^2$ be such that $kx+ly = z$ for some $z \in \mathbb{Z}$ and $0 < \varepsilon < d/2$, where d is the distance between two lines, $kx+ly = 0,\ kx+ly = 1$. Using the pigeon hole principle, we get

$$v = (v_1, v_2) = \left(\left(\frac{n_2}{\alpha}\right) - \left(\frac{n_1}{\alpha}\right), \left(\frac{n_2}{\beta}\right) - \left(\frac{n_1}{\beta}\right)\right)$$

with positive integers $n_1,\ n_2\ (n_1 < n_2)$, and $|v| < \varepsilon < \frac{d}{2}$. Since $(k,l,m) = 1$, for any $z \in \mathbb{Z}$, there is a triple (n, z_1, z_2) with $n \in \mathbb{N},\ z_1 \in \mathbb{Z}$ and $z_2 \in \mathbb{Z}$ such that

$$mn + kz_1 + lz_2 = z$$

From $|v| < d$, we have $kv_1 + lv_2 = 0$. Then, we get

$$k\Big(\frac{n}{\alpha} + z_1 + Nv_1\Big) + l\Big(\frac{n}{\beta} + z_2 + Nv_2\Big) = mn + kz_1 + lz_2 = z$$

for any $N \in \mathbb{N}$. Hence, we can find a positive integer N and integers $u_1,\ u_2$ such that

$$\Big(\frac{n}{\alpha} + z_1 + Nv_1, \frac{n}{\beta} + z_2 + Nv_2\Big) \in B(u_1 + x, u_2 + y, \varepsilon)$$

Thus, Lemma 3 follows. □

3. Proof of the Theorem

Let $\alpha > 1$, $\beta > 1$ be irrational numbers satisfying $S(\alpha) \cap S(\beta) = \phi$. $1,\ \frac{1}{\alpha},\ \frac{1}{\beta}$ are either linearly independent over $\mathbb{Q}$ or linearly dependent over $\mathbb{Q}$. The latter case we multiply a nonzero integer to get $\frac{k}{\alpha} + \frac{l}{\beta} = m$, with $k,\ l,\ m$ relatively prime integers, and $l > 0$. Since $k \neq 0$, we divide the latter into two cases $k < 0$, and $k > 0$.

Case 1. $1,\ \frac{1}{\alpha},\ \frac{1}{\beta}$ are linearly independent over $\mathbb{Q}$.

By Lemma 2, $S\big(\frac{1}{\alpha},\frac{1}{\beta}\big)$ is dense in $[0,1]^2$. Then we have

$$S\Big(\frac{1}{\alpha},\frac{1}{\beta}\Big) \cap \Big(0,\frac{1}{\alpha}\Big] \times \Big(0,\frac{1}{\beta}\Big]$$

is an infinite set. This implies

$$\Big(\frac{r}{\alpha}\Big) \le \frac{1}{\alpha},\ \Big(\frac{r}{\beta}\Big) \le \frac{1}{\beta}$$

for infinitely many positive integers r. Using this and Lemma 1, we have $S(\alpha) \cap S(\beta)$ is an infinite set.

Case 2. $\frac{k}{\alpha} + \frac{l}{\beta} = m$, with $k,\ l,\ m$ relatively prime integers, $l > 0$ and $k < 0$.

The set

$$\{(x,y)|kx+ly\in\mathbb{Z}\}\cap\Big(0,\frac{1}{\alpha}\Big]\times\Big(0,\frac{1}{\beta}\Big]$$

contains a line segment of $kx+ly=0$. By Lemma 3, the set

$$S\Big(\frac{1}{\alpha},\frac{1}{\beta}\Big)\cap\Big(0,\frac{1}{\alpha}\Big]\times\Big(0,\frac{1}{\beta}\Big]$$

is an infinite set. This implies $S(\alpha)\cap S(\beta)$ is an infinite set as in Case 1.

Case 3. $\frac{k}{\alpha}+\frac{l}{\beta}=m$ with $k,\ l,\ m$ relatively prime integers, $l>0$ and $k>0$.

Since $S(\alpha)\cap S(\beta)=\phi$, we have

$$S\Big(\frac{1}{\alpha},\frac{1}{\beta}\Big)\cap\Big(0,\frac{1}{\alpha}\Big]\times\Big(0,\frac{1}{\beta}\Big]=\phi \tag{1}$$

by Lemma 1. It follows that

$$\{(x,y)|kx+ly\in\mathbb{Z}\}\cap\Big(0,\frac{1}{\alpha}\Big]\times\Big(0,\frac{1}{\beta}\Big]$$

does not contain any line segment, otherwise it contradicts (1) by Lemma 3. This implies

$$\frac{1}{\beta}\leq-\frac{k}{l}\frac{1}{\alpha}+\frac{1}{l}$$

which is equivalent to $m\leq 1$. Thus, we obtain $m=1$.

By Cases 1–3, we complete the proof of the Theorem.

References

1. Sam, Beatty, *Problem 3173*, Amer. Math. Monthly, 1926, p159.
2. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, Oxford At The Clarendon Press, 1960.