MEAN VALUE ESTIMATES OF $z^{\Omega(n)}$ WHEN $|z| \geq 2$.

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1. Introduction

Let $n = \prod_{i\leq m} p_i^{e_i}$ be the prime factorization of n. We denote $\Omega(n)$ by $\sum_{i\leq m} e_i$. Then, for any fixed complex number z, we obtain a completely multiplicative function $z^{\Omega(n)}$ as a function of n. There are many results on the average order of this function $z^{\Omega(n)}$. There are remarkable differences in the behavior of $\sum_{n\leq x} z^{\Omega(n)}$ for different values of z. If we consider main terms only, it is $Cx(\log x)^{z-1}$ when $|z| < 2$, $Cx(\log x)^2$ when $z = 2$, and we have oscillation when $z > 2$. First two cases are well-known, so in this paper, we will prove upper, lower bound results, and oscillatory behavior of $\sum_{n\leq x} z^{\Omega(n)}$ when $z > 2$. Further, we will also prove oscillatory behavior for all z such that $|z| > 2$, and z is not positive real. However, we will only prove upper bound result for the case $|z| = 2$, and $z \neq 2$.

We briefly state the known results. For $|z| < 2$, Selberg(See [4], a special case of Theorem 2) proved that:

$$\sum_{n\leq x} z^{\Omega(n)} = x(\log x)^{z-1}\left(\frac{f(1,z)}{\Gamma(z)} + O\left(\frac{1}{\log x}\right)\right) \tag{1}$$

, where

$$f(s,z) = \prod_p \left(1 - \frac{z}{p^s}\right)^{-1}\left(1 - \frac{1}{p^s}\right)^z.$$

This has been improved due to Selberg-Delange method(See [1], Theorem 2, p202): for all δ, $0 < \delta < 1$, there exist positive constants $c_1 = c_1(\delta)$, $c_2 = c_2(\delta)$, such that, uniformly for $x \geq 3$, $N \geq 0$, $|z| \leq 2 - \delta$,

$$\sum_{n\leq x} z^{\Omega(n)} = x(\log x)^{z-1}\left(\sum_{k=0}^{N} \frac{\nu_k(z)}{(\log x)^k} + O_\delta(R_N(x))\right) \tag{2}$$

, where $R_N(x) = e^{-c_1\sqrt{\log x}} + \left(\frac{c_2 N+1}{\log x}\right)^{N+1}$, and ν_k are functions depending only on z.

When $z = 2$, Bateman(See [2], (3)) obtained a result:

$$\sum_{n\leq x} 2^{\Omega(n)} = C_0 x(\log x)^2 + C_1 x\log x + O(x) \tag{3}$$

, where C_i are constant, and the error term $O(x)$ is the best possible.

Now, we proceed on our results. Let $|z| \geq 2$, and let $p_1 = 2 < \cdots < p_r \leq |z| < p_{r+1} < \cdots$ be prime numbers. We define functions A and B which we will use throughout this paper:

$$A(s)=\prod_{p\le|z|}\left(1-\frac{z}{p^s}\right)^{-1}=\sum_{n=1}^{\infty}\frac{a_n}{n^s}, \tag{4}$$

$$B(s)=\prod_{p\le|z|}\left(1-\frac{1}{p^s}\right)^{z}\prod_{p>|z|}\left(1-\frac{z}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{z}=\sum_{n=1}^{\infty}\frac{b_n}{n^s}. \tag{5}$$

Also, we denote $d_z(n)$ by the identity:

$$\zeta(s)^z=\prod_{p}\left(1-p^{-s}\right)^{-z}=\sum_{n=1}^{\infty}d_z(n)n^{-s}. \tag{6}$$

The Dirichlet series for $A(s), B(s)$, and $\zeta(s)^z$ are absolutely convergent respectively on $\sigma>\frac{\log|z|}{\log 2}$, $\sigma>\frac{\log|z|}{\log p_{r+1}}$, and $\sigma>1$. Clearly, we have $\frac{\log|z|}{\log p_{r+1}}<1\le\frac{\log|z|}{\log 2}$, since $2\le|z|<p_{r+1}$. Then, the Dirichlet series $F(s)=\sum_{n=1}^{\infty}z^{\Omega(n)}n^{-s}$ satisfies the identity:

$$F(s)=A(s)B(s)\zeta(s)^z \tag{7}$$

, where the series is absolute convergent on $\sigma>\frac{\log|z|}{\log 2}$. In case of $z>2$, we see that $A(s)$ has singularities on $s=\frac{\log z+2k\pi i}{\log 2}$ for all integers k. Thus, using Perron's formula(See [3], Lemma 3.12, p60) directly on $\sum_{n\le x}z^{\Omega(n)}$ is difficult because of the residues from too many singularities. Indeed, we derive Theorem 1,2, and Theorem 3 without using Perron's formula. The first result is the upper and lower bound.

Theorem 1. Let $z>2$ be fixed, and $x\ge1$. Then there exists a constant B_z such that:

$$\frac{1}{z}x^{\frac{\log z}{\log 2}}\le\sum_{n\le x}z^{\Omega(n)}\le B_z x^{\frac{\log z}{\log 2}}. \tag{8}$$

From Theorem 1, we can also derive the oscillatory behavior of $\sum_{n\le x}z^{\Omega(n)}$.

Theorem 2. Let $z>2$ be fixed. Then,

$$\limsup_{x\to\infty}x^{-\frac{\log z}{\log 2}}\sum_{n\le x}z^{\Omega(n)}-\liminf_{x\to\infty}x^{-\frac{\log z}{\log 2}}\sum_{n\le x}z^{\Omega(n)}\ge1. \tag{9}$$

On the other hand, we can extend z to non-real values.

Theorem 3. Let $|z|>2$, and z is not a positive real number. For $x\ge1$, we have:

$$\operatorname{Re}\sum_{n\le x}z^{\Omega(n)}=\Omega_{\pm}\left(x^{\frac{\log|z|}{\log 2}}\right). \tag{10}$$

In the remaining case, we have an upper bound.

Theorem 4. Let $|z|=2$, and $z\neq2$. For $x\ge3$, we have:

$$\sum_{n\le x}z^{\Omega(n)}=O\left(\frac{x\log x}{(\log\log x)^{2-\operatorname{Re}z}}\right). \tag{11}$$

2. Proof of Theorem 2

Now, we prove Theorem 2 from Theorem 1.

Proof of Theorem 2. Theorem 1 implies that, $\beta(x) = x^{-\frac{\log z}{\log 2}} \sum_{n \leq x} z^{\Omega(n)}$ is a bounded function of x. Consider a bounded sequence $\{S_N\}_{N=1}^{\infty}$,

$$S_N = \beta(2^N - 1) = (2^N - 1)^{-\frac{\log z}{\log 2}} \sum_{n \leq 2^N - 1} z^{\Omega(n)}. \tag{12}$$

We can find a subsequence $\{S_{N_i}\}_{i=1}^{\infty}$ which converges to K_z. Then, we have

$$\begin{aligned}\beta(2^{N_i}) &= (2^{N_i})^{-\frac{\log z}{\log 2}} \left(\sum_{n \leq 2^{N_i} - 1} z^{\Omega(n)} + z^{N_i} \right) \\ &= \left(\frac{2^{N_i} - 1}{2^{N_i}} \right)^{\frac{\log z}{\log 2}} S_{N_i} + 1.\end{aligned}$$

Since $\beta(2^{N_i}) \to K_z + 1$ as $i \to \infty$, the difference between $\limsup_{x\to\infty} \beta(x)$, and $\liminf_{x\to\infty} \beta(x)$ is at least 1. Hence, Theorem 2 is proved.

3. Proof of Theorem 1

A simple observation gives the lower bound,

$$\sum_{n \leq x} z^{\Omega(n)} \geq \sum_{2^e \leq x} z^e \geq z^{\lfloor \log x / \log 2 \rfloor} \geq z^{-1} x^{\log z / \log 2}. \tag{13}$$

We remark that $\sum_{n \leq x} a_n = \sum_{p_1^{e_1} \cdots p_r^{e_r} \leq x} z^{e_1 + \cdots + e_r}$, and derive the following lemma.

Lemma 1. For $x \geq 1$,

$$\sum_{n \leq x} a_n = O\left(x^{\frac{\log z}{\log 2}} \right). \tag{14}$$

Proof. We use induction on r.

When $r = 1$, note that $\sum_{2^e \leq x} z^e = \frac{z^{\lfloor \frac{\log x}{\log 2} \rfloor + 1} - 1}{z - 1} \leq \frac{z}{z-1} x^{\frac{\log z}{\log 2}}$.
Let $r > 1$, and assume the result for $r - 1$, namely,

$$\sum_{p_1^{e_1} \cdots p_{r-1}^{e_{r-1}} \leq x} z^{e_1 + \cdots + e_{r-1}} \leq C_{z,r-1} x^{\frac{\log z}{\log 2}}. \tag{15}$$

Then, we have

$$\begin{aligned}\sum_{p_1^{e_1} \cdots p_r^{e_r} \leq x} z^{e_1 + \cdots + e_r} &= \sum_{p_r^{e_r} \leq x} z^{e_r} \sum_{p_1^{e_1} \cdots p_{r-1}^{e_{r-1}} \leq x p_r^{-e_r}} z^{e_1 + \cdots + e_{r-1}} \\ &\leq \sum_{p_r^{e_r} \leq x} z^{e_r} C_{z,r-1} \left(\frac{x}{p_r^{e_r}} \right)^{\frac{\log z}{\log 2}} \\ &\leq C_{z,r} x^{\frac{\log z}{\log 2}}\end{aligned}$$

, where $C_{z,r} = C_{z,r-1} \sum_e \left(z^{1 - \log p_r / \log 2} \right)^e = C_{z,r-1} \left(1 - z^{1 - \log p_r / \log 2} \right)^{-1}$. This gives the result for r, and completes the proof of Lemma 1. Further, we can write down Lemma 1 in the form:

$$\sum_{n\le x} a_n \le C_z x^{\frac{\log z}{\log 2}} \tag{16}$$

, where

$$C_z = \frac{z}{z-1}\prod_{2<p\le z}\left(1 - z^{1-\frac{\log p}{\log 2}}\right)^{-1}. \tag{17}$$

□

Now, we are ready to prove Theorem 1.

Proof of Theorem 1 (upper bound). By (7), we have $\sum_{n\le x} z^{\Omega(n)} = \sum_{uvw\le x} b_u d_z(v) a_w$. Then by Lemma 1,

$$\begin{aligned}\sum_{uvw\le x} b_u d_z(v) a_w &= \sum_{uv\le x} b_u d_z(v) \sum_{w\le \frac{x}{uv}} a_w \\ &\le \sum_{uv\le x} |b_u| d_z(v) C_z \left(\frac{x}{uv}\right)^{\frac{\log z}{\log 2}} \\ &\le C_z \sum_u \frac{|b_u|}{u^{\frac{\log z}{\log 2}}} \sum_v \frac{d_z(v)}{v^{\frac{\log z}{\log 2}}} x^{\frac{\log z}{\log 2}}.\end{aligned}$$

The u-sum is convergent, since the Dirichlet series for $B(s)$ is absolutely convergent for $\sigma > \frac{\log z}{\log p_{r+1}}$. Also, the v-sum is just $\zeta(\frac{\log z}{\log 2})^z$. Hence, we can write down Theorem 1 in the form:

$$\sum_{n\le x} z^{\Omega(n)} \le B_z x^{\frac{\log z}{\log 2}} \tag{18}$$

, where

$$B_z = C_z \left(\sum_u \frac{|b_u|}{u^{\frac{\log z}{\log 2}}}\right) \zeta\left(\frac{\log z}{\log 2}\right)^z. \tag{19}$$

4. Proof of Theorem 3

We begin with an oscillation lemma. For the proof, see [1], Theorem 8, p112.

Lemma 2. Let $G(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with real coefficients having a finite abscissa of convergence. Suppose there exists a real number $\sigma_0 > 0$ such that $G(s)$ has an analytic continuation which is regular at all points of the half line $[\sigma_0, \infty)$ and has a pole on the vertical line $\sigma = \sigma_0$. Then the associated summatory function satisfies

$$\sum_{n\le x} a_n = \Omega_{\pm}(x^{\sigma_0}). \tag{20}$$

Proof of Theorem 3. Note that $|z| > 2$ and z is not positive real. Let $F(s) = A(s)B(s)\zeta(s)^z$ as before. Then $G(s) = \frac{F(s)+\overline{F(\bar{s})}}{2}$ has a Dirichlet series $\sum_{n=1}^{\infty} \mathrm{Re}(z^{\Omega(n)}) n^{-s}$. Let $z = |z|e^{i\theta}$, then F has singularities on the set;

$$\left\{\frac{\log|z| + i(2\pi k \pm \theta)}{\log 2} : k \in \mathbb{Z}\right\} \tag{21}$$

Since this set does not contain $\frac{\log|z|}{\log 2}$, the Dirichlet series $G(s)$ satisfies all hypotheses for the Lemma 2 with $\sigma_0 = \frac{\log|z|}{\log 2}$. Hence, by Lemma 2,

$$\sum_{n\leq x} \mathrm{Re}(z^{\Omega(n)}) = \Omega_{\pm}(x^{\frac{\log|z|}{\log 2}}). \tag{22}$$

5. Proof of Theorem 4

Let $A(s)$, $B(s)$ be defined by:

$$A(s) = \left(1 - \frac{z}{2^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{23}$$

$$B(s) = \left(1 - \frac{1}{2^s}\right)^{z} \prod_{p>2} \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{z} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}. \tag{24}$$

Then we have $F(s) = \sum_{n=1}^{\infty} z^{\Omega(n)} n^{-s} = A(s)B(s)\zeta(s)^z$ as before. The Selberg-Delange method(See [1], Theorem 5, p191) implies that

$$\sum_{uv\leq x} b_u d_z(v) = \frac{B(1)}{\Gamma(z)} x(\log x)^{z-1} + O\left(x(\log x)^{\mathrm{Re}z-2}\right). \tag{25}$$

Thus, it follows that

$$\begin{aligned}
\sum_{n\leq x} z^{\Omega(n)} &= \sum_{uvw\leq x} a_u b_v d_z(w) \\
&= \sum_{u\leq x/2} a_u \sum_{vw\leq x/u} b_v d_z(w) + \sum_{x/2<u\leq x} a_u \sum_{vw\leq x/u} b_v d_z(w) \\
&= \sum_{u\leq x/2} a_u \left(\frac{B(1)}{\Gamma(z)} \frac{x}{u} \left(\log\frac{x}{u}\right)^{z-1} + O\left(\frac{x}{u}\left(\log\frac{x}{u}\right)^{\mathrm{Re}z-2}\right)\right) + O(x).
\end{aligned}$$

We treat the second error term first,

$$\begin{aligned}
\sum_{u\leq x/2} \left|\frac{a_u}{u}\right| \left(\log\frac{x}{u}\right)^{\mathrm{Re}z-2} &= \sum_{u\leq x/(2\log x)} \left|\frac{a_u}{u}\right| \left(\log\frac{x}{u}\right)^{\mathrm{Re}z-2} + \sum_{x/(2\log x)<u\leq x/2} \left|\frac{a_u}{u}\right| \left(\log\frac{x}{u}\right)^{\mathrm{Re}z-2} \\
&= O\left(\frac{\log x}{(\log\log x)^{2-\mathrm{Re}z}}\right) + O(\log\log x) = O\left(\frac{\log x}{(\log\log x)^{2-\mathrm{Re}z}}\right).
\end{aligned}$$

Using partial summation, we find that the first term is small compared to the error term above. Let $S(t) = \sum_{u\leq t} \frac{a_u}{u}$, and note that $S(t)$ is bounded function of t.

$$\begin{aligned}
\sum_{u\leq x/2} \frac{a_u}{u} \left(\log\frac{x}{u}\right)^{z-1} &= \int_{1-}^{x/2} \left(\log\frac{x}{t}\right)^{z-1} dS(t) \\
&= \left(\log\frac{x}{t}\right)^{z-1} S(t) \Big|_{1-}^{x/2} + \int_{1-}^{x/2} \frac{1}{t} S(t)(z-1) \left(\log\frac{x}{t}\right)^{z-2} dt \\
&= O(1) + O\left(\int_{1}^{x/2} \frac{1}{t}\left(\log\frac{x}{t}\right)^{\mathrm{Re}z-2} dt\right) \\
&= \begin{cases} O(1) + O\left((\log x)^{\mathrm{Re}z-1}\right) & \text{if } \mathrm{Re}z \neq 1 \\ O(1) + O(\log\log x) & \text{if } \mathrm{Re}z = 1. \end{cases}
\end{aligned}$$

Since Re$z-1<1$, we obtain the result:

$$\sum_{n\le x} z^{\Omega(n)} = O\left(\frac{x\log x}{(\log\log x)^{2-\mathrm{Re}z}}\right). \tag{26}$$

6. Further Remarks

In fact, the upper bound in Theorem 4 can be improved to $O_N(x(\log x)(\log\log x)^{\mathrm{Re}z-N})$ using a better error term in (25)(See [1], Theorem 5, p191). There are still some open problems. In the Theorem 2, we obtained oscillatory behavior of the function $\beta(x) = x^{-\frac{\log z}{\log 2}} \sum_{n\le x} z^{\Omega(n)}$. However, we do not know how to obtain $\limsup_{x\to\infty} \beta_z(x)$, and $\liminf_{x\to\infty} \beta_z(x)$ explicitly as a function of z. Also, in the Theorem 4, we only have upper bound result, and still do not know what the best possible bound is. In case of Theorem 4, the function $F(s) = \sum_{n=1}^{\infty} z^{\Omega(n)} n^{-s}$ does not satisfy the hypothesis of Lemma 2, but the author conjectures that $\mathrm{Re} \sum_{n\le x} z^{\Omega(n)} = \Omega_{\pm}(x)$ holds.

References

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