214A HOMEWORK

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1.1 Let $A = k[[T]]$ be the ring of formal power series with coefficients in a field k. Determine Spec$A$.

Proof. We begin with a claim that $A^\times = \{\sum a_i T^i \in A : a_i \in k, \text{ and } a_0 \in k^\times\}$. Let $\sum a_i T^i \in A^\times$, then there exists $\sum b_i T^i \in A$ such that $(\sum a_i T^i)(\sum b_i T^i) = 1$. The constant term in both side should agree, so we have $a_0 b_0 = 1$, giving that $a_0 \in k^\times$. Conversely, let $a_0 \in k^\times$. By multiplying a_0^{-1}, we can assume that $a_0 = 1$. Let $g = -\sum_{i\geq 1} a_i T^i$, then we have $\sum a_i T^i = 1 - g$. The formal power series for $\sum_{i\geq 0} g^i$ is the desired inverse for $\sum a_i T^i$. Thus, our claim is proved.

Let $P \subset A$ be a nonzero prime ideal in A. Denote the number $min\{n \in \mathbb{N} : T^n \in P\}$ by $n(P)$. By our previous claim, we have $n(P) \geq 1$, and we also have $T^{n(P)} \in P$. $n(P) \geq 2$ cannot happen because P is a prime ideal. Thus, we must have $n(P) = 1$, i.e. $T \in P$. It follows that $P = (T)$. Hence Spec$A = \{(0), (T)\}$. □

1.2 Let $\varphi : A \longrightarrow B$ be a homomorphism of finitely generated algebras over a field k. Show that the image of a closed point under Specφ is a closed point.

Proof. Let $\mathfrak{m}$ be a maximal ideal in B, it suffices to show that Spec$\varphi(\mathfrak{m}) = \varphi^{-1}(\mathfrak{m})$ is a maximal ideal in A. By 1st isomorphism theorem, φ induces an embedding $A/\varphi^{-1}(\mathfrak{m}) \hookrightarrow B/\mathfrak{m}$. Then Weak Nullstellensatz shows that $B/\mathfrak{m}$ is a finite field extension of k. It follows that $A/\varphi^{-1}(\mathfrak{m})$ is an integral domain which is finite dimensional k-vector space. For any $a \in (A/\varphi^{-1}(\mathfrak{m}))\backslash\{0\}$, the set $\{1, a, a^2, \cdots\}$ is linearly dependent over k. Thus, a satisfies a polynomial equation $\sum_{i\leq N} c_i a^i = 0$ with $c_i \in k$ and $c_0 \neq 0$. This implies that $a\sum_{1\geq i\geq N} c_i a^{i-1} = -c_0 \in k^\times$. Hence a is invertible in $A/\varphi^{-1}(\mathfrak{m})$, giving that $A/\varphi^{-1}(\mathfrak{m})$ is a field. □

1.3 Let $k = \mathbb{R}$ be the field of real numbers. Let $A = k[X,Y]/(X^2+Y^2+1)$. We wish to describe Spec$A$. Let x, y be the respective images of X, Y in A.

(a) Let $\mathfrak{m}$ be a maximal ideal of A. Show that there exist $a, b, c, d \in k$ such that $x^2 + ax + b, y^2 + cy + d \in \mathfrak{m}$. Using the relation $x^2 + y^2 + 1 = 0$, show that $\mathfrak{m}$ contains an element $f = \alpha x + \beta y + \gamma$ with $(\alpha, \beta) \neq (0, 0)$. Deduce from this that $\mathfrak{m} = fA$.

Proof. Note that $A/\mathfrak{m} = \mathbb{C}$. This is because it is a finite extension of k by Weak Nullstellensatz, and it contain x, y with $x^2 + y^2 + 1 = 0$. Thus, it is clear that x, y satisfy some quadratic equation over k. We add up those quadratic equations to obtain

$$ax + cy + b + d - 1 = 0 \in A/\mathfrak{m}.$$

(Case1) $(a, c) \neq (0, 0)$: We may take $f = ax + cy + b + d - 1 \in \mathfrak{m}$.
(Case2) $a = c = 0$, and hence $b+d = 1$: Either $x, y \in A/\mathfrak{m}$ are both pure imaginary, or one of them is real. Note that x, y are not both 0. Thus, we can find α, β, γ such that $\alpha x + \beta y + \gamma = 0 \in A/\mathfrak{m}$.

We now consider A/fA, assume that $f = \alpha x + \beta y + \gamma$ with $\alpha \neq 0$. Then A/fA becomes

$$k[X,Y]/(X^2+Y^2+1,f) = k[X]/(X^2+((-\beta/\alpha)x-\gamma/\alpha)^2+1) = \mathbb{C}.$$

Therefore, fA is a maximal ideal in A, and combined with $fA \subset \mathfrak{m}$, we have $\mathfrak{m} = fA$. □

(b) Show that the map $(\alpha,\beta,\gamma) \mapsto (\alpha x+\beta y+\gamma)A$ establishes a bijection between the subset $\mathbb{P}(k^3)\backslash\{(0,0,1)\}$ of the projective space $\mathbb{P}(k^3)$ and the set of maximal ideals of A.

Proof. The same argument that we showed $\mathfrak{m} = fA$ in part (a), shows that the map is well-defined. Moreover, part (a) itself shows surjectivity. Suppose that two distinct members (α,β,γ) and (α',β',γ') in $\mathbb{P}(k^3)\backslash\{(0,0,1)\}$ map to the same maximal ideal $\mathfrak{m}$ in A. We may consider the following cases without loss of generality,
(Case1) $\alpha = \alpha' = 1$, $\beta = \beta'$, $\gamma \neq \gamma'$: This gives a contradiction, since $\gamma-\gamma' \in k^{\times}\cap\mathfrak{m}$ cannot hold.
(Case2) $\alpha = \alpha' = 1$, $\beta \neq \beta'$: By subtraction, we obtain a real number y_0 such that $y_0 \in A/\mathfrak{m}$. Plugging in $y = y_0$, we obtain a real number x_0 such that $x_0 \in A/\mathfrak{m}$. Furthermore, $x_0^2 + y_0^2 + 1 = 0$ must hold, which is a contradiction.
(Case3) $\alpha = 1$, $\beta = 0$, $\alpha' = 0$, $\beta' = 1$: This also give two real numbers $x_0, y_0 \in A/\mathfrak{m}$ with $x_0^2 + y_0^2 + 1 = 0$, which is a contradiction. Hence, the map is injective. □

(c) Let $\mathfrak{p}$ be a non-maximal prime ideal of A. Show that the canonical homomorphism $k[X] \longrightarrow A$ is finite and injective. Deduce from this that $\mathfrak{p} \cap k[X] = 0$. Let $g \in \mathfrak{p}$, and let $g^n + a_{n-1}g^{n-1} + \cdots + a_0 = 0$ be an integral equation with $a_i \in k[X]$. Show that $a_0 = 0$. Conclude that $\mathfrak{p} = 0$.

Proof. The homomorphism $k[X] \longrightarrow A$ is injective, since its kernel is $k[X]\cap(X^2 + Y^2+1) = 0$. It is clear that A is an integral extension of $k[X]$. It is also true that $A/\mathfrak{p}$ is an integral extension of $k[X]/\mathfrak{p}\cap k[X]$. They are both integral domains. Since $A/\mathfrak{p}$ is not a field, $k[X]/\mathfrak{p}\cap k[X]$ is not a field. In $k[X]$, only non-maximal prime ideal is (0). Thus, $\mathfrak{p}\cap k[X] = 0$.

Suppose $\mathfrak{p}$ contains a nonzero member g. Choose the integral equation for g with minimal degree n. From the integral equation, we have $a_0 \in \mathfrak{p}\cap k[X]$. Thus $a_0 = 0$. Dividing g gives another integral equation for g with degree $n-1$. This contradicts minimality of n. Hence, $\mathfrak{p}$ must be 0. □

1.8 Let $\varphi : A \longrightarrow B$ be an integral homomorphism.
(a) Show that Specφ : Spec$B \longrightarrow$ Spec$A$ maps a closed point to a closed point, and that any preimage of a closed point is a closed point.

Proof. We use a lemma: $A \subset B$ is integral extension, A and B are integral domains. Then A is a field if and only if B is a field.
Let $\mathfrak{p} \subset B$ be a prime ideal. Then φ induces an embedding $A/\varphi^{-1}(\mathfrak{p}) \hookrightarrow B/\mathfrak{p}$. Since $A/\varphi^{-1}(\mathfrak{p})$ and $B/\mathfrak{p}$ are both integral domains, the lemma applies. Thus, $A/\varphi^{-1}(\mathfrak{p})$ is a field if and only if $B/\mathfrak{p}$ is a field. Equivalently, $\varphi^{-1}(\mathfrak{p})$ is a closed point(i.e. maximal ideal) if and only if $\mathfrak{p}$ is a closed point. □

(b) Let $\mathfrak{p} \in$ Spec$A$. Show that the canonical homomorphism $A_{\mathfrak{p}} \longrightarrow B \otimes_A A_{\mathfrak{p}}$ is integral.

Proof. It is enough to show that any simple tensor $b \otimes_A a$ is integral over $A_\mathfrak{p}$. Consider an integral equation for b, namely $\sum_{i \le n} a_i b^i = 0$ with $a_i \in A$, $a_0 \neq 0$, and $a_n = 1$. Then clearly, $(\sum_{i \le n} a_i b^i) \otimes_A a = 0$. This implies $\sum_{i \le n} a_i (b \otimes_A a)^i = 0$. Hence $b \otimes_A a$ is integral over $A_\mathfrak{p}$. □

(c) Let $T = \varphi(A \backslash \mathfrak{p})$. Let us suppose that φ is injective. Show that T is a multiplicative subset of B, and that $B \otimes_A A_\mathfrak{p} = T^{-1}B \neq 0$. Deduce from this that Specφ is surjective if φ is integral and injective.

Proof. Let $\bar{\varphi}$ be the canonical homomorphism in part (b). Note that if $\mathfrak{q}' \in \mathrm{Max}T^{-1}B$, then $\bar{\varphi}^{-1}(\mathfrak{q}')$ is a maximal ideal in $A_\mathfrak{p}$. In fact, $\bar{\varphi}^{-1}(\mathfrak{q}') = \mathfrak{p}A_\mathfrak{p}$, since $A_\mathfrak{p}$ is a local ring. Consider the diagram,

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow \epsilon & = & \downarrow \epsilon \\ A_\mathfrak{p} & \xrightarrow{\bar{\varphi}} & T^{-1}B \end{array}$$

Apply Spec$(-)$ to this diagram,

$$\begin{array}{ccc} \mathrm{Spec}T^{-1}B & \xrightarrow{\mathrm{Spec}\bar{\varphi}} & \mathrm{Spec}A_\mathfrak{p} \\ \downarrow \mathrm{Spec}\epsilon & = & \downarrow \mathrm{Spec}\epsilon \\ \mathrm{Spec}B & \xrightarrow{\mathrm{Spec}\varphi} & \mathrm{Spec}A \end{array}$$

By commutativity, we have

$$\mathrm{Spec}\epsilon \circ \mathrm{Spec}\bar{\varphi}(\mathfrak{q}') = \mathrm{Spec}\varphi \circ \mathrm{Spec}\epsilon(\mathfrak{q}') = \mathfrak{p}.$$

Hence, Specφ is surjective. □

1.9 Let A be a finitely generated algebra over a field k.

(a) Let us suppose that A is finite over k. Show that Spec$A$ is a finite set, of cardinality bounded from above by the dimension $\dim_k A$ of A as a vector space. Show that every prime ideal of A is maximal.

Proof. Suppose that we could find $r = \dim_k A + 1$ distinct maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$. Then, $\mathfrak{m}_i + \mathfrak{m}_j = A$ for $i \neq j$. We can construct a strictly descending chain of ideals $\mathfrak{m}_1 \supsetneq \mathfrak{m}_1\mathfrak{m}_2 \supsetneq \cdots \supsetneq \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r$. This makes a strictly increasing chain of k-vector spaces $A/\mathfrak{m}_1 \subsetneq A/\mathfrak{m}_1\mathfrak{m}_2 \subsetneq \cdots \subsetneq A/\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r$. However, this implies that $r \le \dim_k A/\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r \le \dim_k A = r-1$, which leads to a contradiction. Hence, the number of distinct maximal ideals in A is bounded above by $\dim_k A$.

Now, we show that every prime ideal $\mathfrak{p}$ in A is indeed maximal. $A/\mathfrak{p}$ is an integral domain which is finite dimensional k-vector space. For any $a \in (A/\mathfrak{p})\backslash\{0\}$, the set $\{1, a, a^2, \cdots\}$ is linearly dependent over k. Thus, a satisfies a polynomial equation $\sum_{i \le N} c_i a^i = 0$ with $c_i \in k$ and $c_0 \neq 0$(This is possible, since $A/\mathfrak{p}$ is an integral domain). This implies that $a \sum_{1 \ge i \ge N} c_i a^{i-1} = -c_0 \in k^\times$. Hence a is invertible in $A/\mathfrak{p}$, giving that $A/\mathfrak{p}$ is a field. □

(b) Show that Spec$k[T_1, \cdots, T_d]$ is infinite if $d \ge 1$.

Proof. We begin with proving that Spec$k[T]$ is infinite. If k is infinite, then $(T-a)$ is maximal ideal for any $a \in k$, thus Spec$k[T]$ is infinite. If k is finite, still we have $\bar{k}$ is infinite. This implies that there are infinitely many irreducible polynomials in $k[T]$, thus Spec$k[T]$ is infinite in this case too.

For the case $d \geq 2$, we define an injective function given by

$$\begin{aligned} \mathrm{Spec}k[T_1] &\longrightarrow \mathrm{Spec}k[T_1, \cdots, T_d] \\ (p(T_1)) &\mapsto (p(T_1), T_2, \cdots, T_d) \end{aligned}$$

Hence, Spec$k[T_1, \cdots, T_d]$ is infinite if $d \geq 1$. □

(c) Show that Spec$A$ is finite if and only if A is finite over k.

Proof. $(\Leftarrow)$ This is done in part (a).
$(\Rightarrow)$ Suppose that A is infinite over k. Then A is transcendental over k. Noether Normalization Theorem applies to obtain $T_1, \cdots, T_d$ which are algebraically independent over k, and an integral extension $k[T_1, \cdots, T_d] \subset k[T_1, \cdots, T_d, t_{d+1}, \cdots, t_r] = A$. Similarly as in (b), we can form an injective map

$$\begin{aligned} \mathrm{Spec}k[T_1, \cdots, T_d] &\longrightarrow \mathrm{Spec}k[T_1, \cdots, T_d, t_{d+1}, \cdots, t_r] \\ P &\mapsto P + (t_{d+1}, \cdots, t_r). \end{aligned}$$

By part (b), it follows that Spec$A$ is infinite. □

2.2 Let $\mathcal{F}$ be a sheaf on X. Let $s, t \in \mathcal{F}(X)$. Show that the set of $x \in X$ such that $s_x = t_x$ is open in X.

Proof. Let $x \in X$, and $s_x = t_x$. Then there exists an open neighborhood U_x of x such that $s|_{U_x} = t|_{U_x}$. For any $y \in U_x$, consider the natural map $\mathcal{F}(X) \longrightarrow \mathcal{F}_y$. Since this natural map is given by $a \in \mathcal{F}(X) \mapsto [X, a]$ where $[X, a]$ denotes the equivalence class represented by (X, a). But, $s|_{U_x} = t|_{U_x}$ implies that $s_y = [X, s] = [U_x, s|_{U_x}] = [U_x, t|_{U_x}] = [X, t] = t_y$. Hence, the set of $x \in X$ such that $s_x = t_x$ is open in X. □

2.7 Let $\mathfrak{B}$ be a base of open subsets on a topological space X. Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X. Suppose that for every $u \in \mathfrak{B}$ there exists a homomorphism $\alpha(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ which is compatible with restrictions. Show that this extends in a unique way to a homomorphism of sheaves $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$. Show that if $\alpha(U)$ is surjective(resp. injective) for every $U \in \mathfrak{B}$, then α is surjective (resp. injective).

Proof. Note that every open set $U \subset X$ is a union of members in $\mathfrak{B}$, say $U = \cup_i U_i$, where $U_i \in \mathfrak{B}$. Let $\beta_i \in \mathcal{F}(U_i)$, and $\beta_j \in \mathcal{F}(U_j)$, and $\beta_i|_{U_i \cap U_j} = \beta_j|_{U_i \cap U_j}$ for all i, j, which data uniquely determine $\beta \in \mathcal{F}(U)(\because \mathcal{F}$ is a sheaf). We know that $g_i = \alpha(U_i)(\beta_i) \in \mathcal{G}(U_i)$, $g_j = \alpha(U_j)(\beta_j) \in \mathcal{G}(U_j)$, and $\alpha(U_i \cap U_j)(\beta_i|_{U_i \cap U_j}) = \alpha(U_i \cap U_j)(\beta_j|_{U_i \cap U_j}) \in \mathcal{G}(U_i \cap U_j)$. Since the homomorphism α is compatible with restrictions on members of $\mathfrak{B}$, we have

$$g_i|_{U_i \cap U_j} = \alpha(U_i \cap U_j)(\beta_i|_{U_i \cap U_j}) = \alpha(U_i \cap U_j)(\beta_j|_{U_i \cap U_j}) = g_j|_{U_i \cap U_j}.$$

Thus, this data uniquely determine $g \in \mathcal{G}(U)(\because \mathcal{G}$ is a sheaf). Define $\alpha(U)(\beta) = g$, then this is the unique extension to a homomorphism of sheaves $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$.

We use the fact that sequence of sheaves $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$ is exact if and only if $\mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x$ is exact for each $x \in X$. In fact, $\alpha(U)$ is surjective(resp. injective) for every $U \in \mathfrak{B}$ imply $\mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow 0$(resp. $0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x$) is exact for each $x \in X$. This in turn, implies that $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$(resp. $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$) is an exact sequence in sheaves. Hence α is surjective(resp. injective). $\square$

2.11 Let X, Y be two ringed topological spaces. We suppose given an open covering $\{U_i\}_i$ of X and morphisms $f_i : U_i \longrightarrow Y$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ (as morphisms) for every i, j. Show that there exists a unique morphism $f : X \longrightarrow Y$ such that $f|_{U_i} = f_i$. This is the glueing of the morphisms f_i.

Proof. We define $f : X \longrightarrow Y$ by $f(x) = f_i(x)$ where $x \in U_i$ for some i. The glueing data $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ gives that $f : X \longrightarrow Y$ is a continuous function. Moreover, continuous function with this property has to be unique.

The morphism of sheaves $f_i^{\#} : \mathcal{O}_Y \longrightarrow f_{i*}\mathcal{O}_{U_i}$ gives the ring homomorphism $f_i^{\#}(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_{U_i}(f_i^{-1}(V)) = \mathcal{O}_X(f^{-1}(V) \cap U_i)$. The glueing data gives that

$$f_i|^{\#}_{U_i \cap U_j}(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V) \cap U_i \cap U_j) = f_j|^{\#}_{U_i \cap U_j}(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V) \cap U_i \cap U_j).$$

Together with the compatibility of restriction, $\alpha \in \mathcal{O}_Y(V) \mapsto a_i \in \mathcal{O}_X(f^{-1}(V) \cap U_i)$ has the glueing data $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$. Since $\mathcal{O}_X(f^{-1}(V) \cap -)$ is a sheaf, we obtain a unique $a \in \mathcal{O}_X(f^{-1}(V))$ such that $a|_{U_i} = a_i$ for each i. Define $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ by $\alpha \in \mathcal{O}_Y(V) \mapsto a \in \mathcal{O}_X(f^{-1}(V))$. Then $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is the desired morphism of ringed topological spaces. $\square$

3.19 Let K be a number field. Let $\mathcal{O}_K$ be the ring of integers of K. Using the finiteness theorem of the class group $\mathrm{cl}(K)$, show that every open subset of $\mathrm{Spec}\mathcal{O}_K$ is principal. Deduce from this that every open subscheme of $\mathrm{Spec}\mathcal{O}_K$ is affine.

Proof. Let U be an open subset of $\mathrm{Spec}\mathcal{O}_K$, then $(\mathrm{Spec}\mathcal{O}_K) \backslash U = V(I)$ for some ideal $I \subset \mathcal{O}_K$. This ideal I belongs to an ideal class $[Q] \in \mathrm{cl}(K)$. Thus, there are $a, b \in \mathcal{O}_K \backslash \{0\}$ such that $aI = bQ$. This implies that $I = (b/a)Q$, and there is some $n \geq 0$ such that $I^n = (b/a)^n Q^n = \gamma\mathcal{O}_K$ for some $\gamma \in K$ (In particular, we can take $n = \#\mathrm{cl}(K)$). Since $I^n \subset \mathcal{O}_K$, we have $\gamma\mathcal{O}_K \subset \mathcal{O}_K$. This proves that $\gamma \in \mathcal{O}_K$, and that I^n is a principal ideal. Hence, $V(I) = V(I^n) = V(\gamma\mathcal{O}_K)$ is a principal closed subset, or equivalently, $U = D(\gamma)$ is a principal open set.

Therefore, every open subscheme $U = D(\gamma)$ of $\mathrm{Spec}\mathcal{O}_K$ is isomorphic to $\mathrm{Spec}\mathcal{O}_K[1/\gamma]$, which is affine. $\square$

4.1 Let k be a field and $P \in k[T_1, \cdots, T_n]$. Show that $\mathrm{Spec}(k[T_1, \cdots, T_n]/(P))$ is reduced (resp. irreducible; resp. integral) if and only if P has no square factor (resp. admits only one irreducible factor; resp. is irreducible).

Proof. Let $A = k[T_1, \cdots, T_n]/(P)$, and $\mathfrak{q} \in \mathrm{Spec}A$. Then the following are equivalent:
(1) $\mathcal{O}_{A,\mathfrak{q}} = A_{\mathfrak{q}}$ is reduced for all $\mathfrak{q} \in \mathrm{Spec}A$.
(2) Only nilpotent element in $A_{\mathfrak{q}}$ is 0.
(3) $\cap\{\mathfrak{q}' \in \mathrm{Spec}k[T_1, \cdots, T_n] : (P) \subset \mathfrak{q}' \subset \mathfrak{q}\} = (P)$.
(4) P has no square factor.
Equivalence of (1),(2), and (3) is direct from definition. We show the equivalence of (3), and (4). This follows from

$$\cap\{\mathfrak{q}' \in \mathrm{Spec}k[T_1, \cdots, T_n] : (P) \subset \mathfrak{q}' \subset \mathfrak{q}\} = (P_0),$$

where P_0 is the product of all irreducible factors of P.

Remark that $\mathrm{Spec}A \neq \emptyset$ is irreducible if and only if $\sqrt{(0)}$ is prime. Equivalently,

$$\cap\{\mathfrak{q}' \in \mathrm{Spec}k[T_1, \cdots, T_n] : (P) \subset \mathfrak{q}'\} = (P_0)$$

is prime. Similarly as before, P_0 is the product of all irreducible factors of P. Now, (P_0) is prime if and only if there is only one irreducible factor in P.

A is integral if and only if A is both reduced and irreducible. Therefore, A is integral if and only if P has no square factor and admits only one irreducible factor, i.e. P is irreducible.

□