HOMEWORK2

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Theorem 1. (Vector-valued central limit theorem) Let $\vec{X} = (X_1, \cdots, X_d)$ be a random variable taking values in $\mathbb{R}^d$ with finite second moment. Define the covariance matrix $\Sigma(\vec{X})$ to be the $d \times d$ matrix Σ whose ij^{th} entry is the covariance $\mathbb{E}(X_i - \mathbb{E}(X_i))(X_j - \mathbb{E}(X_j))$.

- Covariance matrix is positive semi-definite real symmetric.

Proof. We normalize X_i by replacing X_i with $X_i - \mathbb{E}(X_i)$ to have mean 0. So, we assume that $\mathbb{E}(X_i) = 0$ for all i. Now, ij^{th} entry of the covariance matrix Σ is $\mathbb{E}(X_i X_j)$. We compute $\vec{x} \cdot \Sigma\vec{x}$ for any $\vec{x} = (x_1 \cdots x_n)^T \in \mathbb{R}^d$.

$$\sum_{i,j} x_i x_j \mathbb{E}(X_i X_j) = \mathbb{E} \sum_{i,j} x_i x_j X_i X_j = \mathbb{E}(\sum_i x_i X_i)^2 \geq 0.$$

Thus, Σ is positive semi-definite. □

- Conversely, given any positive definite real symmetric $d \times d$ matrix Σ and $\mu \in \mathbb{R}^d$, the normal distribution $N(\mu, \Sigma)_{\mathbb{R}^d}$, given by the absolutely continuous measure

$$\frac{1}{((2\pi)^d \det\Sigma)^{1/2}} e^{-(x-\mu)\cdot\Sigma^{-1}(x-\mu)/2} dx,$$

has mean μ and covariance matrix Σ, and has a characteristic function given by

$$F(t) = e^{-i\mu\cdot t} e^{-t\cdot\Sigma t/2}.$$

Proof. We use the Spectral Theorem to diagonalize Σ, let $\Sigma = U^T D U$ for some orthogonal matrix $U = (U^1 \cdots U^d)$, and $D = \text{diag}(\lambda_1, \cdots, \lambda_d)$. Let $U^i = (U^i_1 \cdots U^i_d)^T$. As in the first part, we can assume $\mu = 0$. Then we have to show that $\mathbb{E}(X_i X_j) = \Sigma_{ij}$ for each i, j. We make a change of variables $x = U^T y$.

$$\begin{aligned}
\mathbb{E}(X_i X_j) &= \int_{\mathbb{R}^d} \frac{x_i x_j}{((2\pi)^d \det\Sigma)^{1/2}} e^{-x\cdot\Sigma^{-1}x/2} dx \\
&= \int_{\mathbb{R}^d} \frac{U^i y \cdot U^j y}{((2\pi)^d \det\Sigma)^{1/2}} e^{-y^T D^{-1} y/2} dy \\
&= \int_{\mathbb{R}^d} \frac{\sum_r U^i_r U^j_r y_r^2}{((2\pi)^d \det\Sigma)^{1/2}} e^{-(\sum_r \lambda_r^{-1} y_r^2)/2} dy \\
&= \sum_{r=1}^d U^i_r U^j_r \lambda_r = \Sigma_{ij}.
\end{aligned}$$

For the characteristic function,

$$\mathbb{E}(e^{it\cdot\vec{X}}) = \int_{\mathbb{R}^d} \frac{e^{it\cdot x}}{((2\pi)^d \det\Sigma)^{1/2}} e^{-x\cdot\Sigma^{-1}x/2}dx$$
$$= \int_{\mathbb{R}^d} \frac{e^{it\cdot U^T y}}{((2\pi)^d \det\Sigma)^{1/2}} e^{-y^T D^{-1}y/2}dy$$

After completing the square of the form,

$$-\frac{y^2}{2\lambda} + iay = -\frac{1}{2\lambda}(y - ia\lambda)^2 - \frac{a^2\lambda}{2},$$

Together with the substitution $a_r = t_1 U_r^1 + \cdots + t_d U_r^d$, we obtain

$$\int_{\mathbb{R}^d} \frac{e^{it\cdot U^T y}}{((2\pi)^d \det\Sigma)^{1/2}} e^{-y^T D^{-1}y/2}dy = \int_{\mathbb{R}^d} \frac{1}{((2\pi)^d \det\Sigma)^{1/2}} e^{-\sum_r\left((y_r - ia_r\lambda_r)^2/(2\lambda_r) + a_r^2\lambda_r/2\right)}dy$$
$$= e^{-(\sum_r a_r^2\lambda_r)/2} = e^{-t^T U^T DUt/2} = e^{-t\cdot\Sigma t/2}.$$

□

• (Degenerate Case) We define the normal distribution $N(\mu, \Sigma)_{\mathbb{R}^d}$ as below, then we still have the characteristic function

$$F(t) = e^{-i\mu\cdot t}e^{-t\cdot\Sigma t/2}.$$

Proof. Again, we normalize and assume $\mu = 0$. As before, we use the change of variable $\vec{Y}^T = (Y_1 \cdots Y_d)^T = U\vec{X}^T$ where $\Sigma = U^T DU$, and $D = \text{diag}(\lambda_1, \cdots, \lambda_d)$. Further, we assume that there exists $K < d$ such that $\lambda_r > 0$ for $r \le K$, and $\lambda_r = 0$ for $K + 1 \le r \le d$. There does not exist a probability density function in this case, instead we use cumulative distribution function for $\vec{Y}$ defined by the measure

$$\prod_{r=1}^{K} \frac{1}{(2\pi\lambda_r)^{1/2}} e^{-\lambda_r^{-1}y_r^2/2}dy_r \prod_{r=K+1}^{d} \delta(y_r)dy_r.$$

, where δ is the Dirac Delta.

Clearly this distribution has mean 0, and satisfies $\mathbb{E}(Y_iY_j) = \delta_{ij}\lambda_i$. This implies

$$\mathbb{E}(X_iX_j) = \sum_r U_r^i U_r^j \mathbb{E}(Y_r^2) = \sum_r U_r^i U_r^j \lambda_r = \Sigma_{ij}.$$

For the characteristic function, define a_r as before, then we have

$$\mathbb{E}e^{it\cdot\vec{X}} = \mathbb{E}e^{it\cdot U^T\vec{Y}} = \mathbb{E}e^{i\sum_r a_r Y_r}.$$

After completing square for $r \le K$, we obtain

$$\mathbb{E}e^{i\sum_r a_r Y_r}$$
$$= \prod_{r\le K} \int_{\mathbb{R}} \frac{1}{(2\pi\lambda_r)^{1/2}} e^{-\left((y_r - ia_r\lambda_r)^2/(2\lambda_r) + a_r^2\lambda_r/2\right)}dy_r \prod_{r=K+1}^{d} \int_{\mathbb{R}} e^{ia_r y_r}\delta(y_r)dy_r$$
$$= e^{-(\sum_{r\le K} a_r^2\lambda_r)/2} = e^{-(\sum_r a_r^2\lambda_r)/2} = e^{-t^T U^T DUt/2} = e^{-t\cdot\Sigma t/2}.$$

Thus, our claim is proved. □

- If $\vec{S_n} := \vec{X}_1 + \cdots + \vec{X}_n$ is the sum of n iid copies of $\vec{X}$, then $\vec{Z_n} = \frac{\vec{S_n} - n\mu}{\sqrt{n}}$ converges in distribution to $N(0, \Sigma(X))_{\mathbb{R}^d}$.

Proof. The Taylor's Theorem gives

$$\begin{aligned} F_{\vec{X}}(t) &= \mathbb{E}e^{it\cdot\vec{X}} \\ &= 1 + \mathbb{E}it\cdot\vec{X} + \frac{1}{2}\mathbb{E}(it\cdot\vec{X})^2 + o(|t|^2) \\ &= \exp\left(-\frac{1}{2}\mathbb{E}(t\cdot\vec{X})^2 + o(|t|^2)\right). \end{aligned}$$

Now, using independence condition, we have

$$F_{\vec{Z_n}}(t) = (\mathbb{E}e^{it\cdot\vec{X}/\sqrt{n}})^n = F_{\vec{X}}\left(\frac{t}{\sqrt{n}}\right)^n$$

Using $\mathbb{E}(t_1X_1 + \cdots t_dX_d)^2 = \mathbb{E}\sum_{i,j} t_it_jX_iX_j = \sum_{i,j} t_it_j\mathbb{E}X_iX_j = t\cdot\Sigma t$, and letting $n \to \infty$, we obtain

$$F_{\vec{X}}\left(\frac{t}{\sqrt{n}}\right)^n \to \exp\left(-\frac{1}{2}\mathbb{E}(t\cdot\vec{X})^2\right) = \exp\left(-\frac{1}{2}t\cdot\Sigma t\right).$$

By the second part, it follows that

$$F_{\vec{Z_n}}(t) \to F_{N(0,\Sigma)}(t).$$

□