EXERCISE 7.4.1. #9

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Problem 1. *Show that the mean value estimate* $\sum_{n\leq x}\tau(n) \sim x\log x$ *is due to the numbers* $n \leq x$ *for which* $|\omega(n) - 2\log\log x| \ll \sqrt{\log\log x}$.

Theorem 1.

$$\sum_{\substack{n\leq x\\ |\omega(n)-2\log\log x|\leq C\sqrt{\log\log x}}} \tau(n) \sim \frac{x\log x}{2\sqrt{\pi}}\int_{-C}^{C} e^{-\frac{t^2}{4}}\,dt.$$

Thus, we achieve the mean value by taking $C\to\infty$. For example,

Corollary 1.

$$\sum_{\substack{n\leq x\\ |\omega(n)-2\log\log x|\leq (\log\log x)^{0.6}}} \tau(n) \sim x\log x.$$

Corollary 2. *Let* $\mathbb{P}_x$ *be a probability measure defined on a weighted sum* $\sum\limits_{n\leq x}\tau(n)$, *i. e.*

$$\mathbb{P}_x(A) := \frac{\sum\limits_{n\leq x,\quad n\in A}\tau(n)}{\sum\limits_{n\leq x}\tau(n)}.$$

Then we have

$$\mathbb{P}_x\left(\frac{\omega(n)-2\log\log x}{\sqrt{2\log\log x}} \leq C\right) \to \Phi(C) \quad \text{as} \quad x\to\infty$$

where Φ *is the cumulative distribution function of the standard normal distribution.*

To prove the theorem, we first prove the following:

Lemma 1.

$$\sum_{\substack{n\leq x\\ |\omega(n)-2\log\log x|\leq C\sqrt{\log\log x}}} 2^{\omega(n)} \sim \frac{6}{\pi^2}\frac{x\log x}{2\sqrt{\pi}}\int_{-C}^{C} e^{-\frac{t^2}{4}}\,dt.$$

Let large $K>0$ be fixed. Let c be a real number. We find the contribution of $n\leq x$ with

$$2\log\log x - (c-\frac{1}{K})\sqrt{\log\log x}+1 < \omega(n) \leq 2\log\log x - c\sqrt{\log\log x}+1. \tag{1}$$

Let k be any integer in the above interval. Then by 7.4.1.3 (c), we have

$$\sum_{\substack{n\leq x\\ \omega(n)=k}} 1 \sim \frac{3}{\pi^2}\frac{x}{\sqrt{2\pi(k-1)}}e^{k-1+(k-1)\log_3 x-(k-1)\log(k-1)-\log_2 x}$$

Let $k = 2\log\log x - t\sqrt{\log\log x}+1$ so that $c-\frac{1}{K} < t \leq c$. Then

$$\sum_{\substack{n\leq x\\ \omega(n)=k}} 1 \sim \frac{3}{\pi^2}\frac{x}{2(\log x)^{2\log 2}\sqrt{\pi\log_2 x}}e^{\log_2 x + t\sqrt{\log_2 x}\log 2-\frac{t^2}{4}} \sim \frac{3}{\pi^2}\frac{x\log x}{2(\log x)^{2\log 2}\sqrt{\pi}\sqrt{\log_2 x}}2^{t\sqrt{\log_2 x}}e^{-\frac{t^2}{4}}.$$

Considering the weight $2^{\omega(n)}$, we have

$$\sum_{\substack{n\le x\\ \omega(n)=k}} 2^{\omega(n)} \sim \frac{6}{\pi^2}\frac{x\log x}{2\sqrt{\pi}\sqrt{\log_2 x}}e^{-\frac{t^2}{4}}.$$

The number of integers in the interval (1) is $\frac{1}{K}\sqrt{\log_2 x}+O(1)$. Thus,

$$\sum_{\substack{n\le x\\ n\in(1)}} 2^{\omega(n)} \sim \frac{6}{\pi^2}\frac{x\log x}{2\sqrt{\pi}\sqrt{\log_2 x}}e^{-\frac{(c+O(1/K))^2}{4}}\frac{1}{K}\sqrt{\log_2 x} = \frac{6}{\pi^2}\frac{x\log x}{2\sqrt{\pi}}\frac{1}{K}e^{-\frac{(c+O(1/K))^2}{4}}.$$

Let $C>0$ and taking the subdivision of interval $(-C,C)$ by the subintervals of length $\frac{1}{K}$, we obtain that

$$LS(e^{-\frac{t^2}{4}},(-C,C),\frac{1}{K}) \le \left(\frac{6}{\pi^2}\frac{x\log x}{2\sqrt{\pi}}\right)^{-1}\sum_{\substack{n\le x\\ |\omega(n)-2\log_2 x-1|\le C\sqrt{\log_2 x}}} 2^{\omega(n)} \le US(e^{-\frac{t^2}{4}},(-C,C),\frac{1}{K}).$$

Letting $K\to\infty$, we obtain the lemma.

To prove the theorem, we use the elementary identity

$$\tau(n)=\sum_{d^2m=n} 2^{\omega(m)}.$$

Then we have

$$\sum_{\substack{n\le x\\ |\omega(n)-2\log_2 x|\le C\sqrt{\log_2 x}}} \tau(n) = \sum_{d\le\sqrt{x}}\sum_{\substack{m\le \frac{x}{d^2}\\ |\omega(d^2m)-2\log_2 x|\le C\sqrt{\log_2 x}}} 2^{\omega(m)}.$$

Now, we split the sum into two parts, $d\le\log_2 x$ and and $d>\log_2 x$. The contribution of the latter is negligible since

$$\sum_{d>\log_2 x}\sum_{m\le\frac{x}{d^2}} 2^{\omega(m)} \ll \sum_{d>\log_2 x}\frac{x}{d^2}\log x \ll \frac{x\log x}{\log_2 x}.$$

Since $\omega(d^2m)=\omega(m)+O(\log_3 x)$ when $d\le\log_2 x$, we have

$$\begin{aligned}\sum_{d\le\log_2 x}\sum_{\substack{m\le \frac{x}{d^2}\\ |\omega(d^2m)-2\log_2 x|\le C\sqrt{\log_2 x}}} 2^{\omega(m)} &\sim \sum_{d\le\log_2 x}\frac{x}{d^2}\frac{\log x}{2\sqrt{\pi}}\frac{6}{\pi^2}\int_{-C}^{C} e^{-\frac{t^2}{4}}\,dt\\ &\sim x\log x\frac{1}{2\sqrt{\pi}}\int_{-C}^{C} e^{-\frac{t^2}{4}}\,dt.\end{aligned}$$

Therefore, Theorem 1 follows.