Stirling Number of Second Kind Identities

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Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all natural numbers and $k, m \in \mathbb{N}$. Denote by $[n]$ the set $\mathbb{N} \cap [1, n]$. Let *m-coloring* on a set X be a function from X to $[m]$.

Definition 0.1. $S(k, m)$ is the number of ways to partition $[k]$ into m sets. For convention, we put $S(n, 0) = \delta(n, 0)$.

$S(k, m)m!$ is the number of m-colorings of $[k]$ which all colors are used, that is, the number of surjective functions from $[k]$ to $[m]$.

Theorem 0.1. For any $x \in \mathbb{C}$,

$$x^k = \sum_{m=0}^{k} S(k, m)(x)_m$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ is the falling factorial.

Proof. It suffices to prove the identity for all natural number x. The LHS is the number of x-colorings of $[k]$. The summand in the RHS,

$$S(k, m)m!\binom{x}{m},$$

is the number of ways to select m colors from x colors and color $[k]$ using all selected colors. □

Corollary 0.1. For any $x \in \mathbb{C}$,

$$(-x)^k = \sum_{m=0}^{k} S(k, m)m!(-1)^m \binom{x+m-1}{m}.$$

Proof. Put $-x$ into x in Theorem 0.1 and note that $\binom{-x}{m} = (-1)^m \binom{x+m-1}{m}$. □

Corollary 0.2. For $|x| < 1$, we have

$$\sum_{\nu=0}^{\infty} \nu^k x^\nu = \frac{1}{1-x} \sum_{m=1}^{k} S(k, m)m! \left(\frac{x}{1-x}\right)^m.$$

Proof. Expanding

$$\sum_{m=1}^{k} S(k, m)m! x^m (1-x)^{-m-1}$$

by the binomial series, the coefficient of x^ν is

$$\sum_{m=1}^{k} \sum_{\substack{w \geq 0 \\ m+w=\nu}} S(k, m)m! \binom{m+w}{w} = \sum_{m \geq 1} S(k, m)m! \binom{\nu}{m}.$$

The result follows by $S(k, 0) = 0$ and Theorem 0.1. □

Theorem 0.2. Let m be an integer with $1 \le m \le k$. We have

$$S(k,m)m! = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-i)^k.$$

Proof. Apply Inclusion-Exclusion Principle. Let A_j be the set of functions from $[k]$ to $[m]$ such that j is excluded from the image. The term $\binom{m}{i}(m-i)^k$ is the number of ways to select i members of $[m]$ and exclude them from the image, that is, the number of elements in i-fold intersections of A_j. □

Corollary 0.3. Let r be a nonnegative integer. Then we have

$$S(k,r+1)(r+1)! = \sum_{m=1}^{k} S(k,m)m!(-1)^{m-k}\binom{m-1}{r}.$$

Proof. By Theorem 0.2 and Corollary 0.1,

$$\begin{aligned} S(k,r+1)(r+1)! &= \sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}(r+1-i)^k \\ &= \sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}\sum_{m=1}^{k} S(k,m)(-1)^{m-k}m!\binom{r+m-i}{m} \\ &= \sum_{m=1}^{k} S(k,m)m!(-1)^{m-k}\sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}\binom{r+m-i}{m} \end{aligned}$$

The inner sum is $\binom{m-1}{r}$ by noting that
$(-1)^i\binom{r+1}{i}$ is the coefficient of x^i in $(1-x)^{r+1}$,
$\binom{r+m-i}{m}$ is the coefficient of x^{r+1-i} in $x(1-x)^{-m-1}$,
and $(-1)^r\binom{r-m}{r} = \binom{m-1}{r}$ is the coefficient of x^{r+1} of $x(1-x)^{r+1-m-1} = x(1-x)^{r-m}$. □

Note that $r = 0$ case is proved by putting $x = 1$ in Corollary 0.1.

There is a combinatorial proof of Corollary 0.3. The case $r = 0$ also has a combinatorial proof by involution between even number of parts and odd number of parts.

We construct an involution T_k, mapping odd ordered partitions of k-element set to even and vice versa: if partition has part $\{k\}$, take a union with the previous part; otherwise move k into new separate part after itself.

Example: $(\{3,4\},\{\mathbf{5}\},\{1,2\}) \leftrightarrow (\{3,4,\mathbf{5}\},\{1,2\})$.

For ordered partitions of the form $(\{k\},\ldots)$, use previous involution T_{k-1} and so on.

Example: $(\{5\},\{4\},\{1,\mathbf{3}\},\{2\}) \leftrightarrow (\{5\},\{4\},\{1\},\{\mathbf{3}\},\{2\})$.

The only partition without pair will be $(\{k\},\{k-1\},\dots,\{1\})$ which is counted in $S(k,k)k!$. Therefore, we have

$$\sum_{m \text{ is even}} S(k,m)m! = \sum_{m \text{ is odd}} S(k,m)m! + (-1)^k.$$

Then consider an ordered partition of $m \geq r+1$ parts, put r part-dividers among possible $m-1$ positions. We nudge all colors between these dividers to one color. Then we have a surjective function from all m-coloring of $[k]$ using all colors to all $r+1$-coloring of $[k]$ using all colors. Consider a coloring of $[k]$ using all $r+1$ colors with part sizes $a_1,\dots,a_{r+1}$. This particular coloring appears

$$\left(\sum_{j=0}^{a_1} S(a_1,j)j!(-1)^{a_1-j}\right)\cdots\left(\sum_{j=0}^{a_{r+1}} S(a_{r+1},j)j!(-1)^{a_{r+1}-j}\right) = 1$$

times on the RHS. Thus, this shows that Corollary 0.3 for special case $r=0$ imply the full version of Corollary 0.3.

Corollary 0.4 (ec1-Chapter 3, Exercise 141(d))**.** For any $t \in \mathbb{C}$,

$$\sum_{m=1}^{k} S(k,m)m!t^{m-1} = \sum_{m=1}^{k} S(k,m)m!(-1)^{m-k}(t+1)^{m-1}.$$

Proof. $S(k,r+1)(r+1)!$ is the coefficient of t^r on the LHS and $\sum_{m=1}^{k} S(k,m)m!(-1)^{m-k}\binom{m-1}{r}$ is the coefficient of t^r on the RHS. The result follows by Corollary 0.3. □

Corollary 0.5. Let $v_k = 2^k - 1$. The expression

$$\rho(x) = (1-x)^{v_k}\sum_{\nu=0}^{\infty}((\nu+1)^k - \nu^k)x^\nu$$

with the power series converges for $|x|<1$, is a polynomial of degree $v_k - 1$ satisfying

$$\rho(x) = (-1)^{k-1}x^{v_k-1}\rho(\frac{1}{x}).$$

Proof. By Corollary 0.2, we have

$$\sum_{\nu=0}^{\infty}((\nu+1)^k - \nu^k)x^\nu = \frac{1}{x}\sum_{m=1}^{k} S(k,m)m!\left(\frac{x}{1-x}\right)^m.$$

Then

$$\rho(x) = (1-x)^{v_k}\frac{1}{x}\sum_{m=1}^{k} S(k,m)m!\left(\frac{x}{1-x}\right)^m.$$

Putting $1/x$ into $\rho(x)$, we have

$$\rho(1/x) = \left(1-\frac{1}{x}\right)^{v_k} x\sum_{m=1}^{k} S(k,m)m!\left(\frac{1}{x-1}\right)^m.$$

Then

$$x^{v_k-1}\rho(\frac{1}{x}) = (x-1)^{v_k}\sum_{m=1}^{k} S(k,m)m!\left(\frac{1}{x-1}\right)^m$$
$$= -(1-x)^{v_k}\sum_{m=1}^{k} S(k,m)m!\left(\frac{1}{x-1}\right)^m.$$

It suffices to prove that

$$\frac{1}{x}\sum_{m=1}^{k} S(k,m)m!\left(\frac{x}{1-x}\right)^m = -(-1)^{k-1}\sum_{m=1}^{k} S(k,m)m!\left(\frac{1}{x-1}\right)^m.$$

Put $t = \frac{x}{1-x} = -1 - \frac{1}{x-1}$, then we have $x = \frac{t}{t+1}$. Thus, we need to prove

$$\frac{t+1}{t}\sum_{m=1}^{k} S(k,m)m!t^m = -(-1)^{k-1}\sum_{m=1}^{k} S(k,m)m!(-1)^m(t+1)^m.$$

This is in fact Corollary 0.4. □

The polynomial $\rho(x)$ has another expression

$$\rho(x) = \sum_{m=0}^{k}\left(S(k,m)m! + S(k,m+1)(m+1)!\right)(1-x)^{v_k-m}x^m.$$

References

[ec1] R. Stanley, *Enumerative Combinatorics*, volume 1, 2nd ed, Cambridge University Press.

[mse] Grigory M, *An answer to the post 395139*, available at `https://math.stackexchange.com/questions/395139/combinatorial-proof-of-a-stirling-number-identity`