225A DIFFERENTIAL TOPOLOGY FINAL

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Problem 1.

From Whitney's Embedding Theorem, we can assume that N is an embedded submanifold of $\mathbb{R}^K$ for some $K > 0$. Then it is possible to define distance function. Now we use ε-Neighborhood Theorem. There exists an open neighborhood

$$U = M^\varepsilon = \bigcup_{y \in M} \{w \in N : |w - y| < \varepsilon(y)\}$$

where $\varepsilon : M \to \mathbb{R}$ is a smooth positive function on M, and $\pi : U \to M$ is a submersion defined by $\pi(w)$ being the unique closest point from w to M. Then, we claim that the inclusion $i : M \to U$ is proper. For, let $K \subseteq U$ be a compact set in U. We have compactness of $\pi(K)$ by continuity of π, and
$i^{-1}(K) = K \cap M \subseteq \pi(K)$ is a closed subset of a compact set, thus $i^{-1}(K)$ is compact. Hence, $i : M \to U$ is proper.

Problem 2.

First, we remark that $M(n, p; k+)$, the set of matrices in $M(n, p)$ whose rank is at least k, is an open subset of $M(n, p)$. This can be shown by considering the function $f : M(n, p) \to \mathbb{R}$ given by:

$$f(A) = \sum_{B \in A_{k \times k}} (detB)^2$$

where $A_{k\times k}$ is the set of all $k \times k$ submatrices of A. Indeed, $M(n, p; k+) = \{A \in M(n, p) | f(A) > 0\}$, and the continuity of f gives the result.
Define the sets of matrices $M(n, p; k)_1$, and $M(n, p; k+)_1$, whose determinant of the first $k \times k$ submatrix is nonzero. These sets form an open subset of $M(n, p; k)$, and $M(n, p; k+)$ respectively, by the continuity of determinant. We claim that $M(n, p; k)_1$ is a $np - (n - k)(p - k)$ dimensional submanifold of $M(n, p; k+)_1$. Then, the global result will follow from this local result. Now, define a map $g : M(n, p; k+)_1 \to \mathbb{R}^{(n-k)(p-k)}$ by

$$g(A) = (detA_{ij})_{\substack{k+1 \le i \le n \\ k+1 \le j \le p}}$$

where A_{ij} is a $(k + 1) \times (k + 1)$ submatrix obtained by attaching the column vector $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{kj} \end{pmatrix}$ to the right of the first $k \times k$ submatrix A_k of A, the row vector $\begin{pmatrix} a_{i1} & \cdots & a_{ik} \end{pmatrix}$ to the bottom, and (a_{ij}) to the right bottom corner, where $A = (a_{uv})_{\substack{1 \le u \le n \\ 1 \le v \le p}}$. Then, clearly g is a smooth function, and $M(n, p; k)_1 = g^{-1}(0)$. We claim that $0 \in \mathbb{R}^{(n-k)(p-k)}$ is a regular value of g, then the result will follow from the preimage theorem. To show this, we find the jacobian of g.

$$Jg = \left(\frac{\partial}{\partial a_{uv}} detA_{ij}\right)_{\substack{k+1\le i\le n\\ k+1\le j\le p\\ 1\le u\le n\\ 1\le v\le p}}.$$

For each (i,j) with $k+1 \le i \le n$, $k+1 \le j \le p$, we have

$$\frac{\partial}{\partial a_{uv}} detA_{ij} = \begin{cases} detA_k \neq 0 & \text{if } u = i, v = j \\ 0 & \text{if } k+1 \le u \le n, \quad k+1 \le v \le p, \quad u \neq i, \quad v \neq j. \end{cases}$$

This shows that the Jg has rank $(n-k)(p-k)$, so 0 is the regular value of g. By the preimage theorem, $M(n,p;k)_1 = g^{-1}(0)$ is a $dimM(n,p;k+)_1 - (n-k)(p-k)$ dimensional submanifold of $M(n,p;k+)_1$. Since $M(n,p;k+)_1$ is an open subset of $M(n,p)$, we have $dimM(n,p;k+)_1 = dimM(n,p) = np$. Hence, we have the global result $M(n,p;k)$ is a $np-(n-k)(p-k)$ dimensional submanifold of $M(n,p)$.

Problem 3.

•(Green's Formula) Let W be a compact domain in $\mathbb{R}^2$ with smooth boundary $\partial W = \gamma$. Then,

$$\int_\gamma f dx + g dy = \int_W \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy.$$

Proof. Let $w = fdx + gdy$, then

$$\begin{aligned} dw &= df \wedge dx + dg \wedge dy \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) \wedge dy \\ &= \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy. \end{aligned}$$

By the generalized Stokes theorem, we have

$$\int_\gamma f dx + g dy = \int_{\partial W} w = \int_W dw = \int_W \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy.$$

•(Stokes Theorem) Let S be a compact oriedted two-manifold in $\mathbb{R}^3$ with boundary, and let $\vec{F} = (f_1, f_2, f_3)$ be a smooth vector field in a neighborhood of S. Then,

$$\int_S (curl\vec{F} \cdot \vec{n}) dA = \int_{\partial S} f_1 dx_1 + f_2 dx_2 + f_3 dx_3.$$

Proof. Let $w = f_1dx_1 + f_2dx_2 + f_3dx_3$, then by the similar calculation above, we have:

$$dw = g_1 dx_2 \wedge dx_3 + g_2 dx_3 \wedge dx_1 + g_3 dx_1 \wedge dx_2,$$

where

$$g_1 = \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \quad g_2 = \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \quad g_3 = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}.$$

We need a lemma: Let $\vec{n} = (n_1, n_2, n_3)$ be the outward pointing normal. Then,

$$dA = n_1 dx_2 \wedge dx_3 + n_2 dx_3 \wedge dx_1 + n_3 dx_1 \wedge dx_2$$

, and

$$n_1 dA = dx_2 \wedge dx_3$$
$$n_2 dA = dx_3 \wedge dx_1$$
$$n_3 dA = dx_1 \wedge dx_2.$$

For, the first formula is equivalent to

$$dA(v,w) = det \begin{pmatrix} v \\ w \\ \vec{n} \end{pmatrix}$$

where $v, w \in T_x S$, and this is precisely the definition of dA. To prove other equations, let $z \in \mathbb{R}^3$. Since $v \times w = \alpha \vec{n}$ for some $\alpha \in \mathbb{R}$, we have

$$\langle z, \vec{n} \rangle \langle v \times w, \vec{n} \rangle = \langle z, \vec{n} \rangle \alpha = \langle z, \alpha \vec{n} \rangle = \langle z, v \times w \rangle.$$

By plugging in $e_1,\ e_2,\ e_3$ in place of z, we obtain above three formulae.
Thus, we have

$$dw = g_1 dx_2 \wedge dx_3 + g_2 dx_3 \wedge dx_1 + g_3 dx_1 \wedge dx_2 = (g_1, g_2, g_3) \cdot \vec{n} dA = (curl \vec{F} \cdot \vec{n}) dA$$

By the generalized Stokes theorem, we have

$$\int_S (curl \vec{F} \cdot \vec{n}) dA = \int_S dw = \int_{\partial S} w = \int_{\partial S} f_1 dx_1 + f_2 dx_2 + f_3 dx_3.$$

•(Divergence Theorem) Let W be a compact domain in $\mathbb{R}^3$ with smooth boundary, and let $\vec{F} = (f_1, f_2, f_3)$ be a smooth vector field on W. Then

$$\int_W (div \vec{F}) dxdydz = \int_{\partial W} (\vec{n} \cdot \vec{F}) dA.$$

Proof. Let $w = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$. Then,

$$dw = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3 = (div \vec{F}) dx_1 \wedge dx_2 \wedge dx_3$$

Now, we use the lemma in the proof of Stokes theorem, we have $w = (\vec{n} \cdot \vec{F}) dA$. By the generalized Stokes theorem, we have

$$\int_W (div \vec{F}) dxdydz = \int_W dw = \int_{\partial W} w = \int_{\partial W} (\vec{n} \cdot \vec{F}) dA.$$

Problem 4.

• Topological and differential structure on $G(k,n)$:
Let $P \in G(k,n)$, and denote $P^\perp$ the orthogonal complement of P in $\mathbb{R}^n$, and let $U_P = \{W \in G(k,n) | W \cap P^\perp = (0)\}$. Let $L(P, P^\perp)$ denote the vector space of linear maps from P to $P^\perp$. Define a map $\phi_P : L(P, P^\perp) \to U_P$ by:

$$A \in L(P, P^\perp) \mapsto W = \{x + Ax | x \in P\}.$$

Then, it follows from the definition of U_P, that ϕ_P is bijective for each P. Now, we claim that $\{(U_P, \phi_P^{-1}) | P \in G(k,n)\}$ forms a topological and smooth structure on $G(k,n)$. We can define a topological structure to be the smallest topology that makes every ϕ_P homeomorphism. In particular, we have

$$\{\phi_P(V) | P \in G(k,n), V \subseteq L(P, P^\perp) \text{ is open}\}$$

as a basis for this topology.
It remains to show that if $U_P \cap U_Q$ is nonempty, then $\phi_Q^{-1} \circ \phi_P : \phi_P^{-1}(U_P \cap U_Q) \to$

$\phi_Q^{-1}(U_P \cap U_Q)$ is smooth. For, let $W \in U_P \cap U_Q$, and $A' = \phi_Q^{-1} \circ \phi_P(A)$. Then, we have $W = \{x + Ax | x \in P\} = \{x' + A'x' | x' \in Q\}$. Let $I_A : P \to \mathbb{R}^n$ denote the map $I_A(x) = x + Ax$, and $\pi_Q : \mathbb{R}^n \to Q$ the projection onto Q. Note that $\pi_Q \circ I_A : P \to Q$ is an isomorphism. If $x' \in Q$ then, we can find $x \in P$ such that $x' + A'x' = x + Ax$. and $\pi_Q \circ I_A(x) = x'$. From this, we have

$$A'x' = I_A \circ (\pi_Q \circ I_A)^{-1}(x') - x'.$$

Now, plugging in the orthonormal basis for Q in place of x' gives the smoothness of $\phi_Q^{-1} \circ \phi_P$. Hence, $\{(U_P, \phi_P^{-1}) | P \in G(k,n)\}$ defines a smooth structure on $G(k,n)$. Since $L(P, P^{\perp})$ is isomorphic to $M(k, n-k) \simeq \mathbb{R}^{k(n-k)}$, it follows that $G(k,n)$ is $k(n-k)$ dimensional smooth manifold.

• Compactness of $G(k,n)$:
Let $O(n)$ be the group of real $n \times n$ orthogonal matrices. We define $\Psi : O(n) \to G(k,n)$ by:

$$A \in O(n) \mapsto span\{A^1, \cdots, A^k\} \in G(k,n),$$

where $A^1, \cdots, A^k$ are the first k columns of A. We claim that Ψ is continuous. Without loss of generality, it suffices to show that there exists an open set $B_I \subseteq O(n)$ containing I with $\phi_I^{-1} \circ \Psi|_{B_I} : B_I \to U_{I_k} \to L(I_k, {I_k}^{\perp})$ is continuous at I where $I_k = span\{I^1, \cdots, I^k\}$. This follows from the continuity of the determinant of the first $k \times k$ block. Thus, we have proved the claim. Since $O(n)$ is a compact manifold and Ψ is continuous, we have $\Psi(O(n))$ is compact. By Gram-Schmidt process, it follows that Ψ is surjective. Hence, $G(k,n)$ is compact manifold.

• $G(k,n)$ is diffeomorphic to $G(n-k,n)$:
Define $\Phi : G(k,n) \to G(n-k,n)$ by $\Phi(P) = P^{\perp}$. Consider the following compositions,

$$L(P, P^{\perp}) \xrightarrow{\phi_P} G(k,n) \xrightarrow{\Phi} G(n-k,n) \xrightarrow{\phi_{P^{\perp}}^{-1}} L(P^{\perp}, P)$$
$$A \mapsto W \mapsto W^{\perp} \mapsto A'.$$

If $W = \{x + Ax | x \in P\}$, then we have $W^{\perp} = \{x' - A^T x' | x' \in P^{\perp}\}$. Thus, $\Phi(A) = A' = -A^T$ is an isomorphism of $L(P, P^{\perp})$ and $L(P^{\perp}, P)$. Hence, $G(k,n)$ is diffeomorphic to $G(n-k,n)$.

Problem 5.

We need the following two propositions:
Proposition1: Suppose $f : X \to Y$ is transversal to Z. If $X = \partial W$, W is compact, $f : X \to Y$ extends to $F : W \to Y$, and F is transversal to Z, then $I(f, Z) = 0$.
Proposition2: Homotopic maps always have the same intersection numbers.

•(i) Under our assumptions, we can extend $u : M \times N \to S^{n-1}$ to $v : D \times N \to S^{n-1}$ defined by:

$$v(x,y) = \frac{F(x) - g(y)}{\|F(x) - g(y)\|}.$$

Since we have $\partial(D \times N) = (\partial D) \times N = M \times N$, we can use proposition1 with $Z = \{y\} \subseteq S^{n-1}$. Hence, by proposition1, $I(u, \{y\}) = deg(u) = L(f,g) = 0$.

•(ii) Let f_t, g_t be homotopy of f_0, and g_0 respectively. In addition, suppose we have $im(f_t)$ does not intersect $im(g_t)$ for each $t \in [0,1]$. Then,

$$u_t = \frac{f_t(x) - g_t(y)}{\|f_t(x) - g_t(y)\|}.$$

is a homotopy of u_0. Hence, by proposition2, we have $L(f_1, g_1) = I(u_1, \{y\}) = I(u_0, \{y\}) = L(f_0, g_0)$.

Problem 6.

•(i)
•(ii)
•(iii)

Problem 7.

• (I) We claim that there exists a natural orientation on some neighborhood of the diagonal Δ in $X \times X$.
Proof. We can cover a neighborhood of Δ by local parametrizations $\phi \times \phi : U \times U \to X \times X$, where $\phi : U \to X$ is a local parametrization of X. We can give a product orientation on $X \times X$ by the orientation of $\phi \times \phi$, and this does not depend on a specific choice of orientation of X.
• (II) Let Z be a compact submanifold of Y, both oriented, with $dimZ = \frac{1}{2}dimY$. We have $I(Z, Z) = I(Z \times Z, \Delta)$, where Δ is the diagonal of Y.
Proof. $I(Z, Z) = I(i, Z) = I(i, i) = (-1)^{dimZ} I(i \times i, \Delta) = (-1)^{dimZ} I(Z \times Z, \Delta)$, and if $dimZ$ is odd, then $I(Z, Z) = (-1)^{dimZ} I(Z \times Z, \Delta) = 0$.
• (III) If Z is a compact submanifold of Y with $dimZ = \frac{1}{2}dimY$, and Z is not oriented. By (I) we have an open neighborhood Y of the diagonal Δ in $Z \times Z$. Then, we have $dimZ = \frac{1}{2}dimY$. We define the Euler Characteristic $\chi(Z) = I(Z \times Z, \Delta)$. This is well defined, since Y is orientable by (I). Also, this definition fits in orientable case, by (II).

Problem 8.

• (I) We claim that if a vector field $\vec{v}$ on $\mathbb{R}^l$ has finitely many zeros, and the sum of the indices of its zeros is 0. Then there exists a vecter field that has no zeros, yet equals $\vec{v}$ outside a compact set. To prove this, we need a series of lemmas:
∘(1) Let $f : U \to \mathbb{R}^k$ be any smooth map defined on an open subset U of $\mathbb{R}^k$, and let x be a regular point, with $f(x) = z$. Let B be a sufficiently small closed ball centered at x, and define $\partial f : \partial B \to \mathbb{R}^k$ to be the restriction of f to the boundary of B. Then $W(\partial f, z) = +1$ if f preserves orientation at x and $W(\partial f, z) = -1$ if f reverses orientation at x.
Proof. For simplicity, take $x = 0 = z$, and set $A = df_0$. By regularity, A is bijective. Write $f(x) = Ax + \epsilon(x)$, where $\epsilon(x)/|x| \to 0$ as $|x| \to 0$(This is possible by multivariable Taylor's Theorem). Take B small enough that the map $F : \partial B \times I \to \mathbb{R}^k$ defined by $F(x, t) = (Ax + t\epsilon(x))/|Ax + t\epsilon(x)|$ is a homotopy. Since homotopic maps have the same intersection numbers, we have $W(A, 0) = W(\partial f, 0)$. Here, $W(A, 0) = +1$ if $detA$ is positive, and -1 otherwise, which is precisely the same conditions whether f preserves orientation, or otherwise.
∘(2) Let $f : B \to \mathbb{R}^k$ be a smooth map defined on some closed ball B in $\mathbb{R}^k$. Suppose that z is a regular value of f that has no preimages on the boundary sphere on the boundary sphere ∂B, and consider $\partial f : \partial B \to \mathbb{R}^k$. Then the number of preimages of z, counted with our usual orientation convention, equals the winding number $W(\partial f, z)$.
Proof. Circumscribe small balls B_t around each preimage points. then the degree of the directional map u on the boundary of $B' = B - \cup_t B_t$ is zero(since u extends to all of B'). i.e. $W(f|_{\partial B'}, z) = 0$. Thus, $W(\partial f, z) = \sum_t W(f|_{\partial B_t}, z_t)$, where z_t are

the center of B_t. By (1), the preimages z_t are counted with our usual orientation convention.

∘(3) Let B be a closed ball in $\mathbb{R}^k$, and let $f : \mathbb{R}^k - Int(B) \to Y$ be any smooth map defined outside the open ball $Int(B)$. If the restriction $\partial f : \partial B \to Y$ is homotopic to a constant, then f extends to a smooth map defined on all of $\mathbb{R}^k$ into Y.

Proof. Assume B is centered at 0, let $g_t : \partial B \to Y$ be homotopy with $g_1 = \partial f$, $g_0 = const$. Then a continuous extension of f through B is given by $f(tx) = g_t(x)$, $x \in \partial B$ and $t \in [0, 1]$. To extend smoothly, we use the smooth function $\rho : \mathbb{R} \to \mathbb{R}$ with $\rho(t) = 0$ if $t \leq 1/4$, and $\rho(t) = 1$ if $t \geq 3/4$.

∘(4) Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a smooth map with 0 as a regular value. Suppose that $f^{-1}(0)$ is finite and the number of preimage points in $f^{-1}(0)$ is zero when counted with the usual orientation convention. Assuming the special case in dimension $k-1$, there exists a mapping $g : \mathbb{R}^k \to \mathbb{R}^k - \{0\}$ such that $g = f$ outside a compact set.

Proof. $l = 1$ is a trivial case, for $l > 1$, we use induction. Take a large ball B around the origin that contains all of $f^{-1}(0)$. (2) implies that $\partial f : \partial B \to \mathbb{R}^k - \{0\}$ has winding number zero. The inductive hypothesis(Any smooth map $f : S^l \to \mathbb{R}^{l+1} - \{0\}$ having winding number 0 with respect to the origin is homotopic to a constant, here $l = k - 1$) implies that $\partial f : \partial B \to \mathbb{R}^k - \{0\}$ is homotopic to a constant. By (3), f can be extended to all B. We can use this extended f to prove the case $l = k$.

∘(Proof of the claim) Use $f = v$ in (4).

• (II) We also claim that any compact manifold X there exists a vector field with only finitely many zeros.

Proof. Assume that $X \subseteq \mathbb{R}^N$, and let $T(X)$ be its tangent bundle. Define $\rho : X \times \mathbb{R}^N \to T(X)$ by making $\rho(x, v)$ be the orthogonal projection of the vector v into $T_x(X)$. Then ρ is a submersion since it is a projection. We apply the Transversality Theorem with $S = \mathbb{R}^N$, $Y = T(X)$, and $Z = X \times \{0\}$. For some v, the vector field $x \to \rho(x, v)$ is transversal to $X \times \{0\}$. Thus, its inverse image of $X \times \{0\}$ is 0 dimensional submanifold of X. Since X is compact, it follows that this preimage is finite.

• (III) We use a varient of isotopy lemma(Given points y_i and $z_i, i = 1, \cdots, n$ in a connected manifold Y, we can find a diffeomorphism $h : Y \to Y$ isotopic to identity, with $h(y_i) = z_i$ for $i = 1, \cdots n$), to pull the vector field back to $\mathbb{R}^N$. Here, we require an additional condition that the finitely many zeros of the vector field are all contained in an open set U. Then, we use (I) to obtain a vector field without zero.

• (IV) Now, we construct an isotopy of identity without fixed points using the nonvanishing vector field on M. To do this, we use a theorem about the flow of a given vector field.

Theorem 1.

Let X be a smooth vector field on a smooth manifold M. For each $m \in M$ there exists $a(m)$ and $b(m)$ in $\mathbb{R} \cup \{\pm\infty\}$, and a smooth curve

$$\gamma_m : (a(m), b(m)) \to M$$

such that

(a) $0 \in (a(m), b(m))$ and $\gamma_m(0) = m$.

(b) γ_m is an integral curve of X.

(c) $(a(m), b(m))$ is maximal that γ_m can be extended.
(d) For each $m \in M$, there exists an open neighborhood V of m and an $\varepsilon > 0$ such that the map

$$(t, p) \mapsto X_t(p)$$

is defined and is smooth from $(-\varepsilon, \varepsilon) \times V$ into M.
(e) Let $D_t = \{m \in M | t \in (a(m), b(m))\}$, then D_t is open for each t.
(f) $\cup_{t>0} D_t = M$.
(g) Let $X_t(m) = \gamma_m(t)$, then $X_t : D_t \to D_{-t}$ is a diffeomorphism with inverse X_{-t}.
Since M is compact in our case, we can find some $t_0 > 0$ such that $M = D_{t_0}$, and X_t has no fixed points for each $t < t_0$. Hence, we have $X_{tt_0/2}$ is the desired isotopy of identity without fixed points(We have $X_0 = id$).